Answers to P-Set # 03, (18.353/12.006/2.050)j
MIT (Fall 2024)

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1.1 Statement: Attracting and Liapunov stable v2

Recall the *definitions for the various types of stability* that concern critical points:
Let $\mathbf{x}^*$ be a fixed point of the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$. Then:

1. $\mathbf{x}^*$ is **attracting** if there is a $\delta > 0$ such that $\lim_{t\to\infty} x(t) = \mathbf{x}^*$ whenever $\|\mathbf{x}(0) - \mathbf{x}^*\| < \delta$. That is: any trajectory that starts within δ of $\mathbf{x}^*$ *eventually* converges to $\mathbf{x}^*$. Note that trajectories that start nearby $\mathbf{x}^*$ *need not stay close in the short run*, but *must approach* $\mathbf{x}^*$ *in the long run.*
2. $\mathbf{x}^*$ is **Liapunov stable** if for each $\epsilon > 0$, there is a $\delta > 0$ such that $\|\mathbf{x}(t) - \mathbf{x}^*\| < \epsilon$ for $t > 0$, whenever $\|\mathbf{x}(0) - \mathbf{x}^*\| < \delta$. Thus, trajectories that start within δ of $\mathbf{x}^*$ stay within ϵ of $\mathbf{x}^*$ for all $t > 0$.
   In contrast with attracting, Liapunov stability requires nearby trajectories to remain close for all $t > 0$.
3. $\mathbf{x}^*$ **is asymptotically stable** if it is *both* attracting and Liapunov stable.
4. $\mathbf{x}^*$ is **repeller** if there exist $\epsilon > 0$ and $\delta > 0$ such that: if $0 < \|\mathbf{x}(0) - \mathbf{x}^*\| < \delta$, then (after some critical time) it will be $\|\mathbf{x}(t) - \mathbf{x}^*\| > \epsilon$ (i.e., for $t > t_c$). Repellers are a special kind of *unstable* critical points.

For each of the following systems, decide whether the origin is attracting but not Liapunov stable, Liapunov stable, asymptotically stable, repeller, or unstable but not a repeller.

a) $\dot{x} = y$ and $\dot{y} = -4x$.
b) $\dot{x} = 2y$ and $\dot{y} = x$.
c) $\dot{x} = 0$ and $\dot{y} = x$.
d) $\dot{x} = 0$ and $\dot{y} = -y$.
e) $\dot{x} = -x$ and $\dot{y} = -5y$.
f) $\dot{x} = x$ and $\dot{y} = y$.

1.2 Answer: Attracting and Liapunov stable v2

a. $\dot{x} = y$ and $\dot{y} = -4x \Longrightarrow E = 4x^2 + y^2 =$ constant on orbits. Center: **Liapunov stable, but not attracting.**

b. $\dot{x} = 2y$ and $\dot{y} = x \Longrightarrow E = x^2 - 2y^2 =$ constant on orbits. Saddle: **unstable, but not a repeller.**†

c. $\dot{x} = 0$, $\dot{y} = x$. The solutions are $x \equiv x_0$ and $y = y_0 + x_0 t$. The origin is **unstable; a repeller.**

d. $\dot{x} = 0$ and $\dot{y} = -y$. The solutions are $x \equiv x_0$ and $y = y_0 e^{-t}$. **Liapunov stable, but not attracting.**

e. $\dot{x} = -x$ and $\dot{y} = -5y$. Stable node: **asymptotically stable.**

f. $\dot{x} = x$ and $\dot{y} = y$. **Unstable** node; **a repeller.**

† There are orbits that approach the origin as $t \to \infty$. Which ones?

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2 Classify fixed points #02 (Linearize and find the fixed points type)

2.1 Statement: Classify fixed points #02

Consider the system $\dot{x} = y - y^3$, $\dot{y} = -x - y^2$. Then

a. Find the fixed points.

b. Linearize the equation near each fixed point, and classify the fixed points (saddles, stable nodes, etc.).

2.2 Answer: Classify fixed points #02

a. The fixed points are the solutions to $\boldsymbol{y - y^3 = 0 = x + y^2}$.
Thus the fixed points $\boldsymbol{(x,\ y)}$ are: $\boldsymbol{(0,\ 0)}$, $\boldsymbol{(-1,\ 1)}$, and $\boldsymbol{(-1,\ -1)}$.

b. The Jacobian for the system is $\mathcal{J} = \begin{pmatrix} 0 & 1 - 3\,y^2 \\ -1 & -2\,y \end{pmatrix}$.
Thus:
At $(0,\ 0)$, $\mathcal{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ has eigenvalues $\lambda_\pm = \pm \mathrm{i}$. .... **Linear center.**

At $(-1,\ 1)$, $\mathcal{J} = \begin{pmatrix} 0 & -2 \\ -1 & -2 \end{pmatrix}$ has eigenvalues † $\lambda_\pm = -1 \pm \sqrt{3}$. .... **Saddle.**

At $(-1,\ -1)$, $\mathcal{J} = \begin{pmatrix} 0 & -2 \\ -1 & 2 \end{pmatrix}$ has eigenvalues ‡ $\lambda_\pm = 1 \pm \sqrt{3}$. .... **Saddle.**

† The characteristic equation is $\boldsymbol{0 = \lambda^2 + 2\,\lambda - 2}$.

† The characteristic equation is $\boldsymbol{0 = \lambda^2 - 2\,\lambda - 2}$.

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3 Eliminate the cubic term

3.1 Statement: Eliminate the cubic term

Consider the system

$$\frac{dX}{dt} = RX - X^2 + aX^3 + O(X^4), \quad (3.1)$$

where $\boldsymbol{R \neq 0}$. The objective here is to find a new variable $\boldsymbol{x}$ such that the system transforms into

$$\frac{dx}{dt} = Rx - x^2 + O(x^4). \quad (3.2)$$

This is a big improvement: the cubic term is eliminated and the error term [1] is bumped to fourth order. [2] The procedure (sketched next) can be generalized to higher orders. [3]
Let $\boldsymbol{x = X + bX^3 + O(X^4)}$, where $\boldsymbol{b}$ is chosen later to eliminate the cubic term in the differential equation for x. This is called a **near-identity transformation**, since x and X are almost equal: they differ by a cubic term. [4] Now we need to rewrite the system in terms of x; this calculation requires a few steps.

1. Show that the near-identity transformation can be inverted to yield $\boldsymbol{X = x + cx^3 + O(x^4)}$, and solve for c.
2. Write $\boldsymbol{\dot{x} = \dot{X} + 3bX^2\dot{X} + O(X^4)}$, and substitute for X and $\dot{X}$ on the right hand side, so that everything depends only on x. Multiply the resulting series expansions and collect terms, to obtain $\boldsymbol{\dot{x} = Rx - x^2 + kx^3 + O(x^4)}$, where k depends on a, b, and R.
3. Choose $\boldsymbol{b}$ so that $\boldsymbol{k = 0}$.
4. **Explain where $R \neq 0$ is used.**

[1] Here the error is relative to approximating (3.1) by the saddle-node normal form $\boldsymbol{\dot{x} = R\,x - x^2}$.
[2] Obviously we assume that both X and x are small.
[3] That is, one can successively eliminate all the higher order terms: $O(x^3)$, $O(x^4)$, ..., etc.
[4] We have skipped the quadratic term X^2, because it is not needed — you should check this later.

3.2 Answer: Eliminate the cubic term

We now fill in the steps outlined in the problem statement:

1. Replacing $X = x + cx^3 + O(x^4)$ into $x = X + bX^3 + O(X^4)$ yields:

   $x = (x + cx^3) + b(x + cx^3)^3 + O(x^4) = x + (c + b)x^3 + O(x^4)$. **Thus, it must be $\boldsymbol{c = -b}$.**

   This process can be carried out to any order. If $x = X + aX^2 + bX^3 + cX^4 + \cdots + O(X^N)$, we can find the inverse transformation $X = x + Ax^2 + Bx^3 + Cx^4 + \cdots + O(x^N)$ by successively selecting the coefficients A, B, C, ... to eliminate the coefficients of the powers x^2, x^3, x^4, ... in a substitution like the one above.

2. Write $\dot{x} = \dot{X} + 3bX^2\dot{X} + O(X^4)$, use equation (3.1) to eliminate $\dot{X}$ on the right hand side, and substitute $X = x - bx^3 + O(x^4)$ — as obtained in the first step — to eliminate X. This yields:

$$\begin{aligned}
\dot{x} &= \dot{X} + 3bX^2\dot{X} + O(X^4) \\
&= (RX - X^2 + aX^3) + 3bX^2(RX - X^2 + aX^3) + O(X^4) \\
&= RX - X^2 + (a + 3bR)X^3 + O(X^4) \\
&= R(x - bx^3) - (x - bx^3)^2 + (a + 3bR)(x - bx^3)^3 + O(x^4) \\
&= Rx - x^2 + kx^3 + O(x^4), \quad \text{where } \boldsymbol{k = a + 2\,b\,R}.
\end{aligned}$$

3. Now choose b so that $k = 0$. That is $$\boldsymbol{b = -\frac{a}{2\,R}}. \tag{3.3}$$

4. Equation (3.3) shows that $R \neq 0$ is crucial for all of this to work.
   When $R = 0$, X^2 is the *dominant* term on the right in (3.1), and the proposed form of the expansion does not work. It is still possible to eliminate the $\boldsymbol{O(X^3)}$ term in (3.1) — as well as any other higher order terms — when $\boldsymbol{R = 0}$, but a **different expansion is needed**, which includes logarithmic terms. The first two terms in this expansion are: $\boldsymbol{x = X + a\,X^2\,\ln X + \ldots}$

Remark 3.1 What happens if one starts with a more general form of the transformation relating x and X, including quadratic terms? That is: $x = X + qX^2 + bX^3 + O(X^4)$? Then the second step above yields $\dot{x} = Rx - px^2 + kx^3 + O(x^4)$. The next step then requires that we select q and b so that $p = 1$ and $k = 0$. This yields $q = 0$ and $k = -a/2R$ — the same answer as above. We simplified the algebra by taking $q = 0$ from the start.

4 Find and classify bifurcations problem #02

4.1 Statement: Find and classify bifurcations problem #02

This problem has three parts, and that in each you have to answer the same set of questions.
Part 1 of 3. For equation (4.1) below, find the values of r at which a bifurcation occurs, and classify them as saddle-node, transcritical, supercritical pitchfork, or subcritical pitchfork. Finally, sketch the bifurcation diagram of fixed points x^* versus r.

$$\frac{dx}{dt} = r\,x - \frac{x^3}{1 + 2\,x^2 + x^4}. \tag{4.1}$$

Check the PCS across $\boldsymbol{r = 0}$ (see remark **4.2**). **Does it hold?**
Optional: Something "strange" happens for $\boldsymbol{r = 0}$ in the bifurcation diagram. Is there another bifurcation taking place there? If so, which type? When you add it, does the PCS apply across $\boldsymbol{r = 0}$? **Hint.** *Look at the equation satisfied by $y = 1/x$. What happens near $(y,\, r) = (0,\, 0)$?*

Remark 4.2 *The* **Principle of Conservation of Stability (PCS)** *says: Consider the ode $\dot{x} = f(x, r)$, where f is smooth and r is a parameter. Assign a weight $w = 1$ to each stable critical point, a weight $w = -1$ to each unstable critical point, and a weight $w = 0$ to each semi-stable critical point. Then the sum of the weights (* **stability index $\mathcal{S}$** *) is a constant (independent of $\boldsymbol{r}$).* ♣

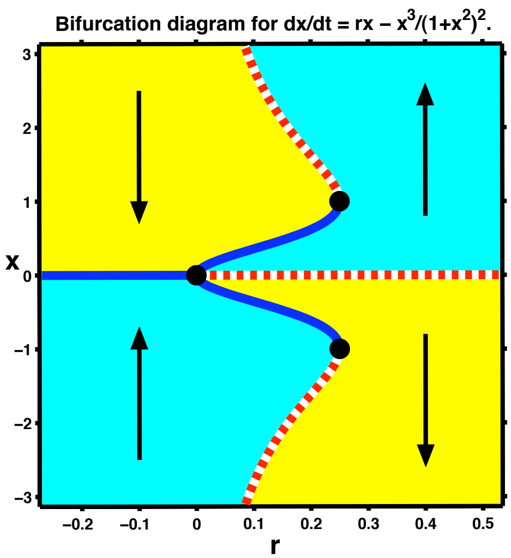
Part 2 of 3. Consider the equation
$$\frac{dx}{dt} = r\,x - \frac{x}{\sqrt{1+x^2}}, \tag{4.2}$$
and repeat the analysis in part 1. **Important:** Be *careful* when doing the transformation to the variable y, as things are not entirely smooth at $y = 0$. It follows that what happens near $(y, r) = (0, 0)$ does not fit the "standard" canonical forms studied in the lectures. Nevertheless, you should be able to do it with minimum effort.

Part 3 of 3. Consider the equation
$$\frac{dx}{dt} = r\,x - x\,\text{sech}(x), \tag{4.3}$$
and repeat the analysis in part 1. **Note:** the situation near $(y, r) = (0, 0)$ is even less "friendly" than the one in part 2. Yet, it is still tractable if you are careful.

4.2 Answer: Find and classify bifurcations problem #02

Part 1. The critical points of (4.1) are given by
$$x = 0 \;\text{ for any }\; r \qquad \text{and} \quad x = \pm\frac{1}{\sqrt{r}} \pm \sqrt{\frac{1}{4\,r} - 1} \;\text{ for any }\; 0 < r \leq \frac{1}{4}. \tag{4.4}$$
The second formula arises from the equation $r = \dfrac{x^2}{(1+x^2)^2}$, with four solutions for $0 < r < 1/4$, one solution $(x = 0)$ for $r = 0$, and two solutions $(x = \pm 1)$ for $r = 1/4$. The stability of the critical points is easy to ascertain by looking at the sign of $\dot{x}$ in each of the four regions that (4.4) splits the plane (r, x) into — this sign is represented by the arrows in figure 4.1.

We conclude that *three bifurcations* occur: **two saddle-nodes** (at $(r, x) = (1/4, \pm 1)$) and **a super-critical (soft) pitchfork** (at $(r, x) = (0, 0)$). The bifurcation diagram is shown in figure 4.1



In each region (yellow or cyan), the black arrows indicate the direction of the flow for the equation $\dot{x} = r\,x - \dfrac{x^3}{(1+x^2)^2}$.

Stable branches of critical points are plotted in solid blue, and unstable branches in dashed red. The black dots indicate the bifurcation points: two saddle-nodes at $\boldsymbol{(r,\, x) = (1/4,\, \pm 1)}$, and a supercritical (soft) pitchfork at $\boldsymbol{r = x = 0}$.

Figure 4.1: Bifurcation diagram for equation (4.1).

The stability index $\mathcal{S}$ satisfies: $\boldsymbol{\mathcal{S} = 1}$ for $\boldsymbol{r < 0}$, and $\boldsymbol{\mathcal{S} = -1}$ for $\boldsymbol{r > 0}$. **This seems to indicate a violation of the PCS**. However, consider the equation satisfied by $y = 1/x$

$$\frac{dy}{dt} = -(r\,y - \frac{y^3}{1 + 2\,y^2 + y^4}). \tag{4.5}$$

This is the same as (4.1) backwards in time. Since $y = 0$ corresponds to $x = \infty$, we conclude:

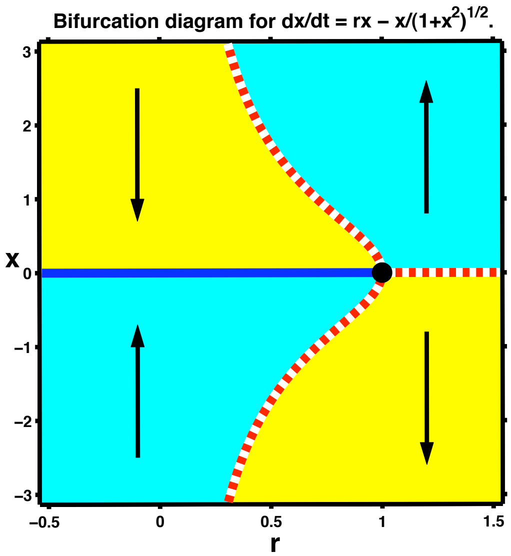
a. $x = \infty$ is a critical point for (4.1), unstable for $r < 0$ and stable for $r > 0$. With this extra critical point: $\boldsymbol{\mathcal{S} \equiv 0}$ and the **PCS holds true.**

b. At $(r,\, x) = (0,\, \infty)$ a **subcritical (hard) pitchfork** bifurcation occurs.

Part 2. The critical points of (4.2) are given by

$$x = 0 \;\text{ for any }\; r \qquad \text{and} \quad x = \pm\sqrt{\frac{1}{r^2} - 1} \;\text{ for any }\; 0 < r \leq 1. \tag{4.6}$$

The second formula arises from the equation $r = \dfrac{1}{\sqrt{1 + x^2}}$, with two solutions for $0 < r < 1$, and one solution ($x = 0$) for $r = 1$. Proceeding as in part 1, we arrive at the results summarized in figure 4.2. In particular: **a subcritical (hard) pitchfork bifurcation occurs** at $(r,\, x) = (1,\, 0)$.



In each region (yellow or cyan), the black arrows indicate the direction of the flow for the equation $\dot{x} = r\,x - \dfrac{x}{\sqrt{1 + x^2}}$.

Stable branches of critical points are plotted in solid blue, and unstable branches in dashed red. The black dot indicates the bifurcation point: a subcritical (hard) pitchfork at $\boldsymbol{(r,\, x) = (1,\, 0)}$.

Figure 4.2: Bifurcation diagram for equation (4.2).

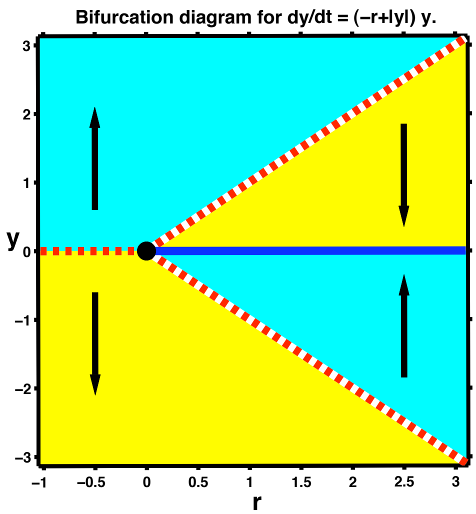
As in part 1, a violation of the PCS seems to occur, with $\mathcal{S} = 1$ for $r < 0$, and $\mathcal{S} = -1$ for $r > 0$. Thus, we inspect the behavior near $x = \infty$ by considering the equation satisfied by $y = 1/x$

$$\frac{dy}{dt} = -r\,y + \frac{y\,|y|}{\sqrt{1 + y^2}} \quad \Longrightarrow \quad \frac{dy}{dt} \approx (-r + |y|)\,y \;\text{ for }\; |y| \ll 1. \tag{4.7}$$

The second equation here yields the (non-canonical) bifurcation diagram in figure 4.3 — note that, *unlike the parabolic shape of the bifurcating branches that occurs for canonical pitchfork bifurcations, here a curve with a corner appears.*

Since $y = 0$ corresponds to $x = \infty$, we conclude that

c. $x = \infty$ is a critical point for (4.2), unstable for $r < 0$ and stable for $r > 0$. With this extra critical point: $\boldsymbol{\mathcal{S} \equiv 0}$ and the **PCS holds true.**



In each region (yellow or cyan), the black arrows indicate the direction of the flow for the equation $\dot{y} = -r\,y + |y|\,y$.

Stable branches of critical points are plotted in solid blue, and unstable branches in dashed red. The black dot indicates the bifurcation point: a (non-canonical) subcritical (hard) pitchfork at $\boldsymbol{r = y = 0}$. Note that the branches of critical points join with a corner at the bifurcation.

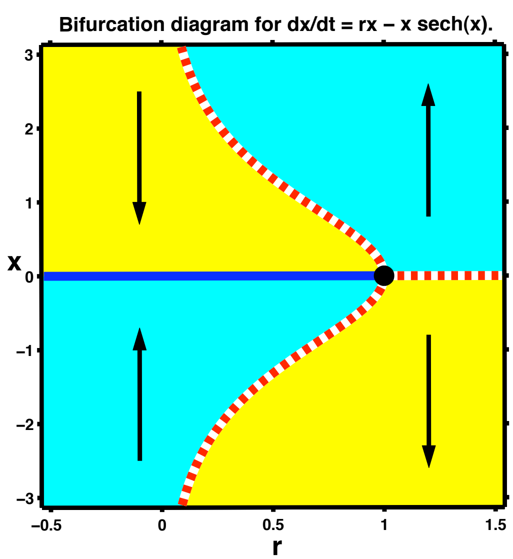
Figure 4.3: Bifurcation diagram for the second equation in (4.7).

d. At $(r,\, x) = (0,\, \infty)$ a **(non-canonical) subcritical (hard) pitchfork** bifurcation occurs.

Part 3. The analysis of (4.3) is entirely similar to that of the two previous cases, with critical points

$$x = 0 \;\text{ for any }\; r \qquad \text{and} \qquad x = \operatorname{sech}^{-1}(r) \;\text{ for any }\; 0 < r \le 1, \tag{4.8}$$

where we note that $\operatorname{sech}^{-1}(r)$ is double valued for $0 < r < 1$. The results summarized in figure 4.4. In particular: **a subcritical (hard) pitchfork bifurcation occurs** at $(r,\, x) = (1,\, 0)$.



In each region (yellow or cyan), the black arrows indicate the direction of the flow for the

equation $\dot{x} = r\,x - x\,\operatorname{sech}(x)$.

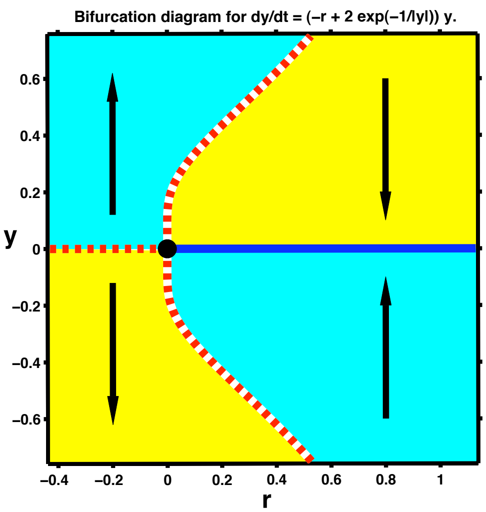
Stable branches of critical points are plotted in solid blue, and unstable branches in dashed red. The black dot indicates the bifurcation point: a subcritical (hard) pitchfork at $\boldsymbol{(r,\, x) = (1,\, 0)}$.

Figure 4.4: Bifurcation diagram for equation (4.3).

Again, we resolve the (apparent) failure of the PCS at $r = 0$ by inspecting the equation satisfied by $y = 1/x$

$$\frac{dy}{dt} = -r\,y + y\,\operatorname{sech}\left(\frac{1}{y}\right) \quad \Longrightarrow \quad \frac{dy}{dt} \approx \left(-r + 2\,\exp\left(-\frac{1}{|y|}\right)\right)\,y \;\text{ for }\; |y| \ll 1. \tag{4.9}$$

Unlike the situation in (4.7), the right hand side here is smooth. However, the perturbation to the linear part $-r\,y$, and all its derivatives, vanish at $y = 0$ — which makes this a very special problem. The second equation here yields the (non-canonical) bifurcation diagram in figure 4.5 — note that, *unlike the parabolic shape of the bifurcating branches that occurs for canonical pitchfork bifurcations, here an "infinitely flat" curve appears.* In conclusion:



In each region (yellow or cyan), the black arrows indicate the direction of the flow for the equation $\dot{y} = -r\,y + 2\,y\,\exp(-1/|y|)$.

Stable branches of critical points are plotted in solid blue, and unstable branches in dashed red. The black dot indicates the bifurcation point: a (non-canonical) subcritical (hard) pitchfork at $\boldsymbol{r = y = 0}$. The unstable branches at the bifurcation point are infinitely flat, without the parabolic shape of a canonical pitchfork.

Figure 4.5: Bifurcation diagram for the second equation in (4.9).

- **e.** $x = \infty$ is a critical point for (4.2), unstable for $r < 0$ and stable for $r > 0$. With this extra critical point: $\boldsymbol{\mathcal{S} \equiv 0}$ and the **PCS holds true.**
- **f.** At $(r,\, x) = (0,\, \infty)$ a **(non-canonical) subcritical (hard) pitchfork** bifurcation occurs.

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5 Ghosts and bottlenecks (Derive critical slowdown characteristic time by scaling)

5.1 Statement: Ghosts and bottlenecks

The aim of this question is to obtain an alternative derivation of the $T_{\text{bottleneck}} = O(r^{-1/2})$ scaling of the critical slowdown time for a system close to a saddle-node bifurcation, with $x(t)$ satisfying $\boldsymbol{\dot{x} = r + x^2}$ and $\boldsymbol{0 < r \ll 1}$. The idea is to do a variable re-scaling that reduces the ode to $\boldsymbol{\dot{u} = 1 + u^2}$. Since u does not depend on r, we can then "read" the bottleneck scaling from how time is transformed. *Proceed as follows:*

- **a.** Suppose x has a characteristic scale $O(r^a)$, where a is unknown for now. Then $x = r^a u$, where $u \sim O(1)$. Similarly, suppose that $t = r^b \tau$, with $\tau \sim O(1)$.
  Show that $\boldsymbol{u(\tau)}$ satisfies $$\boldsymbol{r^{a-b}\frac{\mathrm{d}u}{\mathrm{d}\tau} = r + r^{2a}u^2.} \tag{5.1}$$
- **b.** Assuming that all terms in the equation have the same order with respect to r, derive the values of a and b. Deduce the bottleneck timescale.

5.2 Answer: Ghosts and bottlenecks

- **a.** Using the chain rule, we find that $$\boldsymbol{\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{r^a}{r^b}\frac{\mathrm{d}u}{\mathrm{d}\tau}.}$$
  Substituting into $\dot{x} = r + x^2$ gives the desired result.

b. For all terms to have the same order of r, we need $a - b = 1$ and $2a = 1$. Hence $\boldsymbol{a} = \frac{1}{2}$ and $\boldsymbol{b} = -\frac{1}{2}$. We thus deduce the bottleneck timescale $\boldsymbol{T}_{\textbf{bottleneck}} = \boldsymbol{O(r^{-1/2})}$.

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6 Linear System with Complex eigenvalues

6.1 Statement: Linear System with Complex eigenvalues

Note. Below λ complex means: λ has a non-vanishing imaginary part.

Here we consider, in some detail, the solution (and phase plane portrait) of a linear system where the eigenvalues are complex. The system we will look at is $\dot{x} = x - y$ and $\dot{y} = x + y$, which has the corresponding vector form $\dot{\mathbf{x}} = A\,\mathbf{x}$, where A is a 2×2 matrix.

a. Write A. Then show that it has the eigenvalues $\lambda_1 = 1 + i$ and $\lambda_2 = 1 - i$, with corresponding eigenvectors $\mathbf{v}_1 = (i,\, 1)$ and $\mathbf{v}_2 = (-i,\, 1)$. **Notice: the eigenvalues, and corresponding eigenvectors, are complex conjugates. This is generic (always true) for a real 2×2 matrix A with complex eigenvectors.**#

\# **Proof.** Let A be a real square matrix, with λ a (complex) eigenvalue and corresponding eigenvalue $\mathbf{v}$; i.e.: $A\,\mathbf{v} = \lambda\,\mathbf{v}$. Taking the complex conjugate of this equation shows that $\bar{\lambda}$ is also an eigenvalue of A, with corresponding eigenvector $\bar{\mathbf{v}}$.

b. The general solution is then $\mathbf{x}(t) = c_1\,e^{\lambda_1\,t}\,\mathbf{v}_1 + c_2\,e^{\lambda_2\,t}\,\mathbf{v}_2$, for some constants c_1 and c_2. However, we are not yet done. This way of writing the solution involves complex numbers, and it does not make it clear what real valued solutions (which are the only ones we care about) do, nor what the orbits in the phase plane look like. Hence, **your task:** Express $\mathbf{x}$ purely in terms of real valued functions.† **Hint:** *use $e^{i\,\omega\,t} = \cos\omega t + i\,\sin\omega t$ to rewrite $\mathbf{x}(t)$ in terms of sines and cosines, and then collect the real and imaginary terms of the result.*†

† To get a real valued solution, the coefficients c_j in item **b** must be complex conjugates. **Hint:** use polar notation for the c_j.

Remark. The process described here can be applied to any 2×2 matrix with complex eigenvalues, using the fact that the real and imaginary part of the eigenvector(s) are linearly independent. The example here is specially simple because in this case the real and imaginary parts of $\mathbf{v}_1$ are the cartesian unit vectors.

6.2 Answer: Linear System with Complex eigenvalues

a. The equations can be written in the vector form

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \mathbf{x} = A\,\mathbf{x}, \qquad (6.1)$$

where $\mathbf{x}$ is the (column) vector with components x and y; and A is the 2×2 matrix defined by the equation. The **characteristic equation** for this system is given by $0 = \det(\lambda\,I - A) = \lambda^2 - 2\,\lambda + 2$. Hence the eigenvalues are $\lambda_1 = 1 + i$ and $\lambda_2 = 1 - i$. It is then easy to check that the corresponding eigenvectors $A\,\mathbf{v}_j = \lambda_j\,\mathbf{v}_j$ are given by $\mathbf{v}_1 = (i,\, 1)$, and $\mathbf{v}_2 = (-i,\, 1)$.

b. We can then write the general solution to (6.1) in the form $\mathbf{x}(t) = c_1\,e^{\lambda_1\,t}\,\mathbf{v}_1 + c_2\,e^{\lambda_2\,t}\,\mathbf{v}_2$, for generic (complex) constants c_1 and c_2. However, in order for this solution to be real, c_1 and c_2 must be complex conjugates: $c_1 = r\,e^{i\,\phi}$ and $c_2 = r\,e^{-i\,\phi}$. Thus, in terms of the real part, $\mathbf{e}_2 = (0,\, 1)$, and the imaginary part, $\mathbf{e}_1 = (1,\, 0)$, of $\mathbf{v}_1$, we have:

$$\begin{aligned} \mathbf{x} &= r\,e^t\,\left(\cos(t+\phi) + i\,\sin(t+\phi)\right)(\mathbf{e}_2 + i\,\mathbf{e}_1) + \text{c.c.} \\ &= 2\,r\,e^t\,\left(\cos(t+\phi)\,\mathbf{e}_2 - \sin(t+\phi)\,\mathbf{e}_1\right). \end{aligned} \qquad (6.2)$$

where c.c. = "complex conjugate". That is

$$x = -2\,r\,e^t\,\sin(t+\phi) \quad \text{and} \quad y = 2\,r\,e^t\,\cos(t+\phi). \qquad (6.3)$$

Clearly: the imaginary part of λ_1 gives rise to circular, clock-wise, motion around the origin in the phase plane; while the real part causes exponential growth. In short: **(6.3) describes orbits that spiral out from the origin** (logarithmic spiral).

7 Numerical methods #01 (Test various numerical methods)

7.1 Statement: Numerical methods #01

Goal: test three numerical solutions, $x = x(t)$, for the initial value problem: $\dot{\boldsymbol{x}} = -\boldsymbol{x}$ with $\boldsymbol{x(0) = 1}$.

- **a.** Solve the problem analytically. What is the exact value of $x(1)$?
- **b.** Use the Forward (or Explicit) Euler method to find the numerical approximation $\hat{x}_n(1)$ to $x(1)$ with a timestep of $\Delta t = 10^{-n}$ for $n = 0, 1, 2, 3, 4$. Let $E_n = |\hat{x}_n(1) - x(1)|$ be the error for each timestep. Plot $\log_{10}(\Delta t)$ versus $\log_{10}(E_n)$ and explain the results.
- **c.** Repeat (b) using the Improved Euler method.
- **d.** Repeat (b) using the fourth-order Runge-Kutta method.

The numerical methods above are displayed in Strogatz's book; in: §2.8 *Solving equations on the computer.*

7.2 Answer: Numerical methods #01

- **a.** $x(t) = e^{-t}$ and $x(1) = e^{-1} \sim 0.368\ldots$
- **b.** See Figure 7.1, blue curve. The error of the Explicit Euler method is $O(\Delta t)$.
- **c.** See Figure 7.1, red curve. The error of the Improved Euler method is $O(\Delta t^2)$.
- **d.** See Figure 7.1, green curve. The error of the fourth-order Runge-Kutta method is $O(\Delta t^4)$.

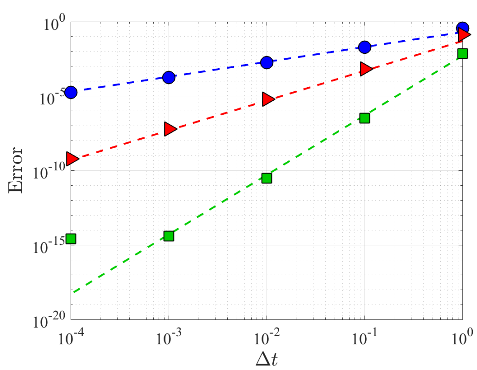

Figure 7.1: Error plots for the methods: Explicit Euler (blue), Improved Euler (red), and Runge-Kutta fourth-order (green). The dash lines have the form $c\Delta t^p$ for $p = 1$ (blue), $p = 2$ (red), and $p = 4$ (green), indicating $O(\Delta t^p)$ convergence. Notice that the error "saturates" at 10^{-15} for the Runge-Kutta fourth-order method ($\Delta t < 10^{-3}$ does not make the error smaller than 10^{-15}). This is a consequence of the floating point accuracy used here; numerically: $1 = 1 + 10^{-16}$.

THE END.