Answers to P-Set # 04, (18.353/12.006/2.050)j
MIT (Fall 2024)

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1.1 Statement: Find a conserved quantity #03 (and sketch phase portrait)

Find a conserved quantity for the system and sketch the phase portrait.

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x + e^{-y} \quad \text{and} \quad \frac{\mathrm{d}y}{\mathrm{d}t} = -y, \qquad (1.1)$$

Include an analysis of any fixed point that occurs.

Hint: write the equation for the orbits, $\frac{\mathrm{d}x}{\mathrm{d}y} = g(x,\, y)$, and solve it.† *It will have a constant of integration, say β. This constant yields the conserved quantity, upon solving $x = X(y, \beta)$ for β as a function of x and y.*

† Note: write the equation for $\frac{dx}{dy}$, not $\frac{dy}{dx}$. The equation cannot be solved by separation of variables; however, upon multiplication by y you should be able to solve it.

1.2 Answer: Find a conserved quantity #03 (and sketch phase portrait)

We begin by writing the equation for the orbits. That is

$$\frac{dx}{dy} = -\frac{x + e^{-y}}{y}. \tag{1.2}$$

Upon multiplying by y, this yields $y\,\frac{dx}{dy} + x = -e^{-y}$. Hence $\frac{d}{dy}(x\,y) = -e^{-y}$, which can be integrated to obtain

$$x = \frac{\beta + e^{-y}}{y}, \tag{1.3}$$

where $\boldsymbol{\beta}$ is a constant. **This gives all the orbits, with the exception of those that satisfy $y \equiv 0$** (missed because to get (1.2) we divided by y). Writing β in terms of x and y gives a conserved quantity

$$\boldsymbol{\beta = x\,y - e^{-y}}. \tag{1.4}$$

Note: it is easy to check directly that β, as in (1.4) is conserved. The system has a **single fixed point at $x = -1$ and $y = 0$.** Then $x = -1 + \delta x$ and $y = \delta y$ yields

$$\beta = -1 + \left(\delta x - \tfrac{1}{2}\delta y\right)\delta y + \cdots = -1 + (\delta x - \frac{1}{4}\delta y)^2 - (\delta x - \frac{3}{4}\delta y)^2 + \ldots, \tag{1.5}$$

which shows that **the critical point is a saddle.**

The **phase portrait** follows from the level curves for $\boldsymbol{\beta}$, given for any $\beta \neq -1$ by (1.3); with either $-\infty < y < 0$ or $0 < y < \infty$. The **level line $\boldsymbol{\beta = -1}$ is special,** because it is the one going through the saddle, and has four components: Two components are given (again) by (1.3), and correspond to the *stable manifolds of the saddle*. The other two components are given by $y = 0$, with either $-\infty < x < -1$ or $-1 < x < \infty$, and correspond to the *stable manifolds of the saddle.*‡ This is **all summarized in figure** 1.1**,** where we have also added the nullclines ($x = -e^{-y}$ where $\dot{x} = 0$, and $y = 0$, where $\dot{y} = 0$).

‡ It is interesting that, while in (1.3) the $y \equiv 0$ orbits are lost (as pointed out earlier), they are recovered in (1.4).

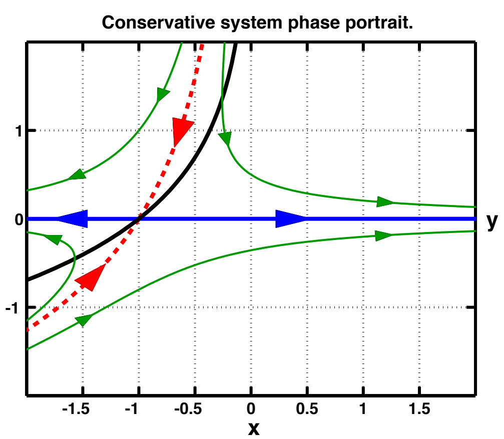

Figure 1.1: Phase portrait for "Find a conserved quantity #03". The arrows indicate the flow direction. Furthermore:

- The saddle stable manifolds are the dashed red lines.
- The saddle unstable manifolds are the solid blue lines.
- The nullcline $\dot{y} = 0$ coincides with the unstable manifold.
- The nullcline $\dot{x} = 0$ is the solid black line.
- Typical orbits are shown by solid green lines.

2 Hamiltonian for the Lotka-Volterra predator-prey model

2.1 Statement: Hamiltonian for the Lotka-Volterra predator-prey model

In non-dimensional variables, the Lotka-Volterra predator-prey model can be written in the form

$$\frac{dx}{dt} = x - x\,y = x\,(1 - y) \qquad \text{and} \qquad \frac{dy}{dt} = -\mu\,y + \mu\,x\,y = \mu\,y\,(x - 1), \tag{2.1}$$

where x is the prey (e.g.: mice), y is the predator (e.g.: owls), and $\mu > 0$ is a constant.

A. Discuss the biological meaning of each of the terms in the model.[1] That is: justify the model. In particular: (i) Comment on unrealistic assumptions. (ii) Explain the meaning of the non-dimensional constant μ.

B. Show that, in the first quadrant[2] $(x,\, y > 0)$ the equations can be written in the form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \alpha\, H_y \qquad \text{and} \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = -\alpha\, H_x, \tag{2.2}$$

for some functions $H = H(x,\, y)$ and $\alpha = \alpha(x,\, y) > 0$. Thus, upon an appropriate re-scaling of time along the solutions — given by $\boldsymbol{\alpha}\, \mathbf{d}\boldsymbol{t} = \mathbf{d}\boldsymbol{t}_{\text{new}}$, the system becomes Hamiltonian.

C. Show that, in the first quadrant, $\boldsymbol{H}$ **is concave** $(H_{xx} + H_{yy} < 0)$, with a **unique strict global maximum at** $\boldsymbol{x = y = 1}$. Furthermore: $\boldsymbol{H \to -\infty}$ **along the edges of the quadrant, or as** $\boldsymbol{x^2 + y^2 \to \infty}$**. What does this imply about the solutions? Is this biologically realistic?**

Hint for part B. Since H must be a conserved quantity, the first step here is to find the conserved quantities. This you can do by looking at the equation $\frac{\mathrm{d}y}{\mathrm{d}x} = g(x,\, y)$, which can be integrated (you can get g from (2.1)). The integration involves a constant of integration — which thus is constant on each orbit. Hence, expressing this constant of integration as a function of x and y gives you a conserved quantity. Finally: note that if E is a conserved quantity, then so is $f(E)$ for any f with a non-zero derivative; thus you can re-cast an awkward looking E into a more convenient form.

Hint for part C. First show that H is concave in the quadrant, and that it goes to $-\infty$ at the edges and at infinity. It is a consequence of this that it has a unique strict global maximum.

2.2 Answer: Hamiltonian for the Lotka-Volterra predator-prey model

The Lotka-Volterra predator-prey model is popular with textbook writers because it is simple, but it should not be taken too seriously (since it involves rather drastic simplifying assumptions). Further, it is not structurally stable: Because it is conservative, it predicts a continuous family of neutrally stable periodic cycles in the populations. On the other hand, real predator-prey cycles typically have a characteristic amplitude. In other words, realistic models should predict a single closed orbit, or perhaps finitely many, but not a continuum of them.

Answer to item A

A1. The **first term in the first equation represents** the *growth of the prey in the absence of the predators* (due to the balance between the natural birth and death ratios, which are assumed to be proportional to the current population). The growth rate has been normalized to one, by selecting an appropriate unit of time (we assume that the prey has enough food, and it is healthy, so that it actually grows in the absence of predators).

The assumption that the prey growth rate, in the absence of predators, is a constant is rather drastic. The prey will in fact interfere with each other, as they compete for the same food and space resources. Only if the prey population is small (relative to the availability of food and space) does this make sense.

A2. The **first term in the second equation represents** the *net growth by the predators in the absence of prey* (negative because they have no food). Again, it is not clear why this rate should be a constant. Worse still: why should the decay of predator numbers be exponential in the absence of food? They will starve, and this is a process that will reach a conclusion in a finite (and fairly short) time.

Since in these equations the growth rate of the prey in the absence of predators has been normalized to one, the **coefficient $\boldsymbol{\mu}$ is** simply *the ratio of the (dimensional) growth rate of the predators to the (dimensional) growth rate of the prey.*

A3. The **second term term in the first equation represents** the *deaths of prey as they are hunted by the predators.* The assumption here is that this process is directly proportional to the number of prey and predators. Perhaps

[1] Note that the model makes some rather drastic simplifying assumptions.
[2] The only region of biological interest.

too drastic a simplification for a complicated process (the hunt) that entails the active involvement of both prey and predator. It is not just random collisions of balls rolling on a billiard table!

The coefficient for this term should be negative — see **A5** below.

A4. The **second term in the second equation** is the counterpart of the second term in the first equation, except that it should have a positive coefficient, since *the predator population should grow when there is more prey available.*

A5. Finally: **why is the coefficient of the nonlinear term in the 1-st equation -1, and the coefficient of the corresponding term in the 2-nd equation μ?** The answer is that we can scale the variables x and y [$x \to c_x\, x$ and $y \to c_y\, y$, with $c_x, c_y > 0$ constants], and make the coefficients of the nonlinear terms anything we want.[3] The choice made here simplifies the answer to item **B.**

As a further problem of the model, note that only in a controlled laboratory experiment can we hope to even get close to a situation where factors other than the prey and predators do not play a role.

Answer to item B

From (2.1)
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mu\, y\,(x-1)}{x\,(1-y)}.$$

Separating variables:
$$\mathrm{d}\,\{\ln(y) - y\} = \mu\,\mathrm{d}\,\{x - \ln(x)\}.$$

From this the conserved quantity
$$\boldsymbol{H = -y - \mu\, x + \ln(y) + \mu\, \ln(x)}, \qquad (2.3)$$

follows. Then **(2.2) applies with**
$$\boldsymbol{\alpha = x\, y > 0}. \qquad (2.4)$$

Answer to item C

On the first quadrant
$$\boldsymbol{H_{xx} + H_{yy} = -\mu/x^2 - 1/y^2 < 0}.$$

Hence $\boldsymbol{H}$ **is concave there.** Furthermore, it is easy to see that $H \to -\infty$ as either $\boldsymbol{x \to 0}$, or $\boldsymbol{y \to 0}$, or $\boldsymbol{x^2 + y^2 \to \infty}$. It follows that $\boldsymbol{H}$ *has a unique, strict, global maximum in the first quadrant.* To calculate its position, we search for where $\nabla H = 0$. i.e.:
$$H_x = -\mu + \mu/x = 0 \text{ and } H_y = -1 + 1/y = 0.$$

It follows that **the maximum occurs at $\boldsymbol{x = y = 1}$.**

Hence all the level curves of $\boldsymbol{H}$ are closed curves enclosing $\boldsymbol{x = y = 1}$. **It follows that all the solutions in the first quadrant yield periodic closed orbits around the center at $\boldsymbol{x = y = 1}$.** As pointed out at the beginning of the answer, this is not biologically realistic.

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3 Nullclines versus stable manifolds

3.1 Statement: Nullclines versus stable manifolds

In the lectures we considered the example
$$\frac{\mathrm{d}x}{\mathrm{d}t} = x + e^{-y} \quad \text{and} \quad \frac{\mathrm{d}y}{\mathrm{d}t} = -y, \qquad (3.1)$$
for which we drew its phase plane portrait.[4]

There is a somewhat confusing aspect of the phase portrait of this system: The nullcline $\dot{x} = 0$ has a similar shape and location as the stable manifold of the saddle, but they are not the same curve (which makes drawing the curves in the blackboard hard to do, even moderately accurate). **To clarify the relation between the two curves, plot both of them on the same phase portrait**, as follows:

[3] But we cannot change their signs.

[4] You can also find it in Matt Durey's Lectures_9_11_pre.pdf (bottom of page 1 and top of page 2), which is posted in the course web page. This is also Example 6.1.1 in Strogatz's book (First edition: pp. 147-148. Second edition: pp. 148-149).

1 Use a computer to do the plot, generating the stable manifold by numerically solving the equation.†

2 Find an explicit formula for the stable manifold, and then do a sketch of the phase portrait.†

Hint. Write the equation for $\frac{dy}{dx}$*, and then solve it.* A sketch without analytical justification is not acceptable.

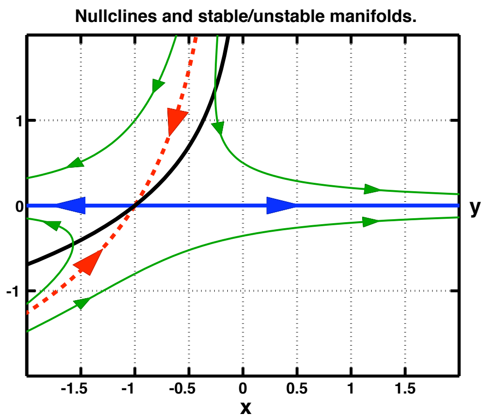
† Note that the nullcline is known analytically.

3.2 Answer: Nullclines versus stable manifolds

See figure 3.1. The nullclines for the system in (3.1) are as follows:

A. The nullcline $\dot{\boldsymbol{y}} = \mathbf{0}$ is the curve $y = 0$. It happens to coincide with the unstable manifold for the saddle.

B. The nullcline $\dot{\boldsymbol{x}} = \mathbf{0}$ is the curve $x = -\exp(-y)$. It has a shape similar to the stable manifold for the saddle, but it does not coincide with it, as the computer plot in figure 3.1 shows.



The arrows indicate flow direction.

- Stable manifold. Thick dashed red line.
- Unstable manifold. Thick solid blue line.
- Nullcline $\dot{\boldsymbol{y}} = \mathbf{0}$. Thick solid blue line.
- Nullcline $\dot{\boldsymbol{x}} = \mathbf{0}$. Thick solid black line.
- Typical orbits. Thin solid green lines.

Figure 3.1: Nullclines versus stable/unstable manifolds for the system in (3.1).

C. From (3.1) we see that
$$\frac{\mathrm{d}x}{\mathrm{d}y} = -\frac{x}{y} - \frac{1}{y}\,e^{-y} \quad \text{if and only if} \quad \frac{\mathrm{d}}{\mathrm{d}y}xy = -e^{-y}. \tag{3.2}$$

Thus the orbits have the form
$$x = \frac{\beta + e^{-y}}{y}, \tag{3.3}$$
where β is a constant — *this misses the unstable manifold* $y \equiv 0$ because dividing by y, as we did to obtain (3.2), requires $y \neq 0$.

From this we can easily obtain the **stable manifold for the saddle.** That is $\boldsymbol{x = \frac{-1+e^{-y}}{y}}$, as follows from the fact that for $\beta = -1$, $x \to -1$ as $y \to 0$ (for any other value of β, $x \to \pm\infty$ as $y \to 0$).

Finally: as expected, *using (3.3) to plot the phase portrait yields no discernible difference with* figure 3.1.

4 Reversible system that is not conservative

4.1 Statement: Reversible system that is not conservative

Give an example of a reversible system that is not conservative.

Hint. *Remember that a phase plane system*
$$\dot{x} = f(x,\, y) \quad \text{and} \quad \dot{y} = g(x,\, y), \tag{4.1}$$
is reversible if, for example, f is odd and g is even in y — that is: $f(x,\,-y) = -f(x,\,y)$ and $g(x,\,-y) = g(x,\,y)$. In this case the change $t \to -t$ and $y \to -y$ leaves the system invariant. Furthermore, we know that conservative systems cannot have sinks or sources. Now, ask yourself: what systems have exactly the opposite property [almost every critical point is either a source or a sink]. Then produce a system of this kind, with f odd and g even.

4.2 Answer: Reversible system that is not conservative

Consider a **gradient system:** $\dot{\boldsymbol{x}} = -\boldsymbol{V_x}$ and $\dot{\boldsymbol{y}} = -\boldsymbol{V_y}$, where $\boldsymbol{V} = \boldsymbol{V}(\boldsymbol{x},\, \boldsymbol{y})$ is some given function. For systems of this type, any local minimum of V is a sink, and any local maximum is a source. Now **take $\boldsymbol{V}$ odd in $\boldsymbol{y}$; then the system is also reversible.** Hence any gradient system where V is odd in y, and where V has either a local maximum or a local minimum, provides the required example. One is $\boldsymbol{V = y\,(1 - y^2)/(1 + x^2)}$, which has a local minimum at $(x,\, y) = (0,\, -1/\sqrt{3})$, and a local maximum at $(x,\, y) = (0,\, 1/\sqrt{3})$.
This example should puzzle/shock you a little bit, at least. A gradient system in 2-D is, basically, reproducing the path followed by water going down a mountain range described by the potential $z = V(x,\, y)$. How can such a system ever be "reversible"? This shows that sometimes (not unusual) the mathematical abstraction of an intuitive notion may not capture all of the intuition, but only some aspect of it. This is why proofs (or, at least, good and careful arguments) are needed before you make conclusions from a mathematical formulation. Intuition often carries with it implicit assumptions that you may not be aware of, and may be missing in the mathematical formulation. It may also be that your intuition is wrong.

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5 Reversible system #03 (a "strange" reversible system)

5.1 Statement: Reversible system #03

Consider the system

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -a\,E_y - b\,E_x \\ a\,E_x - b\,E_y \end{pmatrix} = \begin{pmatrix} -b & -a \\ a & -b \end{pmatrix}\begin{pmatrix} E_x \\ E_y \end{pmatrix} = \boldsymbol{\mathcal{A}}\,\nabla \boldsymbol{E}, \tag{5.1}$$

where $\boldsymbol{E} = \boldsymbol{E}(\boldsymbol{x},\, \boldsymbol{y})$,
$\boldsymbol{a} = \boldsymbol{a}(\boldsymbol{x},\, \boldsymbol{y})$, and
$\boldsymbol{b} = \boldsymbol{b}(\boldsymbol{x},\, \boldsymbol{y})$, are given by

$$E = \frac{1}{2}\,y^2 - x^2 + \frac{1}{2}\,x^4, \quad a = 1, \quad b = \gamma\,x\,(1 - x^2)^2. \tag{5.2}$$

Here γ **is a positive constant.**

Now **do the following:**

- **a.** Show that $\boldsymbol{L} = (-1,\, 0)$, $\boldsymbol{C} = (0,\, 0)$, and $\boldsymbol{R} = (1,\, 0)$, are **fixed points** — and *there are no others.*
  Hint. If you do all the algebraic operations that define the system, and then try to find the zeros, you will get a mess! Instead, notice that the fixed points are (exactly) the points where $\nabla \boldsymbol{E}$ vanishes. Explain why!
- **b.** Show that **the system has a left-right ($x \mapsto -x$) time reversal symmetry.**
  Hint: $\boldsymbol{E}$ and $\boldsymbol{a}$ are even in both $\boldsymbol{x}$ and $\boldsymbol{y}$, while $\boldsymbol{b}$ is odd in $\boldsymbol{x}$ and even in $\boldsymbol{y}$.
- **c.** Show that **both $\boldsymbol{L}$ and $\boldsymbol{R}$ are linear centers,** while **$\boldsymbol{C}$ is a saddle.**
  When computing the Jacobian at the fixed points, it is useful to notice that both $\nabla \boldsymbol{E}$ and $\boldsymbol{b}$ vanish there.
- **d.** Use a computer generated phase portrait to show that **both $\boldsymbol{L}$ and $\boldsymbol{R}$ are actually spirals!** — $\boldsymbol{R}$ is stable, while $\boldsymbol{L}$ is unstable. You should also check that **$\boldsymbol{C}$ is a saddle.** **Use $\gamma = 0.5$ for the portrait.**
- **e.** Recall now the theorem: *linear centers are true nonlinear centers for reversible systems.* **Why is it that items b and d do not contradict this theorem?**
- **f.** **Optional. Prove,** analytically, that: **$\boldsymbol{R}$ is a stable spiral** and **$\boldsymbol{L}$ is an unstable spiral.**
  Hint. Compute $\dot{\boldsymbol{E}}$, and notice that $\boldsymbol{L}$ and $\boldsymbol{R}$ are local minimums of $\boldsymbol{E}$.

5.2 Answer: Reversible system #03

1. $\boldsymbol{\mathcal{A}}$ in (5.1) has determinant $\boldsymbol{a^2 + b^2 \geq 1}$, so it is non-singular.
   Thus a fixed point can happen if and only if $\nabla \boldsymbol{E} = \boldsymbol{0}$. Since
   $$E_x = -2\,x + 2\,x^3 \quad \text{and} \quad E_y = y, \tag{5.3}$$
   the result in item **a** follows.

2. Consider the transformation $(\boldsymbol{x} \mapsto -\boldsymbol{x},\ \boldsymbol{y} \mapsto \boldsymbol{y},\ \boldsymbol{t} \mapsto -\boldsymbol{t})$. Given the symmetry properties of E, a, and b, listed in the hint for item **b**, it is easy to see that (i) $a\,E_y$ and $b\,E_x$ are invariant under the map. (ii) $a\,E_x$ and $b\,E_y$ switch sign under the map. This shows that the system is invariant under map, which proves the result in item **b**.

3. Given that both ∇E and b vanish at the critical points, the Jacobian at any of them has the form

$$\mathcal{J} = a \begin{pmatrix} -E_{yx} & -E_{yy} \\ E_{xx} & E_{yx} \end{pmatrix}. \tag{5.4}$$

Evaluation at the fixed points yields

$$\mathcal{J}_L = \mathcal{J}_R = \begin{pmatrix} 0 & -1 \\ 4 & 0 \end{pmatrix} \quad \text{and} \quad \mathcal{J}_C = \begin{pmatrix} 0 & -1 \\ -2 & 0 \end{pmatrix}. \tag{5.5}$$

Thus the result in item **c** follows.

4. The computer generated plot in Fig. 5.1 shows that ***L*** **is an unstable spiral,** ***C*** **is a saddle, and** ***R*** **is a stable spiral** — i.e.: item **d**.

5. There is no contradiction with the theorem: *linear centers are true nonlinear centers for reversible systems.* **The theorem requires the critical points to be on the symmetry line for the reversibility,** not true here!

6. It is easy to check (chain rule) that (5.1) yields

$$\dot{E} = -b\,(\nabla E)^2. \tag{5.6}$$

Now: **(6a)** $\boldsymbol{L}$ and $\boldsymbol{R}$ are local minimums of $\boldsymbol{E}$,
(6b) $b\,(\nabla E)^2 < 0$ near $\boldsymbol{L}$ (vanishes at L only), and **(6c)** $b\,(\nabla E)^2 > 0$ near $\boldsymbol{R}$ (vanishes at R only). Hence $\boldsymbol{R}$ is an attractor, while $\boldsymbol{L}$ is a repeller. Since the swirling motion around/near a linear center is not destroyed by nonlinearity, it follows that: ***R*** **is a stable spiral** and ***L*** **is an unstable spiral** — i.e.: item **f**.

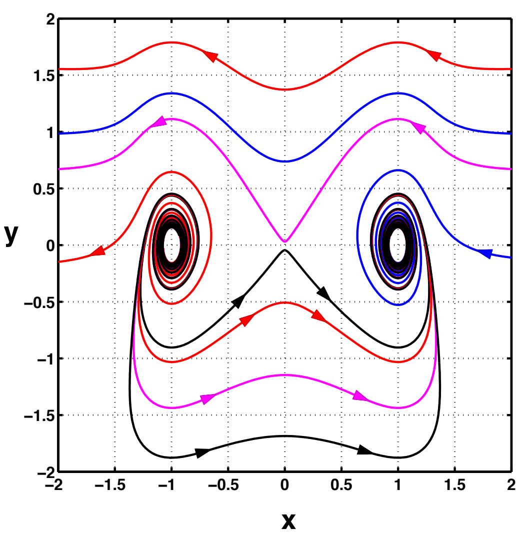

Figure 5.1: Phase portrait for the "strange" reversible system #03, using $\gamma = 0.5$ (as suggested). Clearly: the left fixed point is an unstable spiral, the center fixed point is a saddle, and the right fixed point is a stable spiral. The portrait is clearly invariant under the map: $\boldsymbol{x} \mapsto -\boldsymbol{x}$, $\boldsymbol{y} \mapsto -\boldsymbol{y}$, and $\boldsymbol{t} \mapsto -\boldsymbol{t}$ — as expected.

6 Systems both gradient and Hamiltonian 02

6.1 Statement: Systems both gradient and Hamiltonian 02

Consider a phase plane system which is both gradient and Hamiltonian:

$$\dot{x} = -V_x = \quad H_y, \tag{6.1}$$
$$\text{and} \quad \dot{y} = -V_y = -H_x, \tag{6.2}$$

for some potential $V = V(x,\,y)$ and Hamiltonian $H = H(x,\,y)$.

a. **Show that both V and H satisfy the Laplace equation: $V_{xx} + V_{yy} = H_{xx} + H_{yy} = 0$.**
You may assume that both V and H are twice continuously differentiable.

b. Let $z = x + iy$ and consider the system

$$\frac{\mathrm{d}z^*}{\mathrm{d}t} = i\, e^z, \qquad (6.3)$$

where * denotes the complex conjugate. Show that this system has the form in (6.1–6.2). What are the corresponding V and H?
Recall that $e^z = e^x\,(\cos y + i \sin y)$.

6.2 Answer: Systems both gradient and Hamiltonian 02

a. We have $H_{xx} = (V_y)_x = (V_x)_y = -H_{yy}$, hence $\Delta H = 0$. A similar calculation shows that $\Delta V = 0$.

b. $e^z = e^x\, \cos y + i\, e^x\, \sin y$. Thus (6.3) yields

$$\dot{x} = -e^x\, \sin y \quad \text{and} \quad \dot{y} = -e^x\, \cos y. \qquad (6.4)$$

It follows that $H = e^x\, \cos y$ and $V = e^x\, \sin y$.

Remark. A Hamiltonian system has centers and saddles only, while a gradient system can only have sinks and sources. Hence, *how can a system be both gradient and Hamiltonian?* Answer: by having saddles only; which are allowed by both systems. Or, as in the example in (6.4), by having no critical points at all.
Note: if you replace e^z in (6.4) by any other analytic function (e.g.: $z\, e^z$, or a polynomial), the result will also be a system that is both gradient and Hamiltonian. For, say $z\, e^z$, check that any critical points are saddles.

THE END.