Answers to P-Set # 02, (18.353/12.006/2.050)j MIT (Fall 2024)

Rodolfo R. Rosales (MIT, Math. Dept., room 2-337, Cambridge, MA 02139)

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1.1 Statement: Find and classify bifurcations problem #01

For equation (1.1) below, find the values of r at which a bifurcation occurs, and classify them as saddle-node, transcritical, supercritical pitchfork, or subcritical pitchfork. Finally, sketch the bifurcation diagram of fixed points x^* versus r.

$$\frac{dx}{dt} = r - \frac{x^2}{1+x^2}. \tag{1.1}$$

1.2 Answer: Find and classify bifurcations problem #01

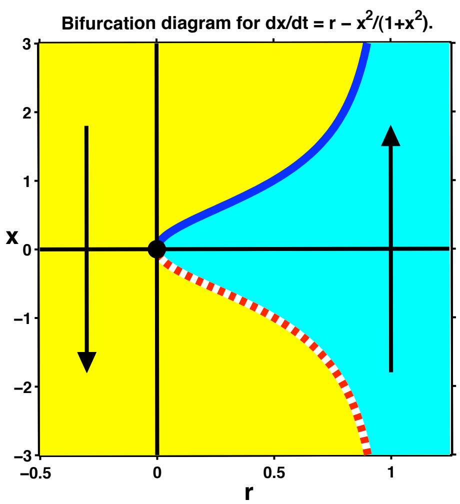
The critical points are given by

$$x = \pm\sqrt{\frac{r}{1-r}} \quad \text{for} \quad 0 \le r < 1, \tag{1.2}$$

with no others. Furthermore $\dot{x} > 0$ for $r > \dfrac{x^2}{1+x^2}$, and $\dot{x} < 0$ for $r < \dfrac{x^2}{1+x^2}$. It follows that the branch $x = \sqrt{\dfrac{r}{1-r}}$ is stable, and the branch $x = -\sqrt{\dfrac{r}{1-r}}$ is unstable, with a **saddle-node bifurcation** at $(r,\, x) = (0,\, 0)$. The bifurcation diagram is shown in figure 1.1. Note, however, that **there is another bifurcation[1] at $r = 1$, apparently not associated with any critical point.** To understand what is going on, we rewrite the equation in terms of $\boldsymbol{y = 1/x}$. Then

$$\frac{\mathbf{dy}}{\mathbf{dt}} = -y^2\left(r - \frac{1}{1+y^2}\right). \tag{1.3}$$

This has $\boldsymbol{y = 0}$ as a critical point for all $\boldsymbol{r}$ (but it is a strange critical point, as it is a double root), and at $\boldsymbol{r = 1}$ two new critical points appear for $0 < r < 1$; $\boldsymbol{y = \pm\sqrt{(1-r)/r}}$. However, **this bifurcation is none of the standard ones that we studied,** because $\boldsymbol{y = 0}$ is a double root. In fact $\boldsymbol{y = 0}$ is always a semi-stable critical point, and it switches the stable and unstable directions at $\boldsymbol{r = 1}$.
Note: you were not expected to find, nor analyze, the $r = 1$ bifurcation. But, if you noticed it, and were able to say something about it, great!



In each region (yellow or cyan), the black arrows indicate the direction of the flow for the equation $\dot{x} = r - \dfrac{x^2}{1+x^2}$.
Stable branches of critical points are plotted in solid blue, and unstable branches in dashed red. The black dot indicates the bifurcation point (saddle-node).

Figure 1.1: Bifurcation diagram for equation (1.1).

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[1] Recall: a bifurcation occurs where there is a qualitative change in the phase diagram. Here the critical points are as follows: none for $r < 0$, two for $\boldsymbol{0 < r < 1}$, and then back to none for $\boldsymbol{r > 1}$.

2 Get equation from phase line portrait problem #02

2.1 Statement: Get equation from phase line portrait problem #02

Consider the ode on the line

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x), \tag{2.1}$$

where f is some function which has (at least) one continuous derivative. Assume that (2.1) has exactly two critical points (i.e.: x_1 and x_2, with $-\infty < x_1 < x_2 < \infty$). Assume also that x_1 is stable and that x_2 is unstable.[2] **Is this possible? Does a function $f = f(x)$ yielding this exist?**
If the answer is no, prove it.
If the answer is yes, prove it by giving an example.

2.2 Answer: Get equation from phase line portrait problem #02

The answer is yes. Example

$$\boldsymbol{f(x) = x^2 - 1}. \tag{2.2}$$

In this case $\boldsymbol{x_1 = -1}$ and $\boldsymbol{x_2 = 1}$.

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3 Get equation from phase line portrait problem #06

3.1 Statement: Get equation from phase line portrait problem #06

Consider the ode on the line

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x), \tag{3.1}$$

where f is some function which has (at least) one continuous derivative. Assume that (3.1) has exactly two critical points (i.e.: x_1 and x_2, with $-\infty < x_1 < x_2 < \infty$). Assume also that both critical points are unstable.[3] **Is this possible? Does a function $f = f(x)$ yielding this exist?**
If the answer is no, prove it.
If the answer is yes, prove it by giving an example.

3.2 Answer: Get equation from phase line portrait problem #06

The answer is no. Because f is continuous, it must either be: $f(x) > 0$ for $x_1 < x < x_2$ (in which case x_2 cannot be unstable), or $f(x) < 0$ for $x_1 < x < x_2$ (in which case x_1 cannot be unstable).

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4 Phase line portrait problem #01

4.1 Statement: Phase line portrait problem #01

Consider the following ode on the line

$$\boldsymbol{\frac{dx}{dt} = f(x) = \mathrm{sech}(x) - \frac{4}{5}}. \tag{4.1}$$

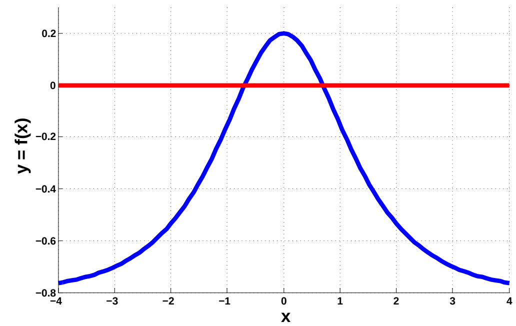
Draw its phase line portrait, indicating the critical points, and their stability properties.

[2] A critical point is unstable if the solutions diverge from the critical point on both sides.
[3] A critical point is unstable if the solutions diverge from the critical point on both sides.

In addition: describe *quantitatively* the behavior of the solutions near the critical points (i.e.: at what rate do they approach or leave them), as well as the behavior of the solutions when $|x|$ is large. **In particular:** *Are there solutions that cease to exist for some finite value of t, or are the solutions valid for all times?*

4.2 Answer: Phase line portrait problem #01

The zeros of f are at $\boldsymbol{x_1 = -\ln(2)}$ and $\boldsymbol{x_2 = \ln(2)}$ (both simple zeros), with f positive between the zeros, and negative outside the interval $\boldsymbol{x_1 \leq x \leq x_2}$. See figure 4.1.



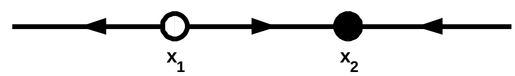


Figure 4.1: Problem 4, equation (4.1).
The zeros of f are at $x_1 = -\ln(2)$ and $x_2 = \ln(2)$.
Left: plot of $\dot{x} = f(x)$ versus x. Red: x-axis; blue: $f(x)$.
Right: phase line portrait.

It follows that x_1 is an unstable critical point, and that x_2 is a stable critical point. The phase line portrait is as in figure 4.1. Because x_1 and x_2 are simple zeros of f, the behavior of the solutions near them is exponential — either approach (stable) or leave (unstable) them exponentially fast — with rate constants[4] $r_j = \pm 12/25$. Since $f(x) \to -4/5$ as $x \to \pm\infty$, when $|x|$ becomes large on a solution, the solution becomes (approximately) linear in time. Thus *the solutions with $x < x_1$ approach $-\infty$ linearly as $t \to \infty$, while the solutions with $x > x_2$ approach ∞ linearly as $t \to -\infty$.* In particular: **the solutions are defined for all times $\boldsymbol{-\infty < t < \infty}$.**

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5 The leaky bucket

5.1 Statement: The leaky bucket

The example here[5] shows that in some physical situations, non-uniqueness is natural and not pathological.

Consider a water bucket with a small hole in the bottom. If you see the bucket with a puddle beneath it, can you figure out when the bucket was full? Of course not! It could have finished emptying[6] 1 min ago, 10 min ago, ... The solution to the corresponding differential equation must be non-unique when integrated backwards in time.

Let us construct a simple model for the situation. Consider a cylindrical bucket, with constant cross-sectional area $\boldsymbol{A}$, and a small hole at the bottom, with area $\boldsymbol{a}$. The hole is small ($\boldsymbol{a \ll A}$), so the water depth $\boldsymbol{h(t) \geq 0}$ (height of the liquid in the bucket at time $\boldsymbol{t}$) goes down slowly. Thus we assume that the bulk of the liquid is at rest, so that the *rate at which the water goes down is controlled by the conversion of potential energy* in the bucket to *kinetic energy* in the exit water jet through the small hole (see remark **5.1**). For this purpose, let $\boldsymbol{v(t) \leq 0}$ be the water velocity through the hole (here we use the *convention* that $v > 0$ corresponds to water entering the bucket, while $v < 0$ if the water exits). Finally, we *neglect* surface tension and dissipation — the hole is not tiny.

[a] Show that $\boldsymbol{a\, v(t) = A\, \dot{h}}$. What physical law do you need to use here?

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[4] The rate constants are given by $f'(x_j)$. The solutions behave like $x = x_j + \text{constant}\, e^{r_j\, t}$ near the critical points.
[5] Hubbard, J. H., and West, B. H. (1991) *Differential Equations: A Dynamical Systems Approach, Part I* (Springer, New York).
[6] Note that, in this problem, evaporation effects are neglected.

[b] To derive an additional equation, use conservation of energy. **First**, write an expression for the potential energy † in the bucket, V, as a function of h, A, g (the acceleration of gravity), and ρ (the density of water). **Second**, write an equation for the rate at which kinetic energy is transported out of the bucket by the escaping water, K_r, as a function of v and m_r, where m_r is the rate at which mass leaves the bucket.‡

† Ignore the effect of air for this (no air pressure change over the depth of the bucket).

‡ You should be able to write m_r in terms of ρ, A, and $\dot{h}$.

Finally, assume that all the potential energy is converted into kinetic energy, to obtain $v^2 = 2\,g\,h$.

[c] Combine **[a]** and **[b]** to obtain the time evolution for h (in the form of a first order ode, with constant(s) written in terms of a, A, and g).
Important, be careful with the signs! Recall that $v,\ \dot{h} \leq 0$. Further, it should be $m_r,\ K_r,\ V \geq 0$.

[d] Given $h(0) = 0$ (bucket empty at $t = 0$), show that the solution for $h(t)$ is **non-unique backwards in time**, i.e., for $t < 0$. **Hint.** Find the solution to the ode, for all $t > t_*$, given $h(t_*) = h_* > 0$. Then use these solutions to produce solutions the backwards in time problem posed here.

[e] The description/derivation above ignores surface tension. Briefly discuss the effect surface tension, if significant (e.g.: tiny hole), would have on the outcome.

Remark 5.1 The problem is set-up in such a way that you do not need to know any fluid dynamics; you only need basic physics notions such as kinetic energy and potential energy. If you have familiarity with fluids, you may be tempted to, say, use Bernoulli's principle to derive the equation for h; however: **do not do it! I want to see a derivation based on energy balance,** like the one outlined in **[a-c].** ♣

5.2 Answer: The leaky bucket

[a] Since water is conserved (*mass conservation*), the rate at which the water exits the bucket, $a\,v(t)$, must equal the rate at which the water in the bucket decreases, $A\,\dot{h}$. Hence $a\,v(t) = A\,\dot{h}$.
Note that this is consistent with the sign convention for v.

[b] The potential energy of the water in the bucket is given by $V = \int_0^h g\,A\,\rho\,z\,\mathrm{d}z = \frac{1}{2}\,g\,A\,\rho\,h^2$.

On the other hand $K_r = \frac{1}{2}\,m_r\,v^2$, where $m_r = -\rho\,A\,\dot{h}$. Hence $K_r = -\frac{1}{2}\rho\,A\,\dot{h}\,v^2$

Conservation of energy requires $K_r + \dot{V} = 0$. Thus $v^2 = 2\,g\,h$.

This last step requires division by ρ, A, and $\dot{h}$. Neither ρ nor A vanishes. As for $\dot{h}$, when it vanishes h and v should also vanish, so the equation remains valid. Further: note that here is where we neglect friction.

An alternative expression for the rate at which kinetic energy is transported out of the bucket by the escaping water is given by $K_r = -\frac{1}{2}\,\rho\,a\,v^3$; since in an infinitesimal time interval $\mathrm{d}t$ the kinetic energy transported out is $\mathrm{d}K = -\frac{1}{2}\,(\rho\,a\,v\,\mathrm{d}t)\,v^2$. Using **[a]**, it is easy to see that this is equivalent to the expression used above.

[c] A simple calculation using **[a-b]** shows that

$$\dot{h} = -C\,\sqrt{h}, \quad \text{where } C = \frac{a}{A}\sqrt{2\,g} > 0, \tag{5.1}$$

subject to the restriction $h \geq 0$ imposed on the solutions.

[d] Given $h(0) = 0$ (empty bucket), then the solution(s) to (5.1) are:

[d_1] Since $\dot{h} \leq 0$, and h must remain non-negative, *the solution is unique for* $t > 0$. That is $h(t) \equiv 0$.

[d_2] Using separation of variables, we can find an infinite number of solutions, valid for $t < 0$. That is

$$h(t) = \begin{cases} 0 & \text{for } t_0 \leq t \leq 0, \\ \left(\frac{C}{2}\,(t_0 - t)\right)^2 & \text{for } t \leq t_0. \end{cases}$$

where $t_0 \leq 0$ is arbitrary.

[e] What about surface tension? Surface tension complicates things in many ways. For example, when surface tension is significant: (i) It effectively decreases the area of the hole, which invalidates **[a]**. (ii) It adds extra terms to the energy, which affects **[b]**. (iii) It may even make the problem non-steady (i.e.: there is no steady, clean, water jet exiting the small hole). This destroys the whole scenario posed above. (iv) On the top surface it creates an extra force that resists the emptying of the bucket. However, by assumption, the top surface is "large", so this is not an important effect [certainly not for a bucket-sized container]. *Let us now examine (at an intuitive level) further scenarios that what can happen when surface tension matters.*

[e_1] The jet that comes out of the hole may be fractured into droplets by surface tension. If this happens "far" from the hole, then it is not relevant for this problem (once the water leaves the bucket, it no longer affects h). But if it happens near the hole, it will make the water flow out of the hole unsteady.

[e_2] If the hole at the bottom is small enough, surface tension may be able to stop the flow before $\boldsymbol{h = 0}$. What happens beyond this depends on whether or not the bucket surface is wetting. If it is not, the interface at the hole will be pinned, with surface tension across it able to support the pressure by a nonzero $\boldsymbol{h}$ in the bucket — a stable, steady, situation. A slightly weaker surface tension will allow some flow, which will produce a drop that grows at the hole, till at some point it breaks out and the process re-starts: "a drip-drip" situation; and extreme case of the situation in [e_1]. On the other hand, If the bucket surface is wetting, the interface will spread, allowing more water to flow out though a thin layer wetting the outside surface of the bucket.

Finally, we have ignored changes in the air pressure from the top level of the bucket to the floor. This is a very small effect for a bucket-size bucket filled with water on the floor of a room. But imagine that the bucket is at the bottom of a pool, filled with a heavier than water, water-immiscible liquid. Then the outside fluid pressure changes matter. We have also assumed a nice, clean, hole at bottom. But, if instead, there is a crack in the bucket wall, likely we would not be able to ignore neither surface tension nor dissipation. Important point, though: **no matter what level of modeling details are required, the conclusion in [d] is unlikely to change: no uniqueness backwards in time.**

6 Toy model for column buckling

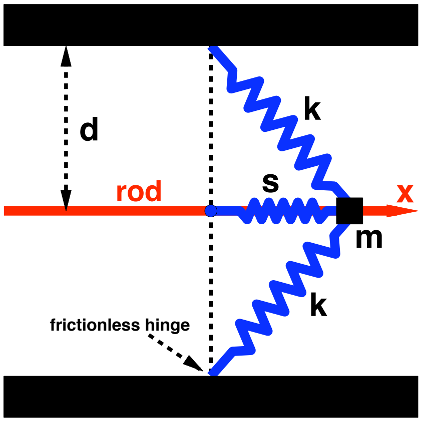
6.1 Statement: Toy model for column buckling

Imagine a vertical cylindrical elastic column, on which you push down along its axis by putting a weight on top. If the load is small enough, the column compresses a little, and the elastic response can balance the weight — with the cylinder staying straight. But, if the load is too large, this configuration is not stable, and the column buckles under the weight. This behavior arises because of the interplay of three forces: (i) the load; (ii) the elastic force along the axis of the cylinder; and (iii) the restoring force that is generated when the cylinder bends. When the axial forces are too large, the bending resistance is not enough to keep the straight state stable.

In this exercise we consider a very simple (1-D) toy model, exhibiting the essentials of the behavior described in the prior paragraph. Note though that it is an over-simplified "toy" model, where all the richness of the original setting is gone, and only the column buckling bifurcation remains.

A sketch depicting the model is shown in figure 6.1. Further assumptions and notation are:

1. The device is restricted to a plane, with the bead moving along a line.
2. Let $\boldsymbol{x}$ **be the distance,** along the rod, of the bead from the vertical line joining the spring supports. Let $x > 0$ if the bead is to the right of the supports and $x < 0$ if to the left. **Note the left-right ($\boldsymbol{x \to -x}$) symmetry of the set-up**.
3. The two main springs are equal, with a rest lengths $L > 0$ and spring constants $k > 0$. Each generates a force along its axis of magnitude (Hook's law) $F = k(\ell - L)$, where ℓ is the spring length. They push if $\ell < L$, and pull if $\ell > L$.
4. The spring aligned with the rod has zero rest length and a spring constant $s > 0$. This spring generates a restoring force $F = -s\,x$ along the rod, pulling the bead towards $x = 0$. *Note: a "better" model would have the restoring force provided by a torsion spring located at the hinge between the two springs on the bead. Such*



A bead of mass m (black square) can slide along a rigid horizontal rod (in red). The bead is connected by two equal springs (in blue), with spring constants k, to two supports placed symmetrically a distance d above and below the rod. A third spring, with spring constant s, pulls the bead towards the middle of the vertical line connecting the supports for the other springs. Everything is frictionless, except for the friction force opposing the motion of the bead along the rod. See the text for further details.

Figure 6.1: Toy model for column buckling.

a spring would generate a restoring torque proportional to the angle between the two main springs. However, there is no qualitative difference between these two set-ups — and the one here yields simpler algebra.[7]

5. When the bead slides along the rod, the motion is opposed by a friction force of magnitude $b\,\dot{x}$, where $b > 0$ is a constant.

6. The distance of the main spring supports from the rod is $\boldsymbol{d > 0}$. *Instead of considering the behavior of the system as a function of an applied compression force, we will consider it as a function of the total "imposed" length* $2\,d$ *of the "column".*

7. Because the rod is rigid, we need to consider only the horizontal components of the various forces that act on the bead. These are the forces provided by the three springs, and friction along the rod.

PROBLEM TASKS:

A. *Derive an ode for the bead position, and write it in appropriate a-dimensional variables.*[8]

B. *Assume that friction is large, so that inertia can be neglected. Exactly which a-dimensional number has to be small for friction to be "large"?*

C. *Analyze the bifurcations that occur for the equation resulting from item B, as the distance* d *changes (with everything else fixed). What type of bifurcation(s) occur?*

D. Consider the model that results from neglecting inertia. *The equation for this model can, with an appropriate scaling, be written in such a way that it contains a single a-dimensional parameter. Exhibit this form.*

To standardize the notation used in the answers, define $\quad \boldsymbol{a = \frac{L}{d}}$ **and** $\boldsymbol{\gamma = \frac{s}{2\,k}}$. (6.1)

6.2 Answer: Toy model for column buckling

Newton's law for the motion of the bead takes the form (see remark **6.1**)

$$m\,\ddot{x} + b\,\dot{x} = 2\,k\,\frac{x}{\sqrt{x^2+d^2}}\left(L - \sqrt{x^2+d^2}\right) - s\,x, \tag{6.2}$$

where the factor $2\,k$ arises because there are two springs, and the factor $x/\sqrt{x^2+d^2}$ is to compute the projection along the rod of the spring's forces. Note also that the signs are correct: when the springs are under compression

[7] Both models are over-simplifications of the situation described in the first paragraph of the exercise. There is no point in worrying about getting small details right, when whooping simplifications occur elsewhere.

[8] Suggestion: to a-dimensionalize use d for length and $b/(2\,k)$ for time.

($\sqrt{x^2+d^2} < L$), and $x > 0$, the springs should be pushing x to the right — with the force sign switching if either $x < 0$ or $\sqrt{x^2+d^2} > L$.

Select a-dimensional variables via $\boldsymbol{x = d\,\tilde{x}}$ and $\boldsymbol{t = \dfrac{b}{2\,k}\,\tilde{t}}$. The equation then becomes

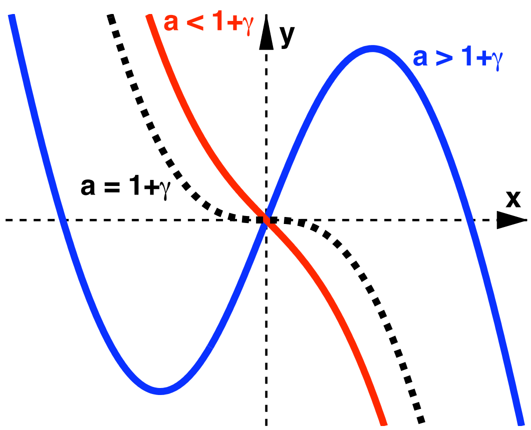
$$\epsilon\,\ddot{x} + \dot{x} = -\gamma\,x + \frac{x}{\sqrt{1+x^2}}\left(a - \sqrt{1+x^2}\right), \tag{6.3}$$

where we have not written the tildes to simplify the notation,

$$\epsilon = \frac{2\,k\,m}{b^2}, \quad a = \frac{L}{d}, \quad \text{and} \quad \gamma = \frac{s}{2\,k}. \tag{6.4}$$

If $\epsilon \ll 1$, we can neglect inertia. Thus we arrive at the final equation (the toy model equation)

$$\dot{x} = -\left(\gamma + \frac{\sqrt{1+x^2}-a}{\sqrt{1+x^2}}\right)x = -\left((1+\gamma)\sqrt{1+x^2} - a\right)\frac{x}{\sqrt{1+x^2}} = f(x,\,a,\,\gamma). \tag{6.5}$$



Plots of the function $f = f(x,\,a,\,\gamma)$ on the right of equation (6.5), with γ fixed. As $a = L/d$ increases (d decreases — see figure 6.1), the critical point at $x = 0$ (straight column) looses stability, and two new (stable) critical points appear at $x = \pm\sqrt{(a/(1+\gamma))^2 - 1}$ — column buckles. A supercritical (soft) pitchfork bifurcation.

Figure 6.2: Bifurcation for the toy model for column buckling.

To understand the bifurcations in equation (6.5), we now study its critical points and their stability — see figure 6.2. Three cases arise:

c1. Case $a < 1+\gamma$. There is a single critical point, which is stable:[9] $\boldsymbol{x_2 = 0}$ (straight column). *Note that, if $d > L$, the column is under tension and the straight state should be stable — which it is, since then $a < 1$.*

c2. Case $a > 1+\gamma$. There are three critical points:

$$\boldsymbol{x_1 = -\sqrt{(a/(1+\gamma))^2 - 1}}, \qquad \boldsymbol{x_2 = 0}, \qquad \text{and} \qquad \boldsymbol{x_3 = \sqrt{(a/(1+\gamma))^2 - 1}}.$$

It is easy to see that **both x_1 and x_3 are stable, while x_2 is unstable** — the column buckles. *Note that x_3 corresponds to the configuration shown in figure* 6.1, while x_1 corresponds to the mirror-image configuration.

c3. Case $a = 1+\gamma$. Only one critical exists: $x_2 = 0$, which is stable. **At $a = 1+\gamma$ a supercritical (soft) pitchfork bifurcation occurs.** As a decreases through $1+\gamma$, x_2 looses stability, and the system moves to either x_1 or x_3. There is no abrupt transition here, because both x_1 and x_3 are small for $0 < a - (1+\gamma) \ll 1$.

Finally, note that, in terms of $\boldsymbol{\tau = (1+\gamma)\,t}$, equation (6.5) takes the form

$$\frac{dx}{d\tau} = \left(r - \sqrt{1+x^2}\right)\frac{x}{\sqrt{1+x^2}}, \tag{6.6}$$

[9] This is a simple consequence of the fact that, in this case, $\left((1+\gamma)\sqrt{1+x^2} - a\right) > 0$ for all values of x.

which has the single non-dimensional parameter

$$r = \frac{a}{1+\gamma} = \frac{L}{d}\,\frac{2\,k}{s+2\,k}. \tag{6.7}$$

This is the single parameter controlling the behavior of the system. Everything else can be absorbed into a scale change (either time or space).

Force balances and a bit of physical intuition

The behavior of the system in this problem is controlled by the following forces:

e1. The force $F_1 = -s\,x$ by the spring lined up with the rod, which pulls always inwards.[10]

e2. The force $F_2 = 2\,k\,\dfrac{x}{\sqrt{x^2+d^2}}\left(L - \sqrt{x^2+d^2}\right)$ resulting from the other two springs. This force can either push outwards (compressed springs) or pull inwards (stretched springs).

Note that, no matter what the system parameters are, this force pulls inwards if $|x|$ is large enough. As we will see, this is what makes the resulting bifurcation "soft".

e3. Large enough friction along the rod, which allows us to simplify the equations by neglecting inertia. After this simplification, friction imposes no further qualitative changes in the system behavior (but it influences the time scale over which things happen).

In figure 6.3 we plot the non-dimensional version of these two forces:

$$F_1 = -\gamma\,x \quad \text{and} \quad F_2 = \frac{x}{\sqrt{1+x^2}}\left(a - \sqrt{1+x^2}\right), \tag{6.8}$$

with γ fixed and two illustrative values for a. The figure shows how the balance of these two forces yields critical points, and bifurcations. As illustrated by the figure, the key facts in producing a supercritical (soft) pitchfork bifurcation are:

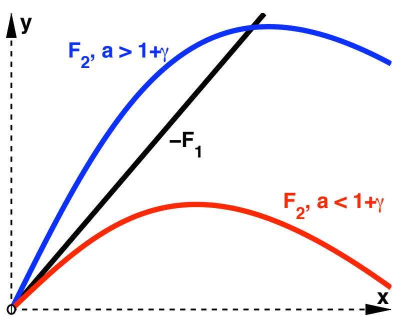


Figure 6.3: Forces in the toy model for column buckling — see equation (6.8). Plot of $y = -F_1(x)$ (for an illustrative value of γ), and of $y = F_2(x)$ (for two illustrative values of a). The plots are for $x \geq 0$ only; $x \leq 0$ follows by reflection across the origin.
Critical points arise when the curves intersect. Their stability can be assessed from the sign of $F_2 + F_1$. A supercritical (soft) pitchfork bifurcation arises because F_2 is concave: when F_2 is above $-F_1$ near the origin (thus $x = 0$ is unstable), a second (stable) critical point occurs at some $x_* > 0$.

k1. The forces are odd.

k2. For $x > 0$, $y = F_2$ is more concave than $y = -F_1$. That is, F_2 acts as a soft-spring, weakening with distance. This is produced by the geometry of the problem.

The bifurcation then occurs as the difference in the effective spring constants at the origin (derivatives at $x = 0$) switches sign.

Remark 6.1 On dimensions and mathematical expressions.
Consider equation (6.2), repeated here for ease of reading

$$m\,\ddot{x} + b\,\dot{x} = 2\,k\,\frac{x}{\sqrt{x^2+d^2}}\left(L - \sqrt{x^2+d^2}\right) - s\,x. \tag{6.9}$$

This is an example of the following: *in formulas involving dimensional variables, any mathematical expression that appear should have an equivalent formulation where all the operations (e.g.: square roots, exponentials, etc.) are done on a-dimensional variables.* For example, if $\boldsymbol{\mathcal{F}} = \boldsymbol{\mathcal{F}}(\boldsymbol{\xi},\,\boldsymbol{\eta})$ is a force that

[10] Here inwards means: towards $x = 0$, while outwards means away from $x = 0$.

depends on the variables ξ and η (e.g.: position, velocity, etc.), then we should be able to write $\mathcal{F} = F_0\, F(\xi/\xi_0, \eta/\eta_0)$, where F_0 is a constant with dimensions of force, ξ_0 and η_0 are constants with the same dimensions as ξ and η, and F is a function that operates on pure numbers (no dimensions). **If this is not possible, you can be certain that there is an error somewhere.**
In the particular case of (6.9), we have

$$\mathcal{F} = 2\,k\,\frac{x}{\sqrt{x^2+d^2}}\left(L - \sqrt{x^2+d^2}\right) - s\,x, \qquad \text{same as} \qquad \mathcal{F} = F_0\left(\frac{\tilde{x}}{\sqrt{\tilde{x}^2+1}}\left(a - \sqrt{\tilde{x}^2+1}\right) - \gamma\tilde{x}\right), \tag{6.10}$$

where $F_0 = 2kd$ and $\tilde{x} = x/d$.

Finally, let me point out that **the use of proper a-dimensional variables is very important in dynamical systems,** as they allow the identification of the key parameters that control the behavior of the system. This is why you find a myriad of "numbers with names" throughout continuum mechanics: Reynolds, Rayleigh, Richardson, Ekman, Prandtl, Lewis, Damkohler, Bond, Froude, Nusselt, Mach, Schmidt, Grashof, Arrhenius, Marangoni, Ursell, ...

7 Taylor–von Neumann–Sedov blast wave radius

7.1 Statement: Taylor–von Neumann–Sedov blast wave radius

Consider what happens when a large amount of energy, E, is released very fast (which we will approximate as "instantaneously") over a very small volume (which we will approximate as "at a point in space"). This in some quiescent gas. The result is the creation of a very strong spherical shock wave (the blast wave) propagating outwards from the energy release location, say $\vec{x} = 0$. We will also assume that the energy release occurs at time $t = 0$.

The shock wave is very strong (at least initially) so that the pressure behind it is much larger than the pressure ahead (i.e.: $p_b \gg p_a$). Hence, for the analysis that follows we will neglect the pressure in the gas ahead of the shock (e.g.: p_a = atmospheric pressure).† On the other hand, we **cannot neglect the gas density ahead of the shock, $\rho_a > 0$.** This is because the density jump across a shock wave is known to be bounded; in other words: while $\rho_b > \rho_a$, it is not true that $\rho_b \gg \rho_a$.

† The argument is that p_a is so small compared to p_b, that we can safely make the approximation $p_a = 0$.

Task #1. What are the dimensions of E and ρ_a?

Task #2. Given the situation described above, use similarity analysis to produce a formula for the blast wave radius as a function of time, $R = R(t)$. This is a formula that involves E and ρ_a, where the only unknown is an a-dimensional multiplicative factor κ. Specifically: R is a length, and there is only one combination of E, ρ_a, and t, that yields a length.
Thus you should obtain a formula of the form

$$R = \kappa\,\mathcal{A}(E,\,\rho_a,\,t), \tag{7.1}$$

where $\mathcal{A}$ is an algebraic expression with no free parameters.

Note. The gas velocity ahead of the blast wave vanishes, so it is not a parameter. On the other hand, the density ρ_b, pressure p_b, and velocity $\vec{u}_b$, of the gas behind can be written [11] once $R(t)$ is known, in terms of $p_a = 0$, ρ_a, and $\vec{u}_a = 0$. Hence p_b, ρ_b and $\vec{u}_b$, **are not** relevant to the similarity analysis that you are asked to do.

Task #3. Suppose that you know the value of κ, and that for some $t_1 > 0$, you are given $R_1 = R(t_1)$. Suppose also that you know ρ_a. Write now a formula for the energy E in terms of R_1, κ, ρ_a, and t_1.

7.1.1 Note on units and dimensions

You should not confuse dimensions with units. For example, a velocity has dimensions of length over time; **not** miles per hour, centimeters per second, leagues per day, ... Other examples: dimension(acceleration) = (length)/(time)^2, dimension(force) = (mass) × times (acceleration) = (mass) × (length)/(time)^2.

[11] Using the "Rankine-Hugoniot" conditions.

Things like miles, pounds, kilograms, meters, etc., are **units**. Units are specific quantities of a particular dimension, selected to measure the dimension. For standard units the selection is quite arbitrary, done for convenience, ease of use (at least for metric), and uniformity.† But in any particular setting, standard units may not be the most appropriate — e.g.: (i) Astronomers use the AU when dealing with solar systems, and light years (or parsecs) when dealing with stars; not meters or feet. (ii) Energy at the atomic level is measured in ev, not joules.

A key part of dimensional analysis is to pick the "right" units for a given problem; and there it must be that the units follow from the parameters that control the situation, *not arbitrary and unrelated things.*‡

† If everyone uses different units, it is hard to communicate, do commerce, or mass produce anything (in particular: tools).

‡ Like the length of the right foot of some king, which is how the foot used to be defined. Every kingdom then had a different foot, or some other part of the ruler's anatomy used to measure length: foot, pie, piede, punto, cana, palm, ...
And, of course, here at MIT we have the smoot.

Finally, **here is a simple example of similarity analysis.** Imagine that you have an object (which we idealize as a point) which, at time $t = 0$, is located at $\vec{x} = \mathbf{0}$ and not moving (zero velocity). Further, for $t > 0$ the object is subject to a constant acceleration $\vec{a}$. Find how the object's position evolves in time (without using calculus, nor even the definition of acceleration beyond its dimensions).

Answer. You need to construct a quantity with dimensions of length, $\vec{x}(t)$, using only an acceleration [with dimensions (length)/(time)^2] and time. Hence it must be

$$x = \kappa\,\vec{a}\,t^2, \quad \textbf{[A]}$$

where κ is a purely numerical constant. If this seems too simple, note that **[A]** is, basically, what Galileo (see **[g]**) hypothesized:

> *A falling body accelerates uniformly: it picks up equal amounts of speed in equaltime intervals, so that, if it falls from rest, it is moving twice as fast after two seconds as it was moving after one second, and moving three times as fast after three seconds as it was after one second.*

and then proved, using his famous *Acceleration Experiment*, where he rolled a ball down an inclined plane (to slow things down, and make measurement possible; he did not have high speed cameras).

Galileo also used a **scaling argument** (a cousin of similarity analysis) to argue that giants cannot exist. The idea is that muscle and bone strength scale (roughly) like their cross-section, while mass scales with the volume. Hence, beyond some size, muscles and bones are not be able to support the weight. *Corroborating evidence and related facts:* (i) elephant bones are much thicker relative their size than, say, rabbit bones. (ii) ants can carry loads much larger (relative to their weight) than, say, humans can. (iii) chitin exoskeletons only work for small creatures; for larger ones you need bones, which are stronger for a given cross-section. If titanium skeletons existed, there would be much larger land animals than elephants. (iv) On water the size limitation is higher, since the weight is then partially supported by the water. (v) How about dinosaurs? Well, a tyrannosaurus rex was roughly the same size as a large African elephant, and the larger dinosaurs seem to have been semi-aquatic — if any of you know more about this topic (dinosaurs), please correct me.

[g] Galileo Galilei Linceo, *Discorsi e Dimostrazioni Matematiche intorno à due nuoue scienze,* published in 1638 in Latin and Italian. For a translation into English see: http://files.libertyfund.org/files/753/0416_Bk.pdf *Dialogues concerning two new sciences, by Galileo Galilei.* Translated from the Italian and Latin by Henry Crew and Alfonso de Salvio. The Macmillan Company, NY, 1914.

7.2 Answer: Taylor–von Neumann–Sedov blast wave radius

Task #1. We have

$$\mathcal{E} = \frac{\mathcal{M}\times\mathcal{L}^2}{\mathcal{T}^2} \text{ and } \mathcal{D} = \frac{\mathcal{M}}{\mathcal{L}^3}, \qquad (7.2)$$

where: $\mathcal{E}$ = dimension(energy); $\mathcal{M}$ = dimension(mass); $\mathcal{L}$ = dimension(length); $\mathcal{T}$ = dimension(time); and $\mathcal{D}$ = dimension(mass density). Since E is an energy and ρ_a is a mass density, the answer follows.

Task #2. From (7.2), E/ρ_a has dimensions $\mathcal{L}^5/\mathcal{T}^2$. Hence it must be

$$R = \kappa\left(\frac{E\,t^2}{\rho_a}\right)^{\frac{1}{5}}, \qquad (7.3)$$

where $\kappa > 0$ is a purely numerical constant, which cannot be obtained by similarity analysis.

Systematic process. In this case the situation is simple enough that one can get (7.3) from (7.2) by "inspection". This may not be so in other cases. In general you would proceed as follows: given that you have $\mathcal{E}$, $\mathcal{D}$, and $\mathcal{T}$, consider (7.2) as two equations for the two unknowns $\mathcal{M}$ and $\mathcal{L}$, and solve for them. Then use the answer to produce (7.3). This shows that similarity analysis works only if you have the "right" number of constants — if, for example, we had two densities, we would not know which one to use! **This is where understanding of the physical problem comes into play** — for example, note how when setting up the problem, we eliminated from consideration the pressure ahead, p_a; as well as all the gas parameters behind the wave.

Task #3. Clearly

$$\boldsymbol{E} = \frac{\rho_a\, \boldsymbol{R_1^5}}{\boldsymbol{\kappa^5\, t_1^2}}. \tag{7.4}$$

7.2.1 Historical context

During World War II there was intense interest (guess why?) in the behavior of the blast wave produced by a strong explosion. Turns out that the whole solution to the relevant pde (not just the radius of the wave, as in this problem) can be written using self-similar variables (which reduce the pde to ode). This was the result of (independent) work by G. I. Taylor (UK), John von Neumann (USA) and Leonid Sedov (Soviet Union). While the analysis above leaves the value of $\boldsymbol{\kappa}$ undetermined, solving the ode yields a value. This was done by Taylor numerically; later on (in 1959, see [d]) Sedov showed that the ode's could be solved analytically.

Related anecdote (this is based on the Wikipedia entry). Taylor calculated the energy of the atomic bomb used in the Trinity nuclear test using similarity (see [a]), on the basis of published photographs (see [b]) of the blast wave that had a length scale and time stamps (see [b]).[†] This calculation of energy caused "much embarrassment" (as stated by Taylor himself) in the USA, since the amount energy released was still classified at the time — although the published photographs in [b] were not. According to [c], "This estimate of the yield of the first atom bomb explosion caused quite a stir... G.I. was mildly admonished by the US Army for publishing his deductions from their (unclassified) photographs."

† And now you know how he did it. Obviously, if Taylor did it, so must have the soviets; they had Sedov's work! By the way, having $\boldsymbol{R}$ for several times helps remove the uncertainty of when $\boldsymbol{t=0}$ happened. You can also recover $\boldsymbol{E}$ from knowing $\boldsymbol{R}$ at two times, and having $\boldsymbol{t_2 - t_1}$, but neither $\boldsymbol{t_1}$ nor $\boldsymbol{t_2}$ accurately.

[a] Taylor, G. I. (1950). *The formation of a blast wave by a very intense explosion.-II. The atomic explosion of 1945.* Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences, 201(1065), 175-186.

[b] Mack, J. E. (1946). *Semi-popular motion-picture record of the Trinity explosion.* (Vol. 221). Technical Information Division, Oak Ridge Directed Operations.

[c] Batchelor, G. K., and Taylor, G. I. (1996). *The life and legacy of G. I. Taylor.* Cambridge University Press.

[d] Whitham, G. B., (1974). *Linear and Nonlinear Waves.* Wiley Interscience. See section 6.6 here.

THE END.