Answers to P-Set # 08, (18.353/12.006/2.050)j
MIT (Fall 2023)

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1.1 Statement: Generalized Cantor sets

Suppose that we construct a new kind of Cantor set, by removing the middle **half** of each subinterval, rather than the middle third.

a. Show that the *length* of the resulting set still vanishes, same as for the regular Cantor set.

b. Find the similarity dimension of the set.

c. Generalize the construction so as to produce a Cantor set with zero length and with a similarity dimension that can be picked as any arbitrary number in $0 < d < 1$.

1.2 Answer: Generalized Cantor sets

Let $0 < r < 1$ be any arbitrary number in the unit interval — in particular, **for parts (a) and (b), take $r = 1/2$.** Consider now the following generalized Cantor construction.

1. Let $I_0 = [0,\, 1)$ be the closed-open unit interval.
2. Let $I_1 = \left[0,\, \frac{1-r}{2}\right) \bigcup \left[\frac{1+r}{2},\, 1\right)$ be the result of **removing a centered closed-open interval of length r** from I_0.
3. Construct I_{n+1} from I_n as follows: take each interval in I_n, and split it in two by removing it's middle $r\times$ **length of interval** closed-open section, as done above to obtain I_1 from I_0. Note that I_{n+1} *is a subset of* I_n, *and that* I_n *is made up of* 2^n *intervals of equal length.*
4. The **generalized Cantor set** is: $C(r) = I_\infty = \bigcap_{n=1}^{\infty} I_n$.

Remark 1.1 Notice that in the construction above we remove, at each stage, intervals which are closed on the left and open on the right. **As far as the length and dimension calculation below, this is an irrelevant detail.** We could remove closed, open, or open-closed intervals at each stage (or even do it selecting the open-closed properties of the removed intervals randomly) and the calculations below in **a**, **b**, and **d** would not be affected. However, doing it this way makes it easier to show that the resulting set has as many points as the unit interval. See remark **1.2** below. ♣

We will **next calculate the "length" and dimension of $C(r)$,** leaving the issue of showing $C(r)$ has as many points as the unit interval to the end.[1]

a. From the construction above, it should be clear that

$$\text{length}(I_n) = (1-r)\,\text{length}(I_{n-1}) = (1-r)^n \to 0 \text{ as } n \to \infty.$$

Thus **$C(r)$ has zero length.**

b. From the construction above, it should be clear that: **I_n is made by 2^n intervals of length $\left(\frac{1-r}{2}\right)^n$, and each of these sub-intervals contains a portion of $C(r)$ which is identical to the full set, except for a scaling factor.** It follows that the fractal similarity dimension d of C is determined by the equality

$$2^n = \left[\left(\frac{1-r}{2}\right)^n\right]^{-d}$$

relating the number of copies of the set with their length. Thus

$$d = \frac{\ln(2)}{\ln(2) - \ln(1-r)} = \frac{\ln(2)}{\ln(2/(1-r))}. \tag{1.1}$$

For $r = 1/2$ this yields $d = 1/2$.

[1] **This calculation is "extra", and not required by the problem statement.**

c. For any $0 < d < 1$, take $$r = 1 - 2^{1-\frac{1}{d}}.$$
Then (1.1) shows that $C(r)$ has fractal similarity dimension d. Notice that, as $d \to 1$, $r \to 0$ and as $d \to 0$, $r \to 1$.

Finally, **we show next that $C(r)$ has as many points as the unit interval.** We do this by generalizing the idea used to show that the regular Cantor set has as many points as the unit interval: introduce an alternative of the base-3 representation of the numbers in the unit interval $[0,\ 1)$.

To any number $p \in [0,\ 1)$ we associate a (unique) representation

$$p = 0.\, a_1\, a_2\, a_3\, a_4\, \ldots \tag{1.2}$$

where a_1, a_2, a_3, etc. are selected from the set of three symbols $\{0, *, 1\}$ using the following algorithm:

1. Let $J_0 = [0,\ 1)$ be the unit closed-open interval, let $L_0 = 1$ be its length, and let $\alpha = \frac{1-r}{2}$.
2. For $n = 0$ to ∞:
   - — Divide the interval J_n into three contiguous closed-open intervals, with J_{n1} (the left-most interval) having length $\alpha\, L_n$, J_{n2} (the middle interval) having length $r L_n$, and J_{n3} (the right-most interval) having length $\alpha\, L_n$,
   - — If $p \in J_{n1}$ then $a_{n+1} = 0$, $J_{n+1} = J_{n1}$, and $L_{n+1} = \alpha L_n$. Further, let $q_{n+1} = 0$.
   - — If $p \in J_{n2}$ then $a_{n+1} = *$, $J_{n+1} = J_{n2}$, and $L_{n+1} = r\, L_n$. Further, let $q_{n+1} = \alpha\, L_n$.
   - — If $p \in J_{n3}$ then $a_{n+1} = 1$, $J_{n+1} = J_{n3}$, and $L_{n+1} = \alpha\, L_n$. Further, let $q_{n+1} = (\alpha + r)\, L_n$.

   end

The role of q_n is clarified in remark 1.3.

It should then be clear that $C(r)$ consists of all the points $p \in [0, 1)$ whose representation above does not involve the symbol $*$. This shows that we can define a bijection from C to $[0,\ 1]$, simply by considering the binary representation of any point in $[0,\ 1]$.

Remark 1.2 If in the definition of $C(r)$ we remove at each stage closed intervals, or open (or any combination of open and/or closed), this only changes I_n by a finite number of points. Thus, this will only affect $C(r)$ by, at most, a countable set of points, so that $C(r)$ will still have as many points as the unit interval. ♣

Remark 1.3 Notice that, in the construction above of the representation (1.2), at the N^{th} stage we have

$$\sum_{n=1}^{N} q_n \leq p < \sum_{n=1}^{N} q_n + L_N,$$

where the left and right ends of this inequality are the left and right ends of the interval J_N.
Since it is clear that $L_N \to 0$ as $N \to 0$ (because $L_{N+1} \leq \max(r,\ \alpha)\, L_N$), it follows that $p = \sum_{n=1}^{\infty} q_n$.
We use this to show that

A. $a_n = *$ for $n > N$ if and only if p is the mid-point of the interval J_N.

B. $a_n = 1$ for $n > N$ (a sequence ending with an infinite string of ones in (1.2)) does not occur.

Proof of **A**. That $a_n = *$ for $n > N$, if p is the mid-point of the interval J_N, is fairly obvious. Let us now consider the reverse. In this case we have, for $n > N$: $q_n = \alpha\, L_{n-1}$ and $L_n = r\, L_{n-1}$. Thus, for $j \geq 1$: $L_{N+j} = r^j\, L_N$ and $q_{N+j} = \alpha\, r^{j-1}\, L_N$. It follows that:

$$p = \sum_{n=1}^{N} q_n + \sum_{j=1}^{\infty} q_{n+j} = \sum_{n=1}^{N} q_n + \sum_{j=1}^{\infty} \alpha\, r^{j-1}\, L_N = \sum_{n=1}^{N} q_n + \alpha\, \frac{1}{1-r}\, L_N = \sum_{n=1}^{N} q_n + \frac{1}{2}\, L_N,$$

which clearly shows that p is the mid-point of J_N. ♣

Proof of **B**. Suppose we had $a_n = 1$ for $n > N$. Then, for $n > N$: $q_n = (\alpha + r)\, L_{n-1}$ and $L_n = \alpha\, L_{n-1}$. Thus, for $j \geq 1$: $L_{N+j} = \alpha^j\, L_N$ and $q_{N+j} = (\alpha + r)\, \alpha^{j-1}\, L_N$. It follows that:

$$p = \sum_{n=1}^{N} q_n + \sum_{j=1}^{\infty} q_{n+j} = \sum_{n=1}^{N} q_n + \sum_{j=1}^{\infty} (\alpha + r)\, \alpha^{j-1}\, L_N = \sum_{n=1}^{N} q_n + \frac{\alpha + r}{1 - \alpha}\, L_N = \sum_{n=1}^{N} q_n + L_N,$$

which clearly shows that p is the end point of J_N. But J_N is open on the right and p is supposed to belong to J_N. This is a contradiction, indicating that a representation ending in an infinite sequence of ones cannot happen (in fact, the true representation for p in a situation like this ends with a sequence of zeros, which is what happens when p is the left end of some J_N). ♣

2 Nonlinear stability of a discrete map, and flip bifurcation

2.1 Statement: Nonlinear stability of a discrete map, and flip bifurcation

Consider a 1-D map, $x_{n+1} = f(x_n)$, where f is smooth. Assume a fixed point $x_f = f(x_f)$, where $f'(x_f) = -1$ — hence linearization does not determine the stability of x_f. Without loss of generality, assume $x_* = 0$, and write

$$f(x) = -x + a\,x^2 + b\,x^3 + O(x^4), \tag{2.1}$$

where a and b are constants. These are **your tasks:**

t1. **Find condition on a and b** that determines wether $x = 0$ is a stable or unstable fixed point. *Hint:*

- t1.a The condition looks like: stability if $h(a,\, b) > 0$, and instability if $h(a,\, b) < 0$, for some function h.
- t1.b Consider what happens upon iterating $g(x) = f(f(x))$, which you can ascertain by expanding g to $O(x^4)$, using (2.1). Then note: if $x_{2\,n+2} = g(x_{2\,n})$ decays/grows, then so does $x_{2\,n+3}$, because f is continuous.

t2. **Answer this question:** *why do you have to expand g up to $O(x^4)$, in item* t1.b*, to determine stability?* Note that here expect the mathematical/technical reason for this.

t3. Let a and b in (2.1) be such that $x = 0$ is stable, i.e.: $h(a,\, b) > 0$, and take a map F such that

$$F(x) = -(1 + \delta)\,x + a\,x^2 + b\,x^3 + O(x^4), \tag{2.2}$$

where $0 < \delta \ll 1$. Then x is a linearly unstable fixed point, and a **period two (stable) solution**

$$x^*_{n+2} = x^*_n, \quad x^*_{n+1} = F(x^*_n), \tag{2.3}$$

appears,‡ where x^*_n has size $O(\sqrt{\delta})$.
This is called a **supercritical (or soft) flip bifurcation.**

‡ Argument: the same we made to explain the scaling behind supercritical pitchfork and Hopf bifurcations. The new solution appears as a balance between the destabilizing linearity, and the stabilizing nonlinearity.

Your task. Pick an example F where this happens, with $a \neq 0 \neq b$, and show a numerically computed picture of cobwebs† converging to the period two stable solution.

† Use two cobwebs (with different colors), one converging from "inside" and the other from "outside".

I suggest that you write a "generic" program for $F(x) = -(1 + \delta)\,x + a\,x^2 + b\,x^3$ and initial data x_0, and then play with the parameters till you get a pretty picture. Further: choose your colors well; e.g.: yellow on a white background is a bad idea! Note: something like $1 < a < 2$, $b \sim -2/3$, and $\delta \sim 0.3^2$, worked for me.

2.2 Answer: Nonlinear stability of a discrete map, and flip bifurcation

The answers to the tasks are below.

t1. We have: $g(x) = -(-x + a\,x^2 + b\,x^3) + a\,(-x + a\,x^2)^2 - b\,x^3 + O(x^4) = x - 2\,(a^2 + b)\,x^3 + O(x^4)$, where we note that **the quadratic terms cancel!**
This can be written as
$$g(x) = \Big(1 - 2\,(a^2 + b)\,x^2 + O(x^3)\Big)\,x = \lambda(x)\,x, \tag{2.4}$$
from which it is clear that:
$$a^2 + b > 0 \Rightarrow 0 < \lambda < 1 \quad \text{for } x \text{ small. Therefore: } x = 0 \text{ is stable.} \tag{2.5}$$
$$a^2 + b < 0 \Rightarrow 1 < \lambda \quad \text{for } x \text{ small. Therefore: } x = 0 \text{ is unstable.} \tag{2.6}$$

t2. We need to compute up to $O(x^4)$ because, as shown in item **t1**, the $O(x^2)$ vanish, and stability is decided by the $O(x^3)$ terms.

t3. See figure 2.1

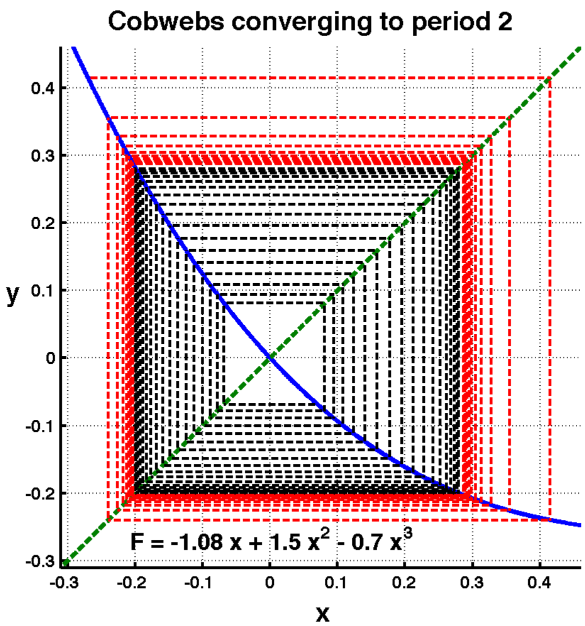

Figure 2.1: Cobwebs converging to period two.
Nonlinear stability of a discrete map, and flip bifurcation. The picture on the left shows two cobwebs, converging towards a stable period two solution (after a supercritical flip bifurcation),
for the map $F(x) = -1.08\,x + 1.5\,x^2 - 0.7\,x^3$.
Note that the amplitude of the period two solution is $O(\sqrt{\delta})$, as expected (here $\sqrt{\delta} \approx 0.283$).
The function F is plotted in blue, while the green dotted line corresponds to $y = x$.

3 Sierpinski gasket

3.1 Statement: Sierpinski gasket

Consider the fractal (a "Sierpinski gasket") in the plane, made in the following **recursive fashion:**

1. Start with an equilateral triangle, with sides of length L.

2. Draw the lines joining the sides mid-points, and divide it into four equal equilateral sub-triangles.

3. Remove the sub-triangle at the center.

4. Repeat the process with each of the other three remaining sub-triangles.

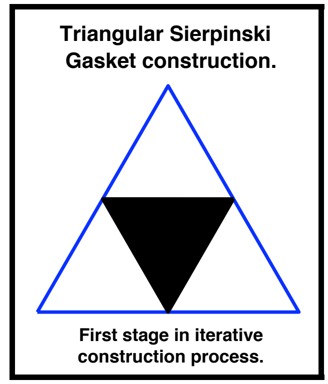

Figure 3.1: The picture on the right illustrates the recursion, showing the result of the first iteration in the process described above.

Now, do the following:

A. Calculate the box dimension of the fractal.

B. Calculate the self-similar dimension of the fractal.

C. Calculate the surface area of the fractal.

D. **Optional.** Show that the fractal has as many points as a full square — **This part is hard(er).**

E. **Optional.** Let d_s be the dimension calculated in part **A**. Modify the construction of the fractal, in such a way that the modified fractal can be selected to have **any** given box dimension $0 < d < d_s$.
Hint: take out bigger chunks at each stage.

F. **Optional.** Construct fractals (subsets of the plane) such that their box dimensions can be selected to have **any** given box dimension $d_s < d < 2$.

3.2 Answer: Sierpinski gasket

We start with the easier questions, and leave part **D** (cardinality of the fractal) to the end.

Part A: *Box dimension of the fractal*

It is clear that the fractal can be covered with either:
1. One equilateral triangle, whose sides are of length L.
2. Three equilateral triangles, whose sides are of length $L/2$.
. . . .
n. 3^n equilateral triangles, whose sides are of length $L/2^n$.
It follows that the **box dimension** is given by:

$$d_b = \lim_{n\to\infty} \frac{\log(3^n)}{-\log(L/2^n)} = \frac{\log(3)}{\log(2)} \approx 1.5850. \tag{3.1}$$

Part B: ***Self-similar dimension of the fractal***

The process is similar to the one used for the box dimension. It is clear that, for any natural number n, the fractal is made of by 3^n identical copies of itself, reduced in size by a factor of 2^n. Thus the **self-similar dimension** is given by:

$$d_s = \frac{\log(3)}{\log(2)} \approx 1.5850. \tag{3.2}$$

Part C: ***Surface area of the fractal***

Since the fractal is included in the starting triangle T_0 — and in all the objects that result from applying the iteration process that defines the fractal — it follows that:

$$A_\infty \leq A_n \quad \text{for every } n = 0,\ 1,\ 2,\ \ldots \tag{3.3}$$

Here A_∞ denotes the area of the fractal, and A_n is the area of T_n, where T_n denotes the set produced by iterating n times the process that leads to the fractal. However, it should be clear that $A_{n+1} = \frac{1}{4} A_n$, so that $A_n \to 0$. Thus **the fractal has no surface area:** $A_\infty = 0$.

Part E: ***Generalizations with a smaller dimension***

Let $d_b = d_s = \log(3)/\log(2)$ be the dimensions calculated in parts **A** and **B**. Here we modify the construction, in such a way that the resulting fractal can have **any** given box (or self-similar) dimension $0 < d < d_s = d_b$.

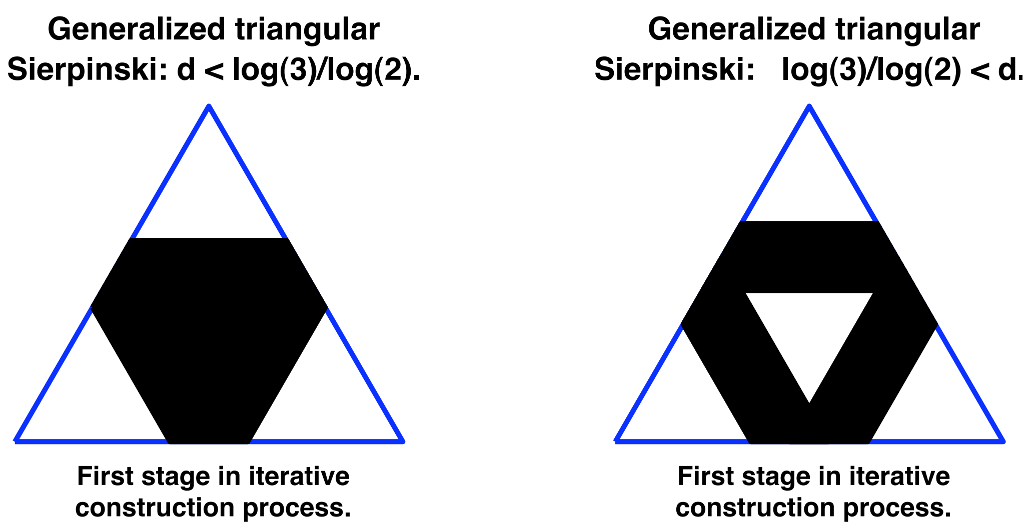

Figure 3.2: Generalizations of the Sierpinski gasket (details explained in the text). **Left:** First step in the recursion defining a fractal with lower dimension than the Sierpinski gasket. Instead of removing just the center triangle, an extra "band" around it is also removed. The three remaining (equal and equilateral) triangles have linear dimensions reduced by some (fixed) factor $0 < s < 1/2$, relative to the starting triangle. **Right:** First step in the recursion defining a fractal with higher dimension than the Sierpinski gasket. Instead of removing a whole chunk at the center of the starting triangle, only three bands (parallel to the sides) are removed. The four remaining (equal and equilateral) triangles have linear dimensions reduced by some (fixed) factor $0 < s < 1/2$, relative to the starting triangle.

The modified construction is illustrated on the left panel in figure 3.2. The change relative to the construction used in figure 3.1 is that, *in addition to removing the center triangle, an extra "band" around it is also removed.* This **band has width:**

$$\text{Width} = w\,h, \quad \text{where } 0 < w < \tfrac{1}{2} \tag{3.4}$$

is an arbitrary parameter (fixed throughout the construction), and h is the height of the starting triangle. What remains is still a set of three equal equilateral triangles, but their linear dimensions are reduced by a factor

$$0 < s = \frac{1}{2} - w < \frac{1}{2}, \tag{3.5}$$

instead of $\frac{1}{2}$ (as before). It follows that the box dimension of the resulting fractal is

$$d = \lim_{n \to \infty} \frac{\log(3^n)}{-\log(s^n L)} = -\frac{\log(3)}{\log(s)} < d_b = d_s. \tag{3.6}$$

Notice also that (for this fractal) the self-similar dimension is equal to the box-dimension.

Part F: *Generalizations with a higher dimension*

In this part we will modify the construction of the fractal, in such a way that the resulting fractal can be selected to have **any** given box (or self-similar) dimension $0 < d < 2$.

The modified construction is illustrated on the right panel in figure 3.2. The change relative to the construction used in figure 3.1 is that, *instead of removing a whole triangle at the center, only three "bands" (parallel to the sides) are removed.* Each **band has a width:**

$$\text{Width} = w\,h, \quad \text{where } 0 < w < \frac{2}{3} \tag{3.7}$$

is an arbitrary parameter (fixed throughout the construction), and h is the height of the starting triangle. One side of each band is a distance

$$\frac{1}{2}\left(1 - \frac{1}{2}w\right) h \tag{3.8}$$

away from the corresponding opposite side, and the other side is a distance $s\,h$ away from the corresponding triangle vertex, where

$$0 < s = \frac{1}{4}(2 - 3w) < 1/2. \tag{3.9}$$

What remains is now a set of **four** equal equilateral triangles, with their linear dimensions reduced by a factor s. It follows that the box dimension for the resulting fractal is given by:

$$0 < d = \lim_{n \to \infty} \frac{\log(4^n)}{-\log(s^n L)} = -\frac{\log(4)}{\log(s)} < 2. \tag{3.10}$$

Notice also that (for this fractal) the self-similar dimension is equal to the box-dimension.

Part D: *The fractal's cardinal is equal to that of a square*

Here we show that the Sierpinski gasket has as many points as the unit square $S_q = \{0 < x,\, y < 1\}$. The proof is not, strictly speaking, mathematically rigorous — it has (roughly) the same level of rigor as the proof in the book that the Cantor set has as many points as the unit interval (see examples 11.2.2 and 11.2.3 in pp. 403–404). For more details, see remark **3.1**.

We begin by showing [2] that

the unit interval $I_u = \{0 < x < 1\}$ has as many points as the unit square S_q.

Proof. For any point $u \in I_u$, consider its decimal representation: $u = 0.u_1\,u_2\,u_3\,u_4\,u_5\ldots$, and use it to construct a correspondence with S_q via:

$$u \mapsto (x,\, y) = (0.u_1\,u_3\,u_5\ldots,\; 0.u_2\,u_4\,u_6\ldots). \tag{3.11}$$

The inverse map is obvious: merge the decimal expansions for x and y into one.

[2] We need this because what we will show later is that the fractal has as many points as I_u.

Next we show that

the points in an equilateral triangle can be represented by 4-adic expansions.[3]

That is, we can write

$$P = 0.d_1\, d_2\, d_3\, d_4\, d_5\, d_6 \dots \quad \text{where } d_n \text{ is one of the digits: } \mathbf{0},\ \mathbf{1},\ \mathbf{2},\ \text{or } \mathbf{3}, \tag{3.12}$$

and P is any point in the triangle. To do this consider figure 3.1. Then: if P is in the middle triangle $d_1 = 0$, if P is in the top triangle $d_1 = 1$, if P is in the right triangle $d_1 = 2$, and if P is in the left triangle $d_1 = 3$. Next, in whichever sub-triangle P is in, the same process is repeated to determine[4] d_2, and so on.

It should be clear that: **in terms of the 4-adic representation above, the Sierpinski gasket is made by those points whose expansion does not include any zeros, i.e.: only the digits 1, 2, and 3 are used.**

Of course, we can re-interpret the three digit expansion (as above) for a point in the fractal, as a ternary representation of a point in the unit interval. It follows that

the fractal has as many points as the unit interval I_u.

Since earlier we had shown that I_u and S_q had the same number of points, we also conclude that:

the fractal has as many points as the unit square S_q.

Remark 3.1 Subtle issues and mathematical rigor.
Earlier on it was mentioned that the proofs here are not quite mathematically rigorous. The reason is that there is a certain amount of ambiguity in the binary/ternary/etc. representations of real numbers (basically: there are numbers for which more than one representation is possible), and this creates some difficulties. In particular: the arguments above, which show one-to-one correspondences between the binary/ternary/etc. representations (but **not** between the numbers themselves) have to be modified to account for this non-uniqueness. Sometimes this is simple, and sometimes it is not — but it always complicates the presentation, to the extent that the main idea becomes hard to follow. ♣

Examples of the issues involved:

— To show that the Cantor set has as many points as the unit interval, the ternary representation of the points in the unit interval (employing only the digits 0, 1, and 2) is used. Then the members of the Cantor set are characterized as those not using the digit 1. But, for example, consider $x = 0.0222\dots$, which can also be written as $x = 0.1000\dots$ Is x in the Cantor set,[5] or is it not? Not a hard problem to fix, though it takes some writing to do so.

— Consider the argument showing that the unit square S_q and the unit interval I_u have the same number of points. Suppose now that we try to avoid trouble by using representations that do not end in an infinite string of 9's. But then, what do we do with numbers whose decimal representation has alternating 9's? Note, for example, that $u = 0.19293959892949\dots$, yields (when the mapping in (3.11) is applied to it) $x = 0.1235824\dots$ and $y = 0.9999999\dots$!

— Consider the 4-adic representation for points in a triangle introduced in (3.12). If the point P happens to lie precisely at the boundary of two sub-triangles, ambiguity arises. Further, once a point shows up at one of the dividing lines (at some stage in the process), at all subsequent stages the point will keep showing up at a dividing line, so that the ambiguity propagates. To avoid this issue a clear rule is needed for how to "pick sides" when a point shows up at the boundary between triangles.

Remark 3.2 ***Fractal definition details.***
When we defined the fractal, we did not specify if the triangle being removed was supposed to be removed with, or

[3] 4-adic means four digits.
[4] When the triangle being analyzed points down, instead of up, assign the digit 1 to the bottom sub-triangle.
[5] This is related to the question of whether, when removing the middle third, the ends are removed or not.

without, its boundaries. None of the earlier arguments in this problem are affected by this, though note that (in fact) each choice gives rise to a **different** fractal.

An interesting point is that, if we choose to remove only open triangles, then showing that the fractal has as many points as the unit square is straightforward. Why? Because in this case the triangle's sides are never removed, hence the fractal has (at least) as many points as an interval. On the other hand, it clearly has less points than a square (since it is a subset of any square large enough to include the original triangle). But a square and an interval have the same number of points, so the fractal must also. Of course: all of this holds only if it turns out that the usual rules for bigger than, smaller than, and equality, apply to cardinals for infinite sets . . . which they (mostly) do, but I have not shown this (nor will I). ♣

Remark 3.3 ***Convergence, completeness, etc.***

If you did not fall asleep when the 4-adic expansion for points in a triangle was introduced in (3.12), the following two questions might have occurred to you:

$\mathbf{Q_1}$. *If I take two different points* $\boldsymbol{P_1}$ *and* $\boldsymbol{P_2}$*, are their 4-adic expansions different?* The answer is: **yes,** and the proof is trivial: Because the points are different, they are a finite distance away. Then, because the sub-triangles the starting triangle is subdivided into keep on getting smaller, eventually the two points end up in different sub-triangles and their 4-adic expansions turn out not equal.

$\mathbf{Q_2}$. *If I just write an arbitrary 4-adic expansion, is there a point on the triangle that gives it?* Again, the answer is **yes,** the proof as follows: What the 4-adic expansion actually does is to describe a sequence of nested sub-triangles inside the starting triangle, whose length scale goes down by a factor of 2 at every stage [6] Thus: consider the *sequence made up by the centers* of these sub-triangles. It should be clear that this is a *Cauchy sequence,* and so it has a limit. This limit is the point in the starting triangle with the desired properties. ♣

The arguments above answer the question:

What is needed to generate a p-adic (p digits) representation of the points in a set?

1. A rule for splitting the set (and each of the resulting sub-sets) into p non-empty parts.
2. A rule for naming the p parts of each split.
3. The size of the parts must go to zero as the number of splits goes to infinity.
4. The starting set must be **complete.** That is, every Cauchy sequence in it must have a limit.

By the way, notice that this (for example) gives a **scheme for generating, and keeping track of the elements in general, non-cartesian, numerical grids**, so it is not completely idle speculation.

THE END.

[6] That is, in the sequence, every sub-triangle is half the size, and inside, the preceding one.