Answers to P-Set # 01, (18.353/12.006/2.050)j
MIT (Fall 2024)

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1.1 Statement: Computer exercises with a 1-D map v2.

The objective of this problem is to give an elementary first introduction to concepts such as: fixed point, stability, bifurcations, and chaos, via "experimental" (i.e., numerical) computation; plus a little bit of simple theory. We will do this by using a very simple mathematical model. This model is highly abstract, but we will argue later in the course that the model is also (to a significant degree) representative of the behavior of many real systems. In addition, this assignment also introduces the kind of computing and/or theoretical problems we shall often, but not always, assign.

In this model we will assume that the system is described by a single (scalar) time-dependent variable, $x(t)$. This variable could represent, say, the globally averaged temperature on the Earth's surface, the size of a particular population of animals on some secluded island (or in a Petri dish), some particular stock market average, etc. Furthermore, we will suppose that we are interested only in the values $x_n = x(t_n)$ at discrete times $t_n = n\,\Delta t$, where Δt is some suitable interval of time (say, a day). We then assume that the evolution of x in time may be written as

$$\boldsymbol{x_{n+1} = F(x_n)}. \qquad (1.1)$$

where F is some function that describes the dynamics. For any of the examples mentioned above, it is obvious that the "true" F would involve complicated equations. Rather than going into that kind of detail, we will consider below two rather simple choices for F. That is:

$$\boldsymbol{F(x) = \mu\, x}, \qquad (1.2)$$

and

$$\boldsymbol{F(x) = L\, p_f\, x\,(1 - x^2)}, \qquad (1.3)$$

where $\boldsymbol{p_f = 1.5\sqrt{3}}$, while $\boldsymbol{\mu}$ and $\boldsymbol{L}$ are parameters (real constants) which vary in some range. **Below, after the remark and definition, are your tasks.**

Remark 1.1 *Notice that, for any function F: given x_n, all x_m with $m > n$ are uniquely determined — the forward "time" evolution is well defined.*†

† This simple observation is the key to the answer for some of the theory-tasks below.

However, unless F is invertible, the backward time evolution is **not** *well defined. There may be no x_{n-1} such that $x_n = F(x_{n-1})$, or there may be many. For example, let $F(x) = x^2$ and consider the cases $x_n < 0$ and $x_n > 0$.* ♣

Definition 1.1 Periodic orbits. *Given (1.1), and some x_0, the sequence $\{x_n\}$ is called an orbit for the dynamical system. An orbit is periodic, of period p ($p > 0$ an integer) if* $\boldsymbol{x_{n+p} = x_n}$, *for all n, and no integer $0 < q < p$ satisfies $x_{n+q} = x_n$.*‡

‡ Obviously, if q is a multiple of p, $x_{n+q} = x_n$. Thus the period is the smallest positive integer satisfying $x_{n+q} = x_n$. Note that $p = 1$ corresponds to a *fixed point.* ♣

Note: In general, as $n \to \infty$, the orbits for (1.1) approach either a periodic or a chaotic # orbit, and periodic orbits are very important in the transition to chaos as some parameter in F varies — say, L in (1.3). For this reason below we pay particular attention to the periodic orbits. # See remark **1.2**.

Theoretical questions.

t1 Assume (1.1) and (1.2). Then, without using the computer, what can you predict about the behavior of x_n as $n \to \infty$, given an initial condition $x_0 \neq 0$?

t1a Qualitatively, what is different about the cases $|\boldsymbol{\mu}| < \mathbf{1}$, $|\boldsymbol{\mu}| > \mathbf{1}$, and $|\boldsymbol{\mu}| = \mathbf{1}$?

t1b What is the qualitative difference between the evolution with $\boldsymbol{\mu} > \mathbf{0}$ and $\boldsymbol{\mu} < \mathbf{0}$?

t1c Check your conclusions numerically (do not hand in any plots for this, you can even use a calculator).

Note: In this case $x = 0$ is a *fixed point* of the evolution (i.e.: $x_n \equiv 0$ if $x_0 = 0$), and your conclusions, in particular, speak to the *stability* of this fixed point (what happens if a small perturbation is applied).

t2 Assume (1.1) and (1.3), with $0 \leq L \leq 1$. Show: F maps the unit interval onto itself — i.e.: $0 \leq F(x) \leq 1$ if $0 \leq x \leq 1$. Thus the unit interval is a suitable *phase space* for this dynamical system (phase space was defined in the lectures).

t3 Assume (1.1) and a generic F. Below: prove true or false.

t3a Assume $\{x_n\}$ is periodic of period p. Then $\{x_n\}$ cycles through p different values. Hint: see remark **1.1**.

t3b Assume $\{x_n\}$ cycles through *exactly* $p > 0$ different values, with every value taken at least twice in the sequence. Then

$\{x_n\}$ is periodic of period p.

Note: This fails for generic sequences; that this is a deterministic dynamical system matters. Examples: Flips of a coin with values head (1) or tail (0). Digits of the decimal representation of an irrational number.

Note: Twice each value matters. Example: $\{3, 1, -1, 1, -1, 1, -1\dots\}$, with $F(x) = -1 + 0.5\,(x-1)^2$, is not periodic if $x_0 = 3$ is included.

Hint: see remark **1.1**.

t3c It is possible to have a non-constant $\{x_n\}$ such that $x_{n+2} = x_n$ and $x_{n+5} = x_n$. Hint: see remark **1.1**.

t3d Optional. Let $\{x_n\}$ have period p, and assume $x_{n+q} = x_n$ for some $q > p$. Then q is a multiple of p.

Computer exploration. The aim is to numerically explore the system in (1.1), with $\boldsymbol{F}$ as in (1.3) and $\mathbf{0} \leq \boldsymbol{x}, \boldsymbol{L} \leq \mathbf{1}$ — see item **t2**. Specifically: **As a function of the parameter $\boldsymbol{L}$, what is the behavior of the orbits $\{\boldsymbol{x_n}\}$ as $\boldsymbol{n \to \infty}$?** Further details (hints and tasks) follow.

c1 For this task a MatLab script, IterateCubiMap, is supplied. Please **read the script description at the top of the IterateCubiMap.m file**, which describes in detail what the script does and how to use it. IterateCubiMap was specifically designed to help with the required task: **How does x_n behave for n large?**

However, **you should not restrict yourself to this script only.** For example, from item **t3** it follows that you can get a lot of information (for a given L) by plotting/marking in the interval $0 \leq x \leq 1$ the points visited by the sequence $\{x_n\}$ for n large (this so the sequence has time to achieve its asymptotic behavior). Then:

c1a If the orbit visits only a finite set of points, then it is periodic, and the period is the number of points.

c1b Chaotic orbits will fill regions, containing an infinite number of points (but see remark **1.2**).

Note. For now we will not attempt to define chaos. Just "looks random" is enough — see remark **1.2**.

Note. Of course, with a computer you cannot plot "an infinite number of points"; but you can detect orbits that visit a very large number of points.

Note. Visiting an infinite number of points is not enough to characterize a discrete dynamical system orbit as chaotic; but for us here this will do. Further, the regions visited by a chaotic orbit are not "intervals"; in fact they can be quite complicated (fractals); though it would be very hard for you to detect their structure with the tools you have.

You should be able to write your own program implementing the idea in this paragraph (not required, though).

c2 The system (1.1/1.3) exhibits many behaviors. For example, $\boldsymbol{x_n}$ may approach a constant (a fixed point) as $\boldsymbol{n \to \infty}$, or it may approach a *periodic cycle*, where $\boldsymbol{x_{n+p} = x_n}$ ($\boldsymbol{p}$ is the period), or the $\boldsymbol{n \to \infty}$ behavior may be chaotic (see remark **1.2**), or it may exhibit *intermittent chaos*, where the sequence alternates (seemingly randomly) between being close to a periodic cycle, and chaotic bursts. [1] These various behaviors correspond to different values of $\boldsymbol{L}$, with the transition from one to another occurring at *critical values* of $\boldsymbol{L}$ (of which there are many; infinitely many, in fact).

A value $\boldsymbol{L = L_c}$ is called a critical value if the long term evolution for $\boldsymbol{x_n}$ changes *qualitatively* as $\boldsymbol{L}$ crosses $\boldsymbol{L_c}$. These qualitative changes are called **bifurcations**. For example, on one side of $\boldsymbol{L_c}$ the system may be attracted to some fixed point $\boldsymbol{x_*}$, and to a different fixed point on the other side. Or maybe the behavior switches from fixed point to periodic of some period $\boldsymbol{p > 1}$, or the period changes across $\boldsymbol{L_c}$, etc.

These are the questions you are asked to investigate:

c2a There is a critical value, $\boldsymbol{L_1}$, such that for $L < L_1$ the orbits converge to $\boldsymbol{x = 0}$, and for $\boldsymbol{L_1 < L < L_2}$ they converge to a constant $\boldsymbol{x = x_* > 0}$. Find L_1 and L_2 (approximately) — *it is very hard to accurately compute the critical values numerically, and you are not being asked to do so.*

Optional. In fact, L_1 and x_* are easy to compute analytically. *Can you do so?*

Hint: Linearize the map for $\boldsymbol{x}$ small, and use item **t1**.

Note that $\boldsymbol{L_1}$ and $\boldsymbol{L_2}$ are *critical values, or bifurcation points.*

c2b What happens for $\boldsymbol{L}$ slightly above $\boldsymbol{L_2}$?

c2c In fact, there is an infinite sequence of critical points $\boldsymbol{L_1 < L_2 < L_3 < \ldots}$, with $\boldsymbol{\lim_{n\to\infty} L_n = L_\infty}$ (where $\boldsymbol{0.88 < L_\infty < 0.89}$). In each "window" $\boldsymbol{L_n < L < L_{n+1}}$, the limit behavior of $\boldsymbol{x_n}$ is very simple — in part **c2b** you should already have discovered what happens for $\boldsymbol{L_2 < L < L_3}$. **Describe the behavior in these windows.** Note: it is very hard to compute in any detail anything beyond L_4,‡ but you can tell what happens for $\boldsymbol{L_4 < L < L_5}$ without knowing L_5. **From these first few windows you should be able to figure out (guess) what the pattern is for $\boldsymbol{L_n < L < L_{n+1}}$.**

‡ The window width $L_{n+1} - L_n$ decreases exponentially with n. Very quickly you will need more digits than you have to tell the L_n them apart.

c2d What happens slightly beyond $\boldsymbol{L_\infty}$. Do your best here; see remark **1.2**.

[1] The MatLab script provided allows you to check for these behaviors.

c2e **What do you see beyond L_∞, as you move up towards $L = 1$.** Let us see how much can you find (there is infinite detail here, so the sky is the limit). Hint. More critical values, more periodic windows, more chaos, intermittent chaos.

Remark 1.2 *For this problem we will not define chaos (will be done later). Here simply check that the behavior is not periodic of any "reasonable" period,† e.g.: $1 \le p \le 16$ (many can be excluded by simple eye-sight).* ♣

† Because you are doing this in a computer, there is only a finite number of values x_n can take, so the computed sequence x_n will always be periodic ... but the period can be huge, for any practical purposes "infinite".

1.2 Answer: Computer exercises with a 1-D map v2

The **problem solution,** with the same item numeration used in the statement, follows below. ***I do not expect you to provide an answer with the amount of detail here;*** the **key things** are that you: (a) Answered the theoretical questions. (b) Answered the computational exploration questions in **c2a-c2c**, and did a reasonable attempt at **c2d-c2e**. (c) Computed the requested critical points with reasonable accuracy (a couple of digits).

Remark 1.3 **MatLab scripts included with the answer.** Two MatLab files are included with the answer:
IterateCubiMapSelections.m This script runs IterateCubiMap for a menu of parameters designed to illustrate the behaviors described in this answer. To run it, just use IterateCubiMapSelections(ns), where ns is a number that picks a selection.
IterateCubiMapAttSensi.m This script, and why it is important, is described in remark **1.6**. ♣

Remark 1.4 **Calculating the various thresholds, "naive" approach.** I will illustrate how to do this, using the MatLab script IterateCubiMap, with the example of L_2 — the value at the boundary between the period 2 and constant x_* attractors. Let us begin by assuming that you have already found values $L_\pm$ above and below L_2 (say, by trial and error, or by doing a coarse sweep of L-values). Then you can check if $L = (L_- + L_+)/2$ is above or below L_2, thus halving the size of the interval where L_2 is known to be. This will quickly narrow down the value of L_2 to the limited accuracy this approach can achieve — see remark **1.5**.
Important: (i) As you get closer to the critical value, you will notice that the dynamical system converges to the attractor slower, so you will need to make the parameter n1 (iterates done before looking at the attractor) larger. (ii) IterateCubiMap includes a parameter, n3, that allows you to check if the answer has period n3. This is critical because near L_2 the period 2 solutions for $L > L_2$ cannot be distinguished "by eyesight" from a constant. (iii) If the period check tells you that the error in satisfying, say, the period 2 condition is 10^{-14} (or some other rather small number, but not zero), it does not mean "it is not period 2". You have limited accuracy, so something that small has to be interpreted as "it likely is period 2" — see remark **1.5**.
Alternative approach: In order to quickly determine the various bifurcations, I used an approach based on the suggestion in **c1** of the statement (see **c1** in the answer below). However, this approach does not remove the need to use IterateCubiMap. ♣

Remark 1.5 **Limits on accuracy.** You may think that, if you are working in double precision (with about 16 digits) you should be able to get the L_n with a lot of digits (if you are patient enough). But this is not true, the approach in remark **1.4** will, generally, not be able to provide you with more than a few significant digits — "why" is explained in some detail in remark **1.7**
Can you do better? Answer: **yes, significantly better,** but this requires theoretical knowledge not yet introduced in the lectures. Using such knowledge, much better approaches to computing the thresholds can be developed.
For that matter, **consider the "period check"** done by the script IterateCubiMap — plot $\Delta_n = x_{n+p} - x_n$ versus n for the computed sequence of iterates $\{x_n\}$. In exact arithmetic it would be $\Delta_n \equiv 0$ for a period-p sequence. But with fixed point calculations, you will know (at best) each x_n with a fixed number of digits (given by the available precision; about 16 for MatLab), or worse (because errors can accumulate). So Δ_n will probably not be zero. Because in the case here the x_n are in the range $0.1 < x_n < 1$ for the periodic solutions, a reasonable practical criteria is that $|\Delta_n| < 10^{-10}$ should be considered as "zero".† ♣

† What I saw in my calculations was either (i) $|\Delta_n| < 10^{-15}$ (quite often $|\Delta_n| \ll 10^{-15}$); or (ii) $|\Delta_n| \ge O(10^{-2})$. Thus the distinction between periodic or not was clear-cut.

Theoretical questions.

t1 The solution to (1.1/1.2) can be written explicitly: $$x_n = \mu^n\, x_0. \qquad (1.4)$$
It follows that:

t1a When $|\mu| < 1$, $x_n \to 0$ as $n \to \infty$. Hence the fixed point $x = 0$ is an attractor, in particular: **stable**.
When $|\mu| > 1$, $|x_n| \to \infty$ as $n \to \infty$. There is no attractor, and $x = 0$ is **unstable**.

When $|\boldsymbol{\mu}| = \mathbf{1}$, $|\boldsymbol{x_n}| \equiv |\boldsymbol{x_0}|$. There is no attractor, but $\boldsymbol{x} = \mathbf{0}$ is **"neutrally" stable** (perturbations neither grow nor decay in size).

t1b When $\boldsymbol{\mu} > \mathbf{0}$ all the iterates have the same sign. No oscillations.
When $\boldsymbol{\mu} < \mathbf{0}$ the iterates alternate sign. Period 2 oscillations (damped if $|\boldsymbol{\mu}| < \mathbf{1}$, growing if $|\boldsymbol{\mu}| > \mathbf{1}$).

t1c No reply was requested here. Numerically check that what is said above is correct.

t2 Done in figure 1.1.

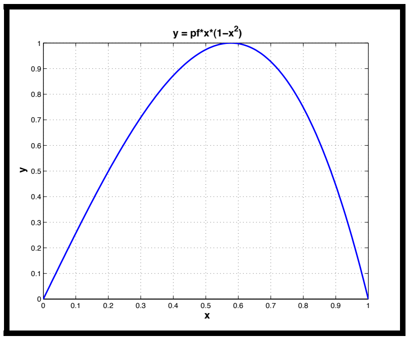

Figure 1.1: Plot of the map F for $L = 1$.
On the left: plot of F for $L = 1$ — i.e.: $\boldsymbol{y = p_f\, x\, (1 - x^2)}$.
Assume (1.1/1.3), with $L > 0$. It is easy to see that $F(x) \geq 0$ for $0 \leq x \leq 1$, and that F reaches a local maximum ($F' = 0$ and $F'' < 0$) at $x = 1/\sqrt{3}$, with $F(1/\sqrt{3}) = L$. Hence F *maps the unit interval onto itself for* $0 \leq L \leq 1$ (the case $L = 0$, not done, is trivial). ♣

t3 Assume (1.1), with F "generic". Then:

t3a $\{x_n\}$ *periodic of period* $p \implies$ *it cycles through* p *different values.*
Proof. Since $x_{n+p} = x_n$, the sequence takes no more than p values. Further, suppose that in one period a value repeats: For some $n \leq n_1 < n_2 < n + p$, $x_{n_1} = x_{n_2}$. Then remark **1.1** would imply $x_{n+q} = x_n$, with $q = n_2 - n_1 < p$, and p would not be the period. ♣

t3b *Let* $\{x_n\}$ *cycle through* exactly $p > 0$ *different values, with every value taken at least twice in the sequence. Then* $\{x_n\}$ *is periodic of period* p.
Proof. If the sequence starts at some $n = n_1$, consider the first repeat of the initial value: $x_{n_1} = x_{n_2}$, where $n_2 > n_1$ — otherwise, take n_1 arbitrary. Then remark **1.1** yields $x_{n+q} = x_n$, where $q = n_2 - n_1$. Thus the sequence is periodic, with some period p_*. Using **t3a**, it follows that the sequence takes p_* different values. Hence it must be $p_* = p$. ♣

t3c **False:** *It is possible to have a non-constant* $\{x_n\}$ *such that* $x_{n+2} = x_n$ *and* $x_{n+5} = x_n$.
Proof. (i) $x_{n+2} = x_n$ yields $x_0 = x_2 = x_4$. (ii) $x_{n+5} = x_n$ yields $x_0 = x_5$. But then $F(x_0) = F(x_2) = F(x_4)$ from (i) and $F(x_4) = x_5 = x_0$ from (ii) so that $F(x_0) = x_0$. Hence the sequence is constant. ♣
Note: here I started the argument with x_0, but it can be done with any x_m, and then it shows $x_{m+q} = x_m$ for all $q \geq 0$.

t3d *Let* $\{x_n\}$ *have period* p, *and assume* $x_{n+q} = x_n$ *for some* $q > p$. *Then* q *is a multiple of* p.
Proof. Let q_L = largest multiple of p with $q_L \leq q$. If $q_L = q$, then we are done. Otherwise $0 < p_* = q - q_L < p$ [A]. Now, since $x_{n+q} = x_n$ and $x_{n+q_L} = x_n$, we have $x_{m+p_*} = x_m$ (take $m = n + q_L$). But then the definition of period requires $p \leq p_*$, which contradicts [A]. ♣

Computer exploration.

c1 Figure 1.2 summarizes the behavior of the dynamical system (1.1/1.3) in $0 \leq x \leq 1$ for all $0 \leq L \leq 0$. It is easy to see that: **(a)** For $\boldsymbol{0 \leq L \leq L_1 \approx 0.4}$, the solutions are attracted to zero ($x_n \to 0$). **(b)** For $\boldsymbol{L_1 \leq L \leq L_2 \approx 0.77}$, the solutions are attracted to a constant $\boldsymbol{0 < x_* < 1}$ ($x_n \to x_*$). **(c)** At $\boldsymbol{L_2}$ another bifurcation occurs, beyond which the solution is attracted to a period 2 solution.† **(d)** Continuing to increase L, more bifurcations occur at values L_n, where period 2^{n-1} sequences‡ become attractors. **(e)** What happens with L_n, n large, and beyond is not clear from this picture. In what follows we will investigate this in more detail by amplifying various regions of this **bifurcation diagram.**

† Use IterateCubiMap.m to check these are period 2 solutions and not separate possible fixed points.

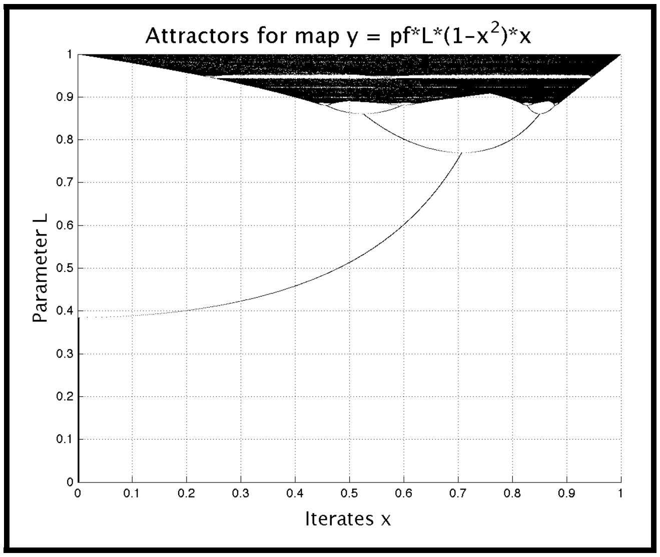

Figure 1.2: Bifurcation diagram for the map F.

How was the picture on the left done? Using the suggestion in the statement item **c1** "However, **you should not restrict yourself to this script only.** For example ...", I wrote a script that, for each L-value, plots the x_n-values as dots in some preselected interval $x_1 \le x \le x_2$. Then it stacks the plots (using many values of L) in a 2-D plot with L in the vertical coordinate, and x in the horizontal. Each plotted sequence $\{x_n\}$ is processed as follows: **(i)** Pick some random initial value $0 < x_0 < 1$. **(ii)** Iterate the map n1 times, where n1 is fairly large (I used n1 = 1000, 3000, and 10000 for the various plots) so any transients are gone. **(iii)** Plot the next n2 iterates (I used n2 = 1000). For L I used an equispaced grid in the desired L-range (with either 1000 or 3000 values).

Note: it is important to control the size of the dots used to plot the x_n. Too small: cannot see very much. Too large: important features are obscured. In MatLab dot size is controlled by the 'MarkerSize' option in the plot statement.

Use the same process for other periodic solutions we will identify.

‡ Here we are relying on **t3a-t3b** to identify periodic solutions.

c2 Below we dive into a more detailed exploration of the dynamical system, using the bifurcation diagram in figure 1.2 as a guide. The answers to the questions in item **c2** of the problem statement can be found below, as follows: For **c2a** see **[A-B]**. For **c2b** see **[B]**. For **c2c** see **[C]**. For **c2d** see **[E]**. For **c2e** see **[F-H]**.

[A] The critical value L_1. Consider the left panel of figure 1.3. There we see that for $\mathbf{0 \le L < L_1 \approx 0.385}$ the solutions are attracted to $x = 0$ ($x_n \to 0$ as $n \to \infty$), while for L above† L_1 they are attracted to a value $\boldsymbol{x_* > 0}$.

† Up to L_2, see the full bifurcation diagram in figure 1.1.

Explanation and analytical calculation of L_1 and x_*. It is easy to see that for L "small", the curve $y = F(x)$ is below $y = x$, so that there is only one fixed point (solution to $x = F(x)$), i.e.: $x = 0$. Furthermore, $x = 0$ is then **stable**, because $\boldsymbol{0 < F'(0) = L\,p_f < 1}$. When L crosses (going up) $1/p_f$ two things happen: (i) $x = 0$ becomes unstable, and (ii) another fixed point appears.[2]

We conclude that

$$\boldsymbol{L_1 = \frac{1}{p_f} = 0.3849001794\ldots} \qquad (1.5)$$

Next, to obtain x_* we must find a positive solution to $x = L\,p_f\,x\,(1 - x^2)$; i.e.: $1 = L\,p_f\,(1 - x^2)$. Hence

$$\boldsymbol{x_* = \sqrt{1 - \frac{L_1}{L}}}, \qquad (1.6)$$

which (of course) requires $L > L_1$ to satisfy $x_* > 0$.

Notice that **there is also a negative, $x = -x_*$, fixed which arises for $L > L_1$.** We do not see it because we are only looking at $0 \le x \le 1$ — but F is odd, so anything that happens for $x > 0$ has a mirror reflection for $x < 0$. Hence **the bifurcation at $L = L_1$ is the discrete analog of a (soft) pitchfork bifurcation.**

[B] The critical value L_2. Consider the right panel of figure 1.3. There we see that at $\boldsymbol{L_2 \approx 0.770}$ another bifurcation happens. The solutions switch from being attracted to the critical point x_*, to being **attracted to a period 2 cycle.**

Note: pointed out before, the situation for $L > L_2$ could correspond to two separate possible (stable) critical points arising. However, using the script IterateCubiMap, you can easily see that this is not the case. Further: the calculation leading to (1.6) shows that

[2] Because then $y = F(x)$ goes above $y = x$ to the right of $x = 0$, but must eventually cross below to reach $F(1) = 0$.

the <u>only</u> critical points are $x = 0$ and $x = \pm x_*$.
Analytical calculation of L_2. The stability of x_* is easy to ascertain from

$$F'(x_*) = 3 - 2\,L\,p_f = \mu. \tag{1.7}$$

We can see that μ starts at $\mu = 1$ at $L = L_1$, and goes down as L increases (making **x_* stable**). However, when $L\,p_f = 2$, $\mu = -1$,
beyond which x_* is unstable. We conclude that

$$L_2 = 2\,L_1 = 0.7698003589\ldots \tag{1.8}$$

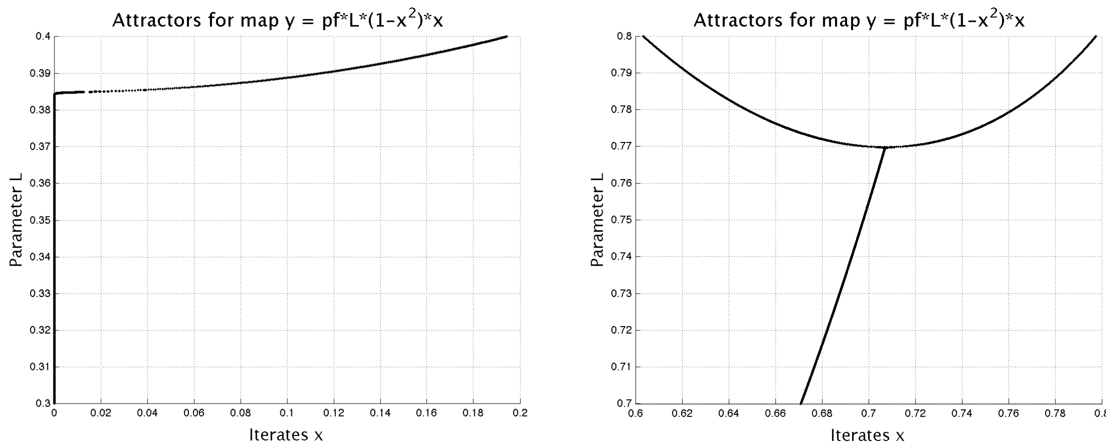

Figure 1.3: Bifurcation diagram details: near L_1 and near L_2. The left panel shows the region $0.3 \leq L \leq 0.4$ and $0 \leq x \leq 0.2$, while the right panel shows $0.7 \leq L \leq 0.8$ and $0.6 \leq x \leq 0.8$.

[C] From L_2 to L_∞. The left panel of figure 1.4 focuses on the bifurcations following L_2, and shows $\boldsymbol{L_3 \approx 0.861}$, $\boldsymbol{L_4 \approx 0.881}$, and $\boldsymbol{L_5 \approx 0.885}$. At each of these bifurcations the attractor doubles[3] its period: from 2 to 4 at L_3, from 4 to 8 at L_4, and from 8 to 16 at L_5. The right panel of figure 1.4 (see also figure 1.5) shows that the pattern **(period doubling cascade)** continues, with an infinite[4] sequence of critical points $\boldsymbol{0 < L_1 < L_2 < L_3 < \ldots}$ such that: **for $L_n < L < L_{n+1}$ the attractor is a periodic solution of period 2^{n-1}.** In addition, *the sequence is bounded, hence it must have a limit* $\boldsymbol{L_\infty = \lim_{n\to\infty} L_n = \sup(\{L_n\})}$.
By focusing on very small regions † of the bifurcation diagram of figure 1.2, one can get the following refined estimates for the bifurcation values: † The process is time consuming (but straightforward), as it requires the use of fairly large values for the parameters n_1 and n_2 introduced in figure 1.2; i.e.: $n_1 = O(10^6)$ and $n_1 = O(10^5)$.

$$\boldsymbol{L_3 = 0.8606625 \;\pm 5\times 10^{-7};\quad L_4 = 0.8806638 \;\pm 3\times 10^{-7};\quad L_5 = 0.8849732 \pm 2\times 10^{-7};}$$
$$\boldsymbol{L_6 = 0.88589737 \pm 4\times 10^{-8};\quad L_7 = 0.88609535 \pm 2\times 10^{-8};\quad \ldots}$$
$$\boldsymbol{L_\infty = 0.88614935 \pm 5\times 10^{-8}.}$$

The accuracy shown here is the best possible with the method used, and double precision fixed point arithmetic (about 16 digits) — for more details see remark **1.7**. Notably, the accuracy in the calculation of L_n seems to improve (a little) as n grows — but I do not know why this is so.

[D] Universality. The differences $\boldsymbol{(\Delta L)_n = L_{n+1} - L_n}$ **decrease exponentially with n.** Rounding to 7 digits:

$$\boldsymbol{\Delta_1 = 0.3849002,\;\; \Delta_2 = 0.0908621,\;\; \Delta_3 = 0.0200013,\;\; \Delta_4 = 0.0043094,\;\; \Delta_5 = 0.0009242,\;\; \Delta_6 = 0.0001980.}$$

Thus $\boldsymbol{\mathcal{F}_1 = 4.236090}$, $\boldsymbol{\mathcal{F}_2 = 4.542812}$, $\boldsymbol{\mathcal{F}_3 = 4.641319}$, $\boldsymbol{\mathcal{F}_4 = 4.662995}$, $\boldsymbol{\mathcal{F}_5 = 4.667997}$, where $\boldsymbol{\mathcal{F}_n = \frac{\Delta_{n+1}}{\Delta_n}}$, for a decrease ratio per n of 4 to 5.

In fact, it can be shown that

$$\lim_{n\to\infty} \mathcal{F}_n = \delta_{F_1}, \tag{1.9}$$

where $\boldsymbol{\delta_{F_1} \approx 4.6692016091029}$ is the **First Feigenbaum universal constant**
(e.g.: see Peitgen, Jurgens, and Saupe; *Chaos and Fractals*, Springer Verlag, 1992).

[3] Again: you should use IterateCubiMap to verify that, indeed, these are all periodic orbits of the stated period.
[4] Numerically we cannot prove the sequence is infinite, but the evidence is strong, and can be backed up by analytical arguments.

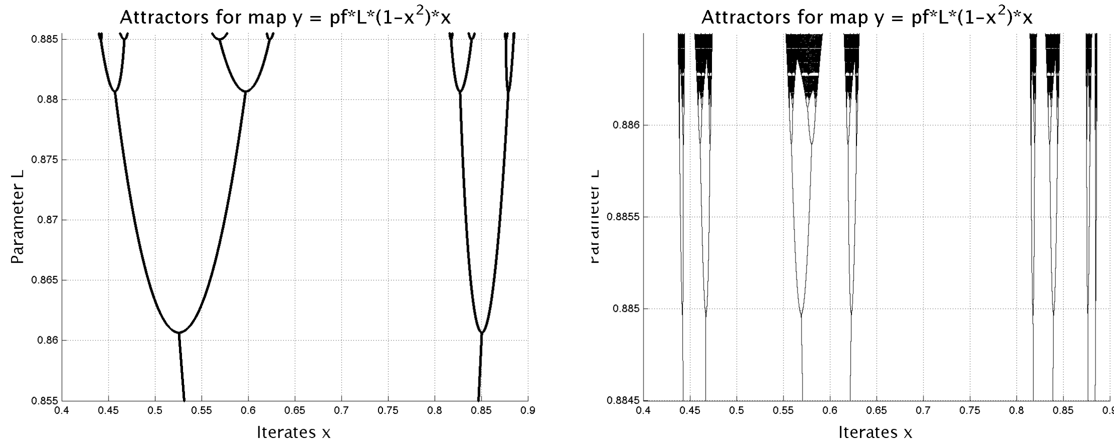

Figure 1.4: Bifurcation diagram beyond L_2. The left panel shows the region $0.8550 \le L \le 0.8855$ and $0.4 \le x \le 0.9$ (zooms on L_3, L_4, and L_5), while the right panel shows $0.8845 \le L \le 0.8865$ and $0.4 \le x \le 0.9$ (starting above L_4, with a period 8 solution, and going past the start of chaos).

Universal refers to the fact that $\boldsymbol{\delta_{F_1}}$ appears for the sequence of bifurcations of any map $\boldsymbol{x_{n+1} = F(x_n,\, \lambda)}$ with a single (unimodal) quadratic maximum that increases with λ ($F_\lambda >$ constant > 0).

[E] Crossing L_∞; chaos. As L crosses L_∞ the attractor stops being a periodic sequence, and switches to being a (seemingly) random walk through an infinite set of points — you can see this in figure 1.5. The MatLab script handed with this answer IterateCubiMapSelections has two values in this range preselected: $\boldsymbol{L = 0.886150}$ (set input ns = 6) and $\boldsymbol{L = 0.886160}$ (set input ns = 7). Of course, this attractor is not random (cannot be, the system is deterministic), but chaotic. For a peek at what chaos is, see remark **1.6**.

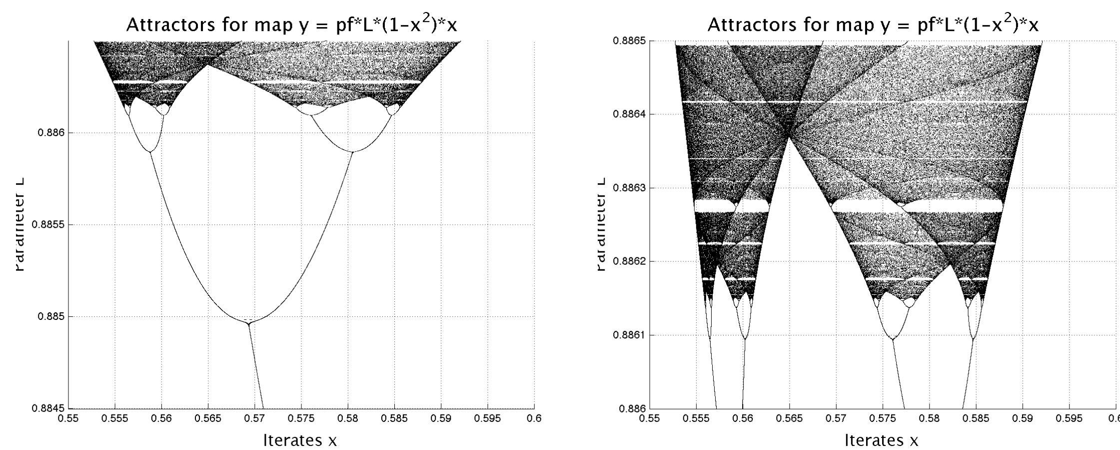

Figure 1.5: Bifurcation diagram from period 8 to beyond L_∞. The right panel in figure 1.4 is very hard to read; hence, using the fact that the picture has an 8-fold symmetry (corresponding to the period 8 solutions chain of bifurcations), we **show here, on the left panel,** the third branch from the left in the right panel of figure 1.4. That is: $0.8545 \le L \le 0.8865$ and $0.55 \le x \le 0.60$. The **right panel** shows further detail, from period 32 and up only, corresponding to $0.8860 \le L \le 0.8865$ and $0.55 \le x \le 0.60$.

[F] Beyond L_∞ (more periodic orbits, more period doubling, more chaos, . . .). Figure 1.1 (better yet, figure 1.5) indicates that chaos fills up large chunks of the region $\boldsymbol{L > L_\infty}$, but that the space is shared with windows where the attractor is a periodic orbit, which undergoes period doubling within the window.‡

‡ There is an infinity of windows (most very small and hard to see), including windows within windows.

Let us now explore this a little.

Period 6 window, left panel of figure 1.6. The window starts with a period 6 attractor at the bottom, which then undergoes a sequence of period doublings (i.e.: periods 6, 12, 24, ...) entirely similar to that of the $\{L_n\}$ we studied before. This includes a limiting "L_∞", followed by chaos and periodic windows within the chaos — and, within these windows, more period doubling, etc., ad infinitum!

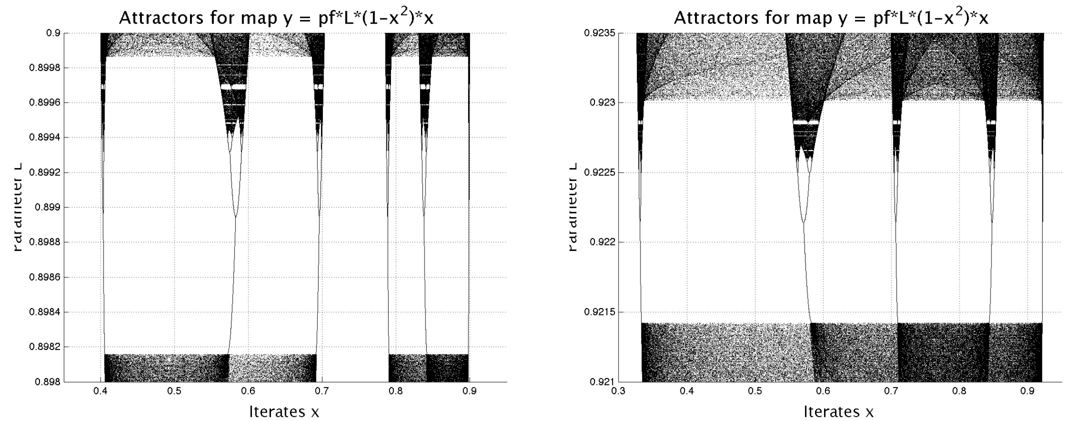

Figure 1.6: Period 6 and period 5 windows. The left panel ($0.898 < L < 0.900$ and $0.35 < x < 0.95$) shows the period 6 window. The right panel ($0.9210 < L < 0.9235$ and $0.30 < x < 0.95$) shows the period 5 window.

Period 5 window, right panel of figure 1.6. The situation is entirely similar to that of the period 6 window.

Period 3 window, left panel of figure 1.7. The situation is entirely similar to that of the period 6 window.

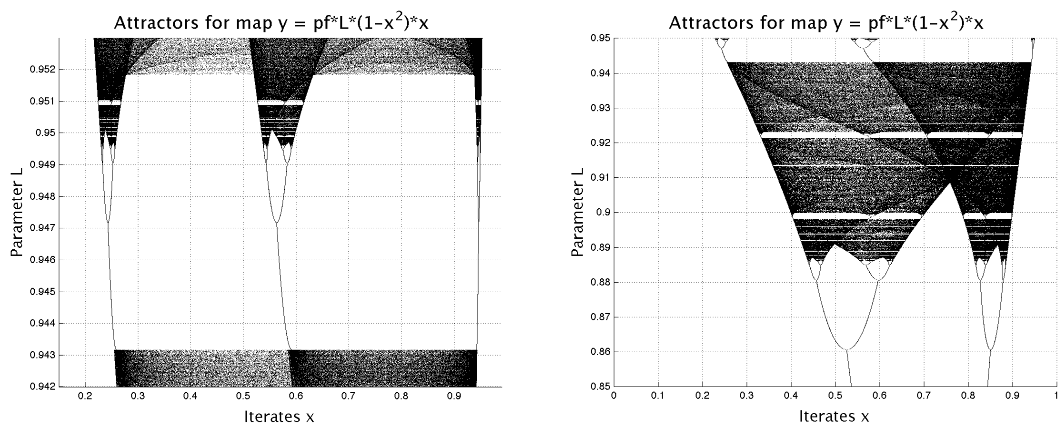

Figure 1.7: Period 3 window and period 2 to 3. The left panel ($0.942 < L < 0.953$ and $0.15 < x < 0.99$) shows the period 3 window. Right panel ($0.85 < L < 0.95$ and $0 < x < 1$) shows: bifurcation diagram from period 2 to period 3.

[G] Intermittent chaos. Within the chaos regions of the bifurcation diagram (see right panel of figure 1.5, right panel of figure 1.7, and figure 1.8) dark lines can be easily seen. These lines corresponds to regions that the orbit visits more frequently, which causes a higher density of dots there. They are caused by a phenomena entirely analogous to the "critical slow down" that we saw in the lectures for the dynamical system $\dot{x} = f(x, \lambda)$ when a saddle node bifurcation is about to happen (i.e.: a "phantom" critical point). In this case, the underlying mechanism is "phantom" periodic orbits. Thus, as the chaotic orbit transits these regions, it more-or-less tracks the "phantom" orbit for a while, and then resumes its chaotic bouncing. The net effect is *intermittent chaos*, whereby the orbit appears to consist of a periodic orbit interrupted by burst of chaos.

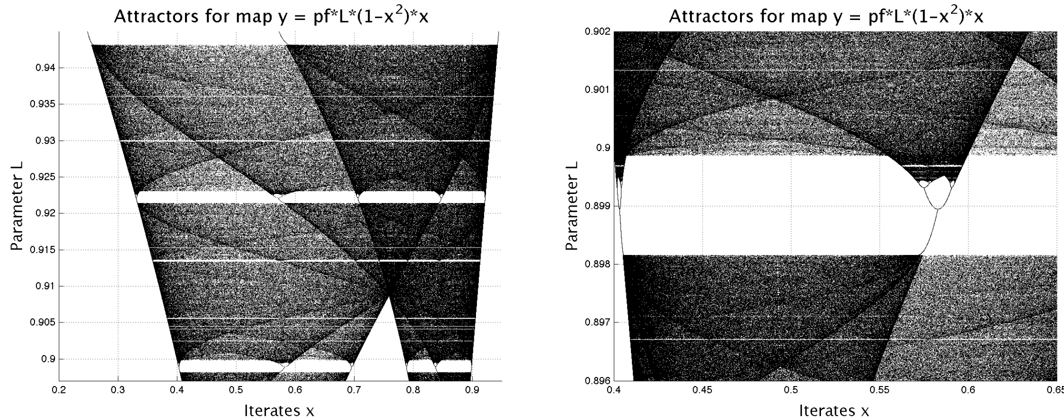

Figure 1.8: Darker lines signaling intermittency. Left ($0.897 < L < 0.945$ and $0.2 < x < 0.95$): bifurcation diagram; period 6 to 3 windows. Right ($0.896 < L < 0.902$ and $0.4 < x < 0.65$); detail: left side of the period 6 window.

For an example of intermittent chaos, see figure 1.9. Note that the value of L used for the figure, $L = 0.898040$, is one of the ones pre-loaded into the MatLab script IterateCubiMapAttSensi — which shows that sensitivity to perturbations occurs for this L-value. By the way: when you run IterateCubiMapAttSensi, you should run it several times for a given input. This because the behavior of the perturbation may depend on the random initial x_0, and what you want to see is the typical behavior.

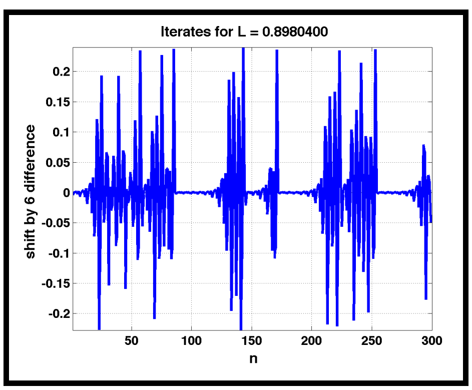

Figure 1.9: Period 6 intermittent chaos.

On the left: plot of $x_{n+6} - x_n$ (as a function of n) for the map iterates $\boldsymbol{x_{n+1} = L\, p_f\, x_n\,(1 - x_n^2)}$, with $\boldsymbol{L = 0.898040}$ — after 3×10^6 iterations, to ensure that any transients are gone. The plot was done using the MatLab script IterateCubiMap.

The value of L used is slightly below the start of the period 6 window, so that there is a "ghost" of a stable period 6 solution under the hood. Via the "critical slowing down" phenomena, this causes the solution to be almost trapped when it gets close to the ghost.

The result of this is a chaotic solution with intervals where it is close to a period 6 sequence, as evidenced by the plot on the left. ♣

[H] Self-similarity. The bifurcation diagram in figure 1.2 is self-similar (fractal): smaller portions of it reproduce the whole pattern — e.g.: see the windows description in **[F]**, as well as: figure 1.5, figure 1.7, and figure 1.8 (all of which display this phenomena prominently). As we will see later, there is also **universality** in this behavior. For example: all the unimodal maps with a quadratic maximum exhibit the same pattern of windows (which windows show up, and how they are ordered).

Remark 1.6 **Chaos and sensitivity to small perturbations.** In the problem statement we were pretty vague as to what chaos is; "defining" it as simply "non-periodic (see remark **1.2**). Naively, as something that "looks" random. Here we introduce a **second property of chaos,** which together with **non-periodic and bounded** (in our case bounded is automatic) offers a **reasonable characterization of chaos.** This second property is .... **the solutions in the attractor are sensitive to small perturbations.**

Important: while periodic attractors consist of just one orbit, chaotic attractors are sets with infinitely many orbits. What this means is that when a chaotic solution is slightly perturbed, the perturbation grows exponentially in time — as long as it remains small. The script IterateCubiMapAttSensi is designed to test this property. This is done as follows: **First.** Iterate the map for $\boldsymbol{1 \le n \le n_1}$, where $\boldsymbol{n_1}$ is large enough that transients decay and $\boldsymbol{x_n}$ would be in the attractor for $\boldsymbol{n > n_1}$ (e.g.: $\boldsymbol{n_1 = 10^6}$). **Second.** Continue the iteration for $\boldsymbol{n_1 < n \le n_1 + n_3}$ (where $\boldsymbol{n_3}$ is also large). In addition, start a second iteration, with

perturbed initial data $y_{n_1} = x_{n_1} + \delta$, where δ is small (e.g.: $\delta = 10^{-5}$) — this also for $n_1 < n \le n_1 + n_3$. **Third.** Plot the difference $y_n - x_n$ for $n_1 < n \le n_1 + n_3$.
When the test is used on a **periodic attractor,** the **perturbation** $|y_n - x_n|$ never grows beyond a few times the initial size, and it eventually **decays**. On the other hand, for a **chaotic attractor** the **perturbation grows** (at first exponentially) till it reaches a size orders of magnitude larger than δ, and then it does not decay (although its size may fluctuate wildly). **You should test these facts by running the script** with its pre-loaded examples, or input your own.†

†The script includes a parameter n_2 — plot the attractor for $n_1 < n \le n_1 + n_2$. Do not take n_2 large, or you will not see much.

Sensitivity to perturbations is a very important property of chaos, and the basis for the "butterfly effect".

The term "butterfly effect" was coined Edward Lorenz, in the 1960's, possibly inspired by the short fiction story "A Sound of Thunder" (Ray Bradbury, Collier's magazine, June 28, 1952; and Bradbury's collection "The Golden Apples of the Sun", 1953). ♣

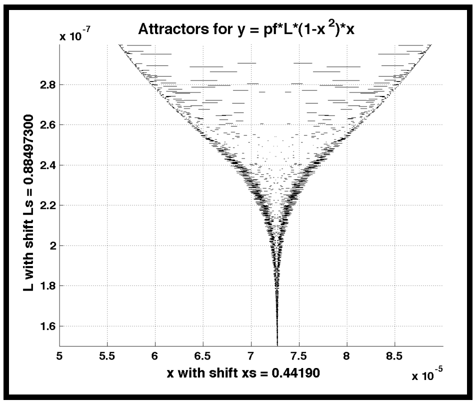

Figure 1.10: Numerical errors make bifurcations fuzzy.

Detail of the bifurcation diagram in figure 1.2 near L_5. Recall that L_5 is the critical value at which the attractor goes from period 8 to period 16 — in the bifurcation diagram it appears as 8 same-L "forks". The picture is a detail near one of the forks; specifically, for the region: $0.88497315 < L < 0.88497330$ and $0.44195 < x < 0.44199$, using 1000 equispaced values of L. For each L a random x_0 was selected and then 10^6 map iterations were performed before starting to plot (which was done for the next 2.5×10^5 iterations).
Because of the mechanism explained in remark **1.7**, the iterates cannot (numerically) quite converge to the attractor when L is close to a critical value, which results in the bifurcation being smeared — thus limiting the precision with which the critical value can be computed.

Remark 1.7 **Errors in computing thresholds.** The first thing to point out is that, as L crosses a L_n going up, the "perturbation" to the prior attractor (e.g: failure to satisfy the period 2 condition) has size $O(\sqrt{L - L_n})$ when $L - L_n$ is small.‡

‡ The script IterateCubiMapSelections includes examples showing this.

Now, if your have only 16 significant digits,[5] you can only tell that you "have crossed L_n" only once $\sqrt{L - L_n} \approx 10^{-16}$, or larger. You cannot tell the effect of smaller changes because they are beyond the resolution you have. This gives you the largest penalty: at best you can hope for 8 digits in L_n.
But this is not the end of it. You also have to consider the fact that in fixed point calculations, the last digit jumps around. The behavior is not "random" (i.e.: it is deterministic[6]) but its effect is quite similar to a random perturbation. Because the approach to the attractor is very slow close to L_n, these perturbations are able to derail it, creating a fuzzy region near L_n, where it is not clear which side of the critical number you are — see figure 1.10. This makes things worse, and more digits may be lost.† ♣

† In figure 1.10 the fuzzy region is small; size $O(10^{-7})$ in L. For other L_n it is larger (or slightly smaller). The stronger the stability of the involved attractors, the smaller the region is.

THE END.

[5] Technically, I should be doing this analysis using binary notation, but at the rough level here this is not important.
[6] If you run a calculation twice, in the same computer, you get the same answer.