

Answers to P-Set # 06, (18.353/12.006/2.050)j

MIT (Fall 2023)

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Contents

1 Homoclinic bifurcation with a computer #01	1
1.1 Statement: Homoclinic bifurcation with a computer #01	1
1.2 Answer: Homoclinic bifurcation using a computer #01	2
1.2.1 Analysis for the Hopf bifurcation; a proof that the nonlinearity stabilizes	3
2 Hopf bifurcation using a computer #01	4
2.1 Statement: Hopf bifurcation using a computer #01	4
2.2 Answer: Hopf bifurcation using a computer #01	4
3 Limit cycle bifurcation with a computer #01	6
3.1 Statement: Limit cycle bifurcation with a computer #01	6
3.2 Answer: Limit cycle bifurcation with a computer #01	6
4 Orbit diagrams for 1-D maps #01	7
4.1 Statement: Orbit diagrams for 1-D maps #01	7
4.2 Answer: Orbit diagrams for 1-D maps #01	8

List of Figures

1.1 Homoclinic bifurcation with a computer #01	2
1.2 Homoclinic bifurcation with a computer #01	3
2.1 Hopf bifurcation using a computer #01. Stable spiral point before a Hopf bifurcation	5
2.2 Hopf bifurcation using a computer #01. Soft (supercritical) Hopf bifurcation	5
3.1 Limit cycle bifurcation with a computer #01	6
4.1 Orbit diagram for the sine map	8
4.2 Orbit diagram for the cubic map	9
4.3 Orbit diagram for the cubic map (extended r-range)	9
4.4 Orbit diagram for the exponential #1 map	10
4.5 Orbit diagram for the tent map	10
4.6 Details of the orbit diagram for the tent map	11
4.7 Iterates of the tent map	11
4.8 Details of iterates of the tent map	12

1 Homoclinic bifurcation with a computer #01

1.1 Statement: Homoclinic bifurcation with a computer #01

Consider the system $\dot{x} = \mu x + y - (2\mu + 1)x^2$ and $\dot{y} = -x + \mu y + (2 - \mu)x^2$, (1.1)

where μ is a parameter. For this system:

(1) Find and analyze/classify the critical points.

(2) Using a computer, show that a supercritical Hopf bifurcation happens at $\mu = 0$.

(3) Using a computer, show that an homoclinic bifurcation happens for $0 < \mu = \mu_c$, where μ_c is very small.

You can get a rough idea of “how small is small” by recalling that the “radius” of the Hopf bifurcation limit cycle grows like the square root of the parameter deviation from the onset. Hence, in this case it should be $C\sqrt{\mu_c} \approx$ distance from the origin to the saddle, where C is a constant. Of course, you do not know C , so this estimate can be off by some factor, probably less than 10. Do this estimate before doing (3).

(4) Find the approximate value of μ_c .

(5) Illustrate your results with phase plane plots in the region $-0.4 < x, y < 0.6$.

Hint: The nonlinear terms determine the nature of the bifurcation: supercritical if they stabilize, subcritical if they de-stabilize. Hence check the behavior of the orbits near the critical point for $\mu = 0$, to see if the nonlinear terms stabilize or de-stabilize.

Suggestions for computation. Use PHPLdemoB2, which allows for easy parameter searches. To get precisely controlled plots, you can use PHPLplot or PHPLplot_v2, which are a bit less user friendly, but allow precise control. Note that, for μ small, the radial dynamics near the origin is slow, thus you may need “large” integration times ... do not use the default integration times and tolerances in the scripts.

1.2 Answer: Homoclinic bifurcation using a computer #01

(1) The critical points are:

1a. The origin, with linearization $\delta\dot{x} = \mu\delta x + \delta y$ and $\delta\dot{y} = -\delta x + \mu\delta y$.

This yields: stable spiral point for $\mu < 0$ and unstable spiral point for $\mu > 0$.

1b. $(x, y) = (1/2, 1/4)$, with linearization $\delta\dot{x} = -(\mu + 1)\delta x + \delta y$ and $\delta\dot{y} = (1 - \mu)\delta x + \mu\delta y$.

This yields a saddle for all μ , with eigenvalues $\lambda = -\frac{1}{2} \pm \sqrt{\frac{5}{4} + \mu^2}$.

Figures 1.1 and 1.2 confirm 1a. and 1b

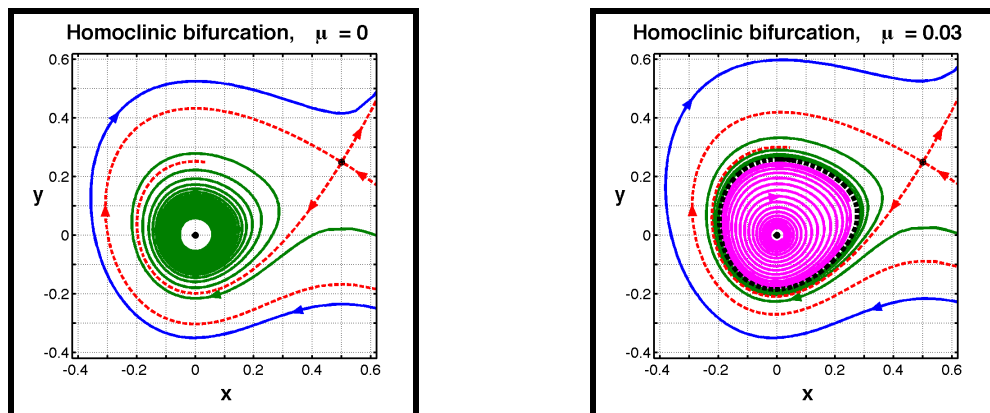


Figure 1.1: Homoclinic bifurcation with a computer #01, for the system in (1.1) — see item (2).

(2) The Hopf bifurcation, figure 1.1. The left panel shows the system’s phase portrait for $\mu = 0$, with the saddle stable/unstable trajectories in dashed red, and two typical orbits in blue and green. The green trajectory approaches the origin as $t \rightarrow \infty$, albeit very slowly (so does one of the unstable trajectories from the saddle). We conclude that the nonlinearity is stabilizing, resulting in a supercritical Hopf bifurcation — however: see § 1.2.1. The right panel shows the limit cycle for $\mu = 0.03$ (dashed black line), with the saddle stable/unstable trajectories in dashed red, and three typical orbits in blue, green, and magenta — the last two approach the limit cycle as $t \rightarrow \infty$.

Next we do a gross estimate for μ_c . The distance from the origin to the saddle is $d = \sqrt{5/16}$, hence $\mu_c \approx (5/16)/C^2 \approx 0.31/C^2$ — as we will see, $C \approx 2$ gives the correct answer.

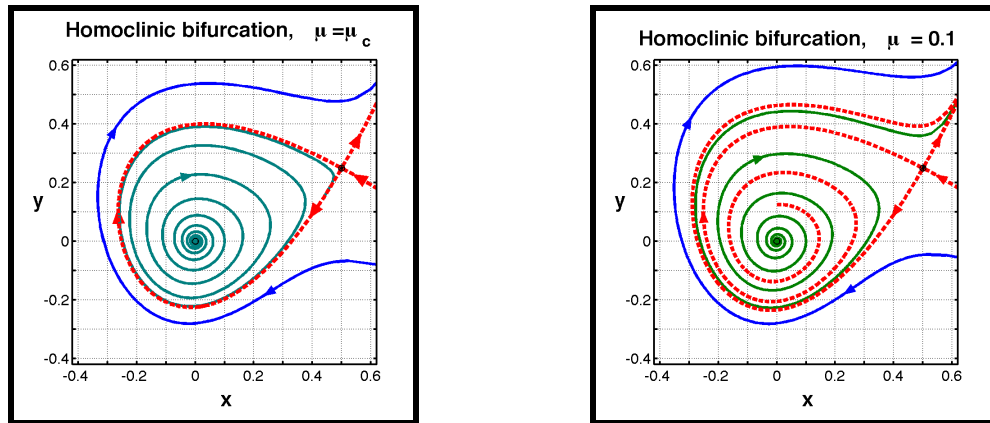


Figure 1.2: Homoclinic bifurcation with a computer #01, for the system in (1.1) — see item (3).

(3) **The homoclinic bifurcation, figure 1.2.** The **left panel** shows the system's phase portrait for $\mu = \mu_c$, where $0.0767 < \mu < 0.0768$. Here the saddle stable/unstable trajectories are shown in dashed red, and two typical orbits in blue and green. This is the precise **value of μ at which the limit cycle vanishes, upon colliding with the saddle, and becoming an homoclinic connection.** The orbits inside the cycle graph, in green, spiral out from the origin towards the homoclinic connection. The **right panel** shows the phase plane portrait for $\mu = 0.1 > \mu_c$, with (again) the saddle stable/unstable trajectories in dashed red, two typical orbits in blue and green, and **no limit cycle.**

(4) **The critical value is $\mu_c = 0.0767\dots$** To obtain this value I computed the stable/unstable manifolds for the saddle (in practice: compute trajectories starting from a point very close to the saddle), and vary μ till an homoclinic orbit is obtained.

(5) **Phase plane plots.** See figures 1.1 and 1.2.

1.2.1 Analysis for the Hopf bifurcation; a proof that the nonlinearity stabilizes

The computer argument in (2), showing that the nonlinearity stabilizes, has a **weakness**: The approach of the green trajectory to the origin (left panel in figure 1.1) gets ever slower as time increases, and it requires a bit of a leap of faith to claim that the trajectory approaches the origin as $t \rightarrow \infty$. Here we prove that

this is correct, using a Liapunov function. The $\mu = 0$ system is $\dot{x} = y - x^2$ and $\dot{y} = -x + 2x^2$. (1.2)

We will show that the origin is (nonlinear) stable spiral for this

system (which then implies that a supercritical Hopf bifurcation occurs for $\mu = 0$). To do so, first we eliminate the variable y by taking the time derivative of the first equation,

and substituting \dot{y} from the second. Hence

$$\ddot{x} + 2x\dot{x} + x - 2x^2 = 0. \quad (1.3)$$

This equation is fully equivalent to (1.2) via the substitution

$y = \dot{x} + x^2$. It can also be written in the form

$$\dot{x} = v \quad \text{and} \quad \dot{v} = -(1 + 2v)x + 2x^2. \quad (1.4)$$

Now, define (see "motivation" below)

$$L = \frac{1}{2}x^2 + \frac{1}{2}v - \frac{1}{4}\ln(1 + 2v) - \frac{2}{3}x^3. \quad (1.5)$$

Then it is straightforward to see that

$$\frac{dL}{dt} = -4x^2v^2/(1 + 2v). \quad (1.6)$$

This is **precisely what we were looking for**. Near the

origin, $\dot{L} < 0$, except along the coordinate lines $x \equiv 0$ and $v \equiv 0$, where $\dot{L} = 0$. However, it is easy to check that non-trivial orbits cross the coordinate axis near the origin.¹ Hence

L is a *strictly decreasing function of time along non-trivial trajectories near the origin*. Further

so that the origin is a local minimum for L . In addition, it

$$L = \frac{1}{2}(x^2 + v^2) + \text{higher order terms}, \quad (1.7)$$

¹For example, if $x = 0$ and $v \neq 0$, $\dot{x} = v \neq 0$.

is easy to see that $\dot{\theta} = -1 + O((x^2 + v^2)^{1/2})$, where θ is the polar angle in the (x, v) plane. **Hence the origin is a (non-linear) stable spiral point.** ♣

Equation (1.6) “explains” why the spiraling down to the origin of the green orbit in figure 1.1 is so slow. If $r = \sqrt{x^2 + v^2}$, then for a linear spiral $\dot{r} = O(r)$, while (1.6) shows that here $\dot{r} = O(r^3)$; massively slower.

Motivation. Equation (1.5) requires explanation; it is not the kind of thing one just guesses. Here it goes

(a) The reduction from (1.2) to (1.3) is because (1.3) is more intuitive: an oscillator in a **potential** $V = \frac{1}{2}x^2 - \frac{2}{3}x^3$, with a typical non-linear driven-damping term.

(b) Upon eliminating the non-linear term $2x^2$ in the “spring force”, we can write $dv/dx = -(1 + 2v)x/v$, using (1.4). This equation is separable, and can be integrated. The constant

of integration then yields an integral of motion

$$E = \frac{1}{2}x^2 + \frac{1}{2}v - \frac{1}{4}\ln(1 + 2v).$$

Hence $\ddot{x} + 2x\dot{x} + x = 0$ is a **conservative system!**

It is then natural to ask: What happens if the “missing” piece of the potential V , i.e.: $-\frac{2}{3}x^3$, is added to E ? The result is L , a Liapunov function.

2 Hopf bifurcation using a computer #01

2.1 Statement: Hopf bifurcation using a computer #01

For the system

$$\dot{x} = y + \mu x \quad \text{and} \quad \dot{y} = -x + \mu y - x^2 y, \quad (2.1)$$

a Hopf bifurcation occurs at the origin when $\mu = 0$.

Using a computer, **plot the phase portrait and determine whether the bifurcation is subcritical or supercritical. For small values of μ , verify that the limit cycle is nearly circular. Then measure the period and radius of the limit cycle, and show that the radius R scales with μ as predicted by theory.**

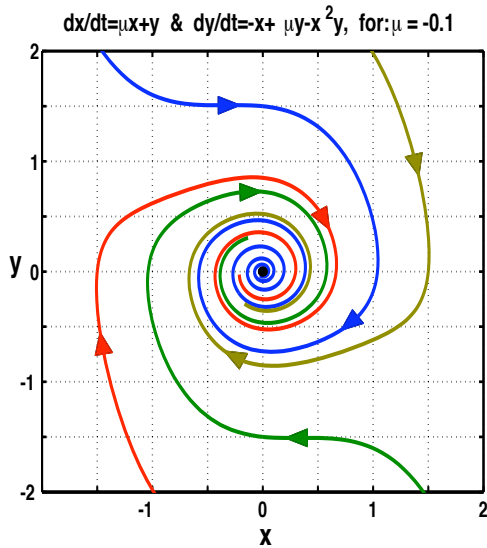
Hints: (a) The nonlinear terms determine the nature of the bifurcation: supercritical if they stabilize, subcritical if they de-stabilize. Hence *check the behavior of the orbits near the critical point for a $|\mu| \ll 1$ to see if the nonlinear terms stabilize or de-stabilize.* (b) In the *subcritical case* the limit cycle is unstable, hence hard to compute forward in time. However, if you *run the system backwards in time the solutions converge to the limit cycle!*

Suggestions for computation. For the calculation of the limit cycles, and measuring the radius and period, PHPLdemoB is not the best tool (because it depends on the mouse to input the initial point on an orbit. For this you should use **PHPLplot** or **PHPLplot.v2**, which are a bit less user friendly, but allow precise control of the orbits. With them you can select the initial data (say, give $x = x_0 > 0$ along $y = 0$) and play with x_0 till you hit the limit cycle — set $BF = 0$, so the orbit is computed forward only. Then x_0 gives you the “radius”. To compute the period, play with `t.f` so the orbit barely closes — alternatively: “hack” the script so it outputs the raw data from the ode solver, `[T, Y]`, which will allow you to compute the period and radius very accurately.

2.2 Answer: Hopf bifurcation using a computer #01

See figure 2.1 for the phase portrait with $\mu = -0.1$. It indicates a stabilizing nonlinearity, thus a **supercritical (soft) Hopf bifurcation**.

Figure 2.2 shows a picture of the phase portrait for $\mu = 0.1$ on the left and the limit cycles for various values of $0 < \mu \ll 1$ on the right. A stable limit cycle appears around the critical point for $0 < \mu \ll 1$. This **confirms that a supercritical (soft) Hopf bifurcation** occurs.



Hopf bifurcation using a computer #01.

System (2.1) for $\mu = -0.1 < 0$.

All the orbits spiral towards the origin, even those $O(1)$ away. Since μ is fairly small, this is a good hint that the nonlinearity is stabilizing, which should lead to a **super-critical (soft) Hopf bifurcation**.

Figure 2.1: Hopf bifurcation using a computer #01. Phase portrait for the system in (2.1) when $\mu = -0.1 < 0$

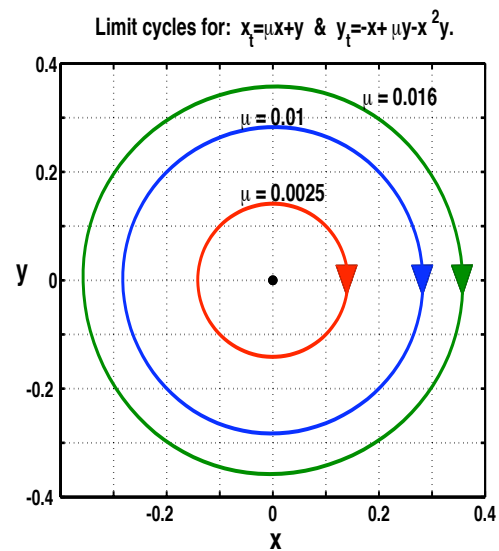
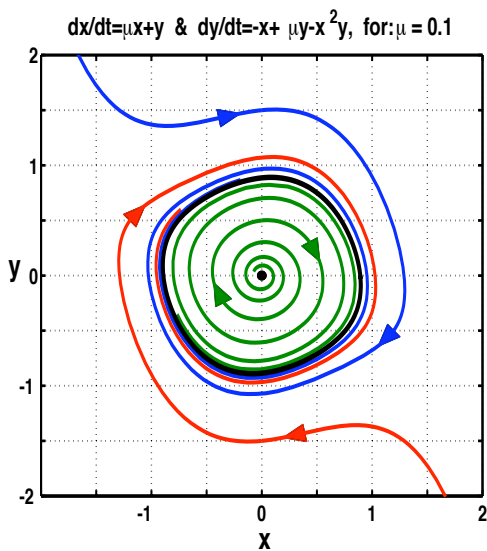


Figure 2.2: Hopf bifurcation using a computer #01. Left: phase portrait for the system in (2.1) when $\mu = 0.1 > 0$. The picture on the right shows the limit cycles for various values of $0 < \mu \ll 1$.

For $0 < \mu \ll 1$ it is easy to see (in figure 2.2) that the limit cycles are nearly circular. The table on the right shows a listing of various parameters for these cycles. It should be clear that the theoretical predictions (e.g.: $\frac{R}{\sqrt{\mu}} \sim \text{constant}$, and period $\sim \text{linear period}$) are satisfied.

Limit Cycle Parameters, $0 < \mu \ll 1$.			
μ	$R = \text{radius.}$	$\frac{R}{\sqrt{\mu}}$	$\frac{\text{Period}}{\pi}$
0.0160	0.35775	2.8283	2.0014
0.0100	0.28285	2.8285	2.0005
0.0025	0.14142	2.8284	2.0000

Furthermore, the limit cycle for $\mu = 0.1$ is not very circular, but if we interpret its radius as the value of x when $y = 0$,

then $R \approx 0.89235$ and $\frac{R}{\sqrt{\mu}} = 2.8219$ (quite close to the values in the table). The period is $P \approx 2.0579\pi$.

3 Limit cycle bifurcation with a computer #01

3.1 Statement: Limit cycle bifurcation with a computer #01

The system

$$\dot{x} = y/\delta \quad \text{and} \quad \dot{y} = (2\mu y - (1 + \mu^2)x + x^2(\mu x - y))/\delta, \quad (3.1)$$

where $\delta = 1 + x^2$, undergoes a

supercritical Hopf bifurcation at $\mu = 0$. Beyond that, as μ increases, while the critical points at $(x, y) = (\pm x_0, 0)$ ($x_0 = \sqrt{\mu + 1/\mu}$) move towards the origin. For $\mu = \mu_c$, $0.3 < \mu_c < 0.4$, **another bifurcation occurs**. Using a computer:

- (1) Describe what happens at $\mu = \mu_c$.
- (2) Find the nature of all the critical points of the system.
- (3) Ascertain the value of μ_c with two significant digits.

Illustrate your conclusions with three phase plane portraits in the region $-3 < x, y < 3$, one for $\mu < \mu_c$, one for $\mu > \mu_c$, and one for $\mu \approx \mu_c$.

Remark #1. Note that the system has the symmetry $(x, y) \rightarrow (-x, -y)$.

Suggestions for computation. Since the problem involves investigating a situation as a parameter varies, with an equation with many terms, PHPLdemoB is not the best tool (impractical). Instead, I suggest that you use **PHPLdemoB2**, which will allow you to do μ “sweeps” efficiently. Once you know well what happens, to produce “neat” final plots, you could use **PHPLplot** or **PHPLplot.v2**, which are less user friendly than PHPLdemoB2, but allow precise control of the orbits.

3.2 Answer: Limit cycle bifurcation with a computer #01.

The critical points of are the **origin (unstable spiral for $\mu > 0$)** and $(x, y) = (\pm x_0, 0)$ ($x_0 = \sqrt{\mu + 1/\mu}$), both of which are **saddles**. This can be seen in all the panels in figure 3.1. For $0 < \mu < \mu_c$, left panel in figure 3.1, the

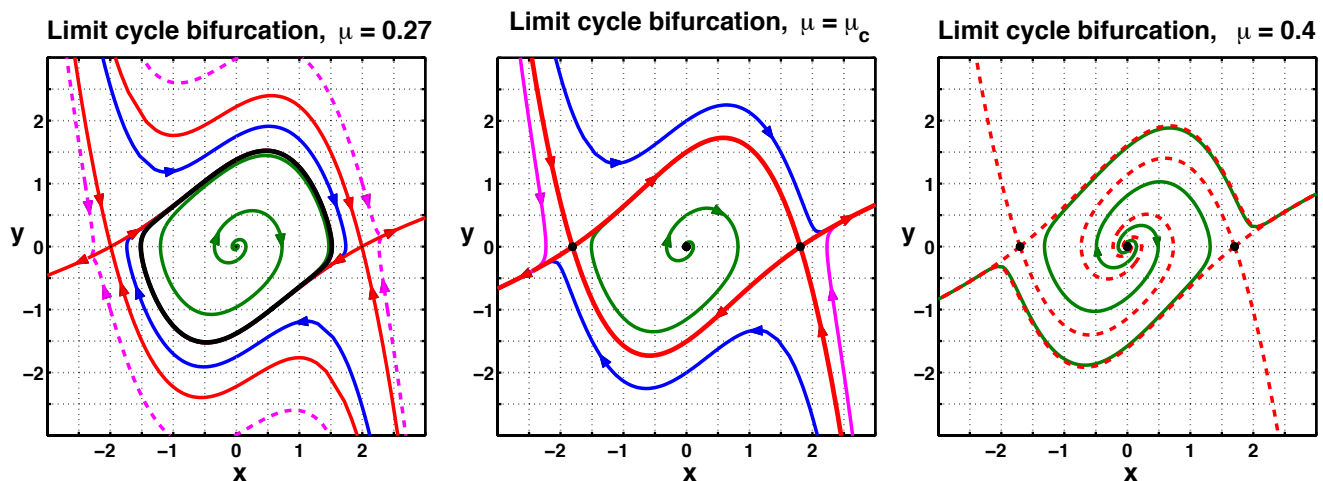


Figure 3.1: Limit cycle bifurcation with a computer #01, for the system in (3.1). **Left panel:** $\mu = 0.27 < \mu_c$ **Middle panel:** $\mu = \mu_c = 0.34106 \dots$ **Right panel:** $\mu = 0.40 > \mu_c$.

limit cycle encloses the origin and it is located in between the two saddles — see item (1) below. As μ increases, the limit cycle gets larger, while the saddles move towards the origin (in this range x_0 is a decreasing function of μ).

When μ reaches μ_c , where $0.34106 < \mu_c < 0.34107$, the two saddles merge with the limit cycle (simultaneously, because of the symmetry in remark #1), as shown in the middle panel of figure 3.1 — see item (2) below. Beyond μ_c , $\mu > \mu_c$, there is no limit cycle, as shown in the right panel of figure 3.1 — see item (3) below.

- (1) Figure 3.1, left panel, $\mu < \mu_c$. Here the stable/unstable trajectories of the saddle (in red) divide the plane into seven regions. The trajectories inside the limit cycle (in green) approach the limit cycle (in black) as $t \rightarrow \infty$, and the origin as $t \rightarrow -\infty$. The trajectories in the regions above and below the limit cycle (in blue) approach the limit cycle as $t \rightarrow \infty$, and diverge to infinity as $t \rightarrow -\infty$. The trajectories in the four remaining regions (in magenta) diverge to infinity as both $t \rightarrow \pm\infty$.
- (2) Figure 3.1, middle panel, $\mu = \mu_c$. The **limit cycle** is destroyed, and **becomes a cycle graph**, made by two heteroclinic orbits connecting the saddles. Thus we have an **heteroclinic bifurcation of the limit cycle**, very **similar to an homoclinic bifurcation** (with one saddle merging with the cycle), **possible only because of the symmetry in remark #1** — any break of symmetry would destroy the simultaneous nature of the “collisions” of the saddles with the limit cycle. The trajectories inside the cycle graph approach the cycle graph as $t \rightarrow \infty$, and the origin as $t \rightarrow -\infty$. The other trajectories (except for the saddle’s stable/unstable manifolds) all diverge to infinity as $t \rightarrow \pm\infty$.
- (3) Figure 3.1, right panel, $\mu > \mu_c$. There is **no limit cycle** and all the trajectories (except for the saddle’s stable manifolds) escape to infinity as $t \rightarrow \infty$. As $t \rightarrow -\infty$ some trajectories approach the origin, while the others diverge to infinity (again, excluding the saddle’s unstable manifolds).

4 Orbit diagrams for 1-D maps #01

4.1 Statement: Orbit diagrams for 1-D maps #01

Compute the orbit diagrams² for the 1-D maps $x_{n+1} = f(x_n)$ below. Furthermore, in all the cases perform the following **task**: *given the range of r selected, justify the selected range for x .*

Meaning of “justify”. Show that the x -region is such that it may contain an attractor, because either: (a) It is trapping; an orbit starting there stays there; or (b) Orbits starting outside the region diverge to infinity, so that any attractor has to be inside.

1. The **sine map** $f(x) = r \sin(x)$, with $-5 \leq r \leq 5$ and $-5 \leq x \leq 5$.
2. The **cubic map** $f(x) = r x - x^3$, with $0 \leq r \leq 3$ and $-2 \leq x \leq 2$.
3. The **exponential #1 map** $f(x) = x e^{r(1-x)}$, with $1 \leq r \leq 4$ and $0 \leq x \leq 0.25 e^3 = 5.0213\dots$
4. The **tent map** $f(x) = r(1 - |2x - 1|)$, with $0 \leq r \leq 1$ and $0 \leq x \leq 1$. At first sight the tent map orbit diagram may look rather plain. However, *do a blow up of the region in the tent map orbit diagram given by:*
 - 4b. $0.5 \leq r \leq 0.6$ and $0.492 \leq x \leq 0.501$.
 - 4c. $0.5 \leq r \leq 0.525$ and $0.4999 \leq x \leq 0.5001$.

Note that doing these blow-ups will require to plot far more iterates than for the main plot, because you will not be plotting the whole attractor, but just a small region within it, which means most points will be outside the plotted region.

Process: Pick a grid in r ; say $r_m = r_0 + m \Delta r$, for some small Δr , covering the desired range for r . Then, for every r_m , take a random starting point x_0 , and calculate the resulting x_n for a large number of iterates $1 \leq n \leq N$. Finally: throw away the first few hundred iterates (to eliminate transients as the attractor is approached) and plot

² An orbit diagram is what figure 10.2.7 in Strogatz’s book shows.

the rest as points (r_m, x_n) in a 2-D diagram.

Note: if you use MatLab, “vectorize” the operation, so that you do **all** the r simultaneously (MatLab will be very slow if you use a “for loop” in both n and m).

Also, in MatLab the command “print -dpng FigureName” will save the figure as a small png file (and it is more reliable than trying to use the GUI in the figure window).

4.2 Answer: Orbit diagrams for 1-D maps #01

Note: the answer below is much more detailed than what you were expected to turn in. I am using this as an opportunity to examine 1-D maps a little deeper.

1. The orbit diagram for the **sine map** $f(x) = r \sin(x)$, with $-5 \leq r \leq 5$ and $-5 \leq x \leq 5$, is shown on the left panel in figure 4.1. **Justification of the x -range:** since $|f(x)| \leq r$, the attractor must be contained within $|x| \leq r$. Note that the figure shows that the chaotic attractors explore the whole range $-r < x < r$.

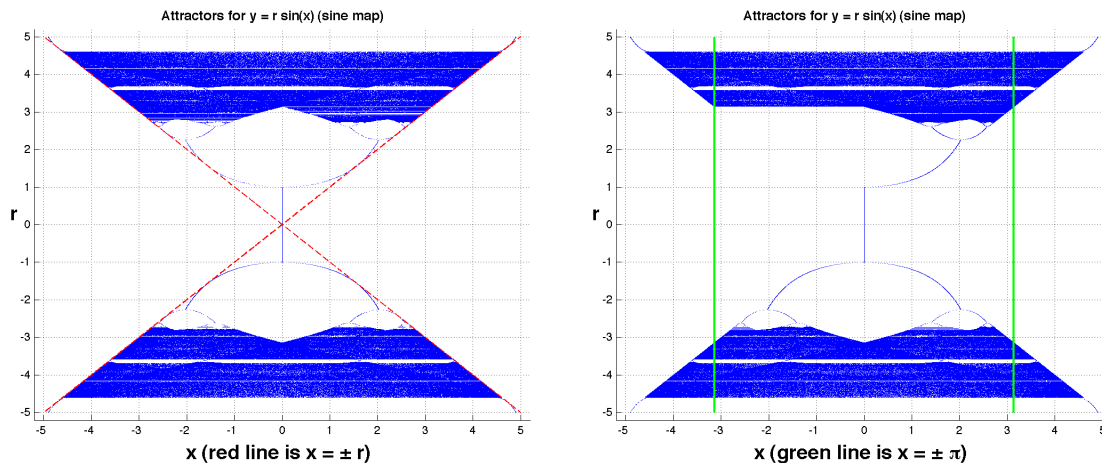


Figure 4.1: The **left panel** shows the orbit diagram for the sine map, where (for every value $-5 \leq r \leq 5$) x_0 is a random point in $-5 \leq x_0 \leq 5$. Here the dashed red line indicates the limits $x = \pm r$. For the **right panel** the initial data is restricted by $0 \leq x_0 \leq r$. Here the green lines correspond to $x = \pm \pi$.

Remark 1a: For $0 \leq r \leq \pi$, the interval $[0, r]$ is mapped onto itself by the sine map — and the same happens for $[-r, 0]$. Hence, for $0 \leq r \leq \pi$, the left and right sides of the orbit diagram correspond to separate, independent, attractors. Only for $r > \pi$ that the two sides “mix” — see the right panel in figure 4.1. ♣

Remark 1b: The sine map is odd. Hence the orbit diagram is invariant under $x \mapsto -x$ (left panel in figure 4.1). However: why is the orbit diagram shown in the figure also invariant under $r \mapsto -r$? The reason is as follows: consider an orbit $\{x_n\}_{n=0}^{\infty}$ for the map, corresponding to some value $r = r_0$. Then $\{(-1)^n x_n\}_{n=0}^{\infty}$ is an orbit of the map corresponding to $r = -r_0$ (the proof is trivial). In addition, $\{-x_n\}_{n=0}^{\infty}$ is also an orbit corresponding to $r = r_0$, so that $\{-(-1)^n x_n\}_{n=0}^{\infty}$ is also an orbit for $r = -r_0$. It follows that the attractors for $r = r_0$ and $r = -r_0$ have the same points (even though the orbit’s even and odd terms map separately). ♣

2. The left panel in figure 4.2 shows the orbit diagram for **$f(x) = r x - x^3$ (cubic map)**, with $0 \leq r \leq 3$ and $|x| \leq 2$. **Justification of the x -range:** Let $x^2 = r + 1 + \delta$ ($\delta > 0$), then $|f(x)| = |r - x^2| |x| = (1 + \delta) |x|$. This shows that: when $|x_0| > \sqrt{r + 1}$, $|x_n| \rightarrow \infty$. Hence the attractor must be contained within $|x| \leq \sqrt{r + 1}$.

Remark 2a: The figure shows³ that the attractor is contained within $|x| \leq x_\mu$, where $x_\mu = 2(r/3)^{1.5}$ is the value of f (local maximum) at $x = \sqrt{r/3}$. ♣

³I do not have a proof of this. I could not think of an “elegant” way to do it, and did not want to get involved into a lot of messy algebra. If you can think of an elegant way to do so, I will appreciate it.

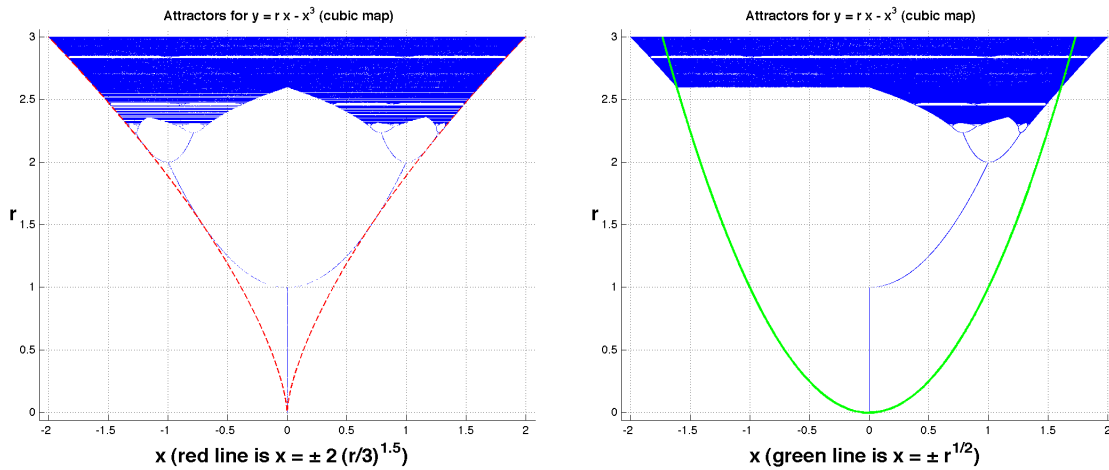


Figure 4.2: The **left panel** shows the orbit diagram for the cubic map, where (for every value $0 \leq r \leq 3$) x_0 is a random point in $-2 \leq x_0 \leq 2$. Here the dashed red line indicates the curves $x = \pm 2 (r/3)^{1.5}$. For the **right panel** the initial data is restricted by $0 \leq x_0 \leq \sqrt{r}$. Here the green line corresponds to $r = x^2$.

Remark 2b: For $0 \leq x \leq \sqrt{r}$, $0 \leq f(x) \leq x_\mu$. Thus if $x_\mu \leq \sqrt{r}$ (i.e.: $r \leq 1.5 \sqrt{3}$), the interval $0 \leq x \leq x_\mu$ is mapped onto itself by the cubic map — and the same happens for the interval $-x_\mu \leq x \leq 0$. Hence, for $0 \leq r \leq 1.5 \sqrt{3}$, the left and right sides of the orbit diagram correspond to separate, independent, attractors. It is only for $r > 1.5 \sqrt{3}$ that the two sides “mix” — as shown by the right panel in figure 4.2. ♣

Remark 2c: The cubic map is odd. Hence the orbit diagram should be invariant under $x \mapsto -x$, which the left panel in figure 4.1 confirms. However, unlike what happens for the sine map (see remark 1b) there is no $r \mapsto -r$ symmetry: for $r < -1$, $|f(x)| = |r - x^2||x| > |r||x|$, so that all non-zero orbits diverge to infinity (while $x = 0$ is unstable). In fact, figure 4.3 shows that, for $-1 < r < 1$ the attractor is $x = 0$. ♣

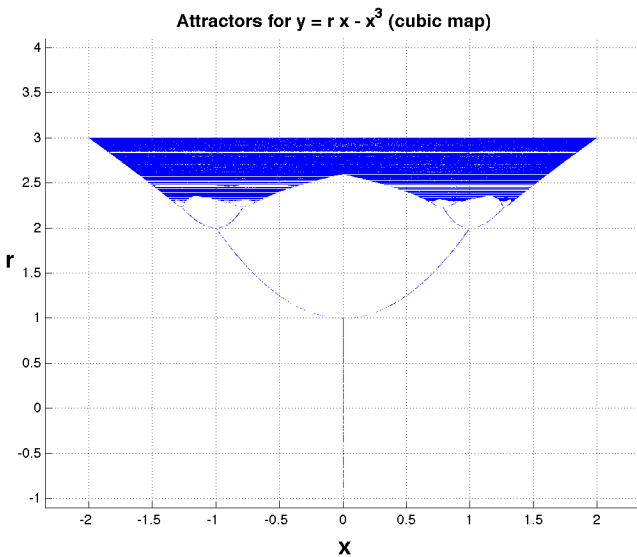


Figure 4.3: Cubic map.

Orbit diagram for $f(x) = rx - x^3$ (cubic map), for the range $-1 < r < 4$. Note that for $r > 3$ there is no attractor (orbits are either unstable or diverge to infinity), and the same happens for $r \leq -1$ (this is shown in the text). For $-1 < r < 1$ the attractor is just $x = 0$. At $r = 1$ a pitchfork bifurcation happens, followed by a period doubling cascade.

Remark 2d: The computations show that, beyond $r = 3$ there is no attractor — see figure 4.3. I do not have a proof for this. If you can figure out an argument for this, I would like to know it. ♣

- The orbit diagram for the **exponential #1 map** $f(x) = x e^{r(1-x)}$, with $1 \leq r \leq 4$ and $0 \leq x \leq f_M$ (where $f_M = 0.25 e^3 = 5.0213 \dots$), is shown on the left panel in figure 4.4.

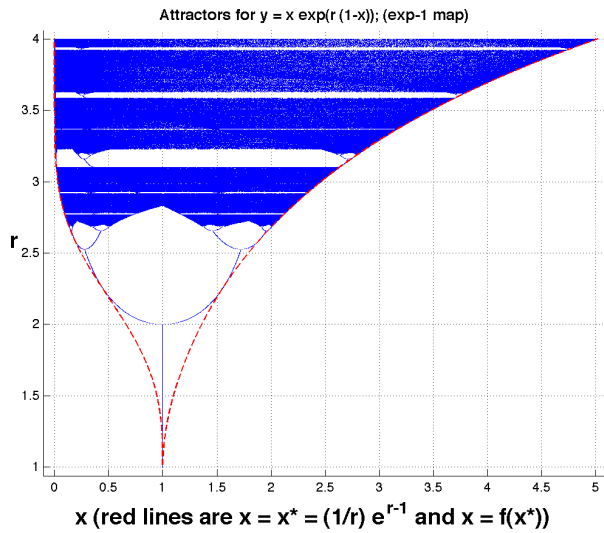


Figure 4.4: Exponential #1 map. Orbit diagram for $f(x) = x e^{r(1-x)}$ (exponential #1 map). Here, for every value $1 \leq r \leq 4$, x_0 is a random point in $0 \leq x_0 \leq f_m$. The dashed red curves are: $x = f_m = \frac{1}{r} e^{r-1}$ on the right, and $x = f(f_m)$ on the left.

Justification of the x -range: for $x \geq 0$, $0 \leq f(x) \leq f_m = \frac{1}{r} e^{r-1}$ (with f_m achieved⁴ at $x = 1/r$). In turn, the maximum of f_m over the given range of r is achieved at $r = 4$, where $f_m = f_M$. Thus the attractor must be contained within $0 \leq x \leq f_M$. Further point: because f is concave for $x < 0$ ($f'' < 0$), with $f'(0) > 1$, a cobweb shows that any orbit with $x_0 < 0$ diverges to $-\infty$.

Note that figure 4.4 shows that the attractors are contained † within $f(f_m) \leq x \leq f_m$, and that the chaotic attractors explore the whole range. Note that $f(f_m) = \frac{1}{r} e^{2r-1-e^{r-1}}$ vanishes very fast as r grows.

† The reason why the attractors must be within $x \leq f_m$ was explained earlier. The additional restriction $f(f_m) \leq x$ arises as follows: It is easy to see that any point $0 \leq x \leq 1$ satisfies $f(x) > x$ ($x = 0$ and $x = 1$ are the fixed points of the map) so the map pushes these points to the right — do the cobweb. The only way that these points can be in the attractor is if they are mapped back there from points beyond the maximum of f at $x = 1/r$, where f is decreasing. Clearly the smallest point thus achievable is the image of the largest x possible, i.e.: $x = f_m$, where f reaches its minimum value in the allowed range.

- The orbit diagram for the **tent map** $f(x) = r(1 - |2x - 1|)$, with $0 \leq r \leq 1$ and $0 \leq x \leq 1$, is shown on the left panel in figure 4.5. **Justification of the x -range:** follows because $0 \leq f(x) \leq r$ for $0 \leq x \leq 1$.

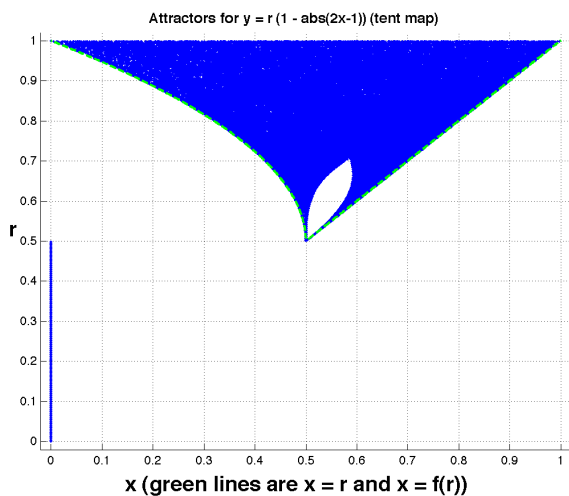


Figure 4.5: Tent map. Orbit diagram for $f(x) = r(1 - |2x - 1|)$ (tent map). Here, for every value $0 \leq r \leq 1$, x_0 is a random point in $0 \leq x_0 \leq 1$. The dashed green curves are: $x = 2r(1 - r)$ on the left (that is: $x = f(r)$ for $r > 0.5$), and $x = r$ on the right. Note that for $0 \leq r < 0.5$, $x = 0$ is a global attractor. It is only for $r > 0.5$, where a new (and unstable) fixed point at $x = 2r/(1 + 2r)$ appears, that “the show” begins. This diagram appears quite featureless, except for the football shaped gap for $0.5 \lesssim r \lesssim 0.7$. However: see figure 4.6.

⁴ Note that $f' = (1 - rx) e^{r(1-x)}$, and $f'' = -r(2 - rx) e^{r(1-x)}$.

Notice that *the attractor is contained within the lines $x = f(r)$ and $x = r$ (for $r > 0.5$)*. The explanation for this is exactly analog to the reasons given for the bounds on the attractor for the exponential #1 map.

Note also that *the orbit diagram shows nothing for $r = 0.5$* . This is because in this case every point $0 \leq x \leq 0.5$ is a fixed point. Hence if the initial random point satisfies $x_0 \leq 0.5$, then $x_n = x_0$ for all n . Else $x_n = x_1 < 0.5$ for all $n > 0$. This means that, at most, there are two points plotted for $r = 0.5$, which (of course) do not show up in the png file. Even in the original (full data) figure they are very hard to spot.

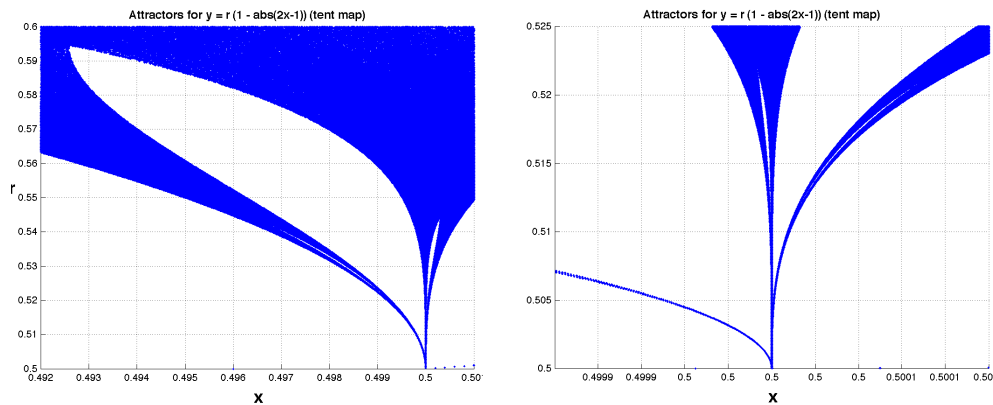


Figure 4.6: Details of the orbit diagram for the tent map $f(x) = r(1 - |2x - 1|)$. The **left panel** shows the region $0.5 \leq r \leq 0.6$ and $0.492 \leq x \leq 0.501$, while the **right panel** shows $0.5 \leq r \leq 0.525$ and $0.4999 \leq x \leq 0.5001$. These plots indicate that whole (distorted) copies of the full diagram exist within the tongues to the left and right of the gap in figure 4.5.

The tent map orbit diagram in figure 4.5 seems to have no structure. However, a blow of the “tongue” to the left of the gap in the attractor (specifically: the region $0.5 \leq r \leq 0.6$ and $0.492 \leq x \leq 0.501$) shows that this is not so — see the left panel in figure 4.6. A (distorted) whole copy of the diagram appears in the blow up. A further blow up of the right “tongue” of the picture on the left panel in figure 4.6 (specifically: the region $0.5 \leq r \leq 0.525$ and $0.4999 \leq x \leq 0.5001$) shows again the same phenomena — see the right panel in figure 4.6. This shows that *this orbit diagram has fractal structure, just like the prior ones*.

Finally, we “explain” *how/why the gaps in figures 4.5–4.6 form*. We do this by observing how the iterates of the tent map behave, where **the iterates are defined by $f_1(x) = f(x)$ and $f_{n+1}(x) = f(f_n(x))$ — thus an orbit is given by $x_n = f_n(x_0)$** .

Figure 4.7 shows plots of iterates of the tent map for values of r corresponding to the main gap shown in

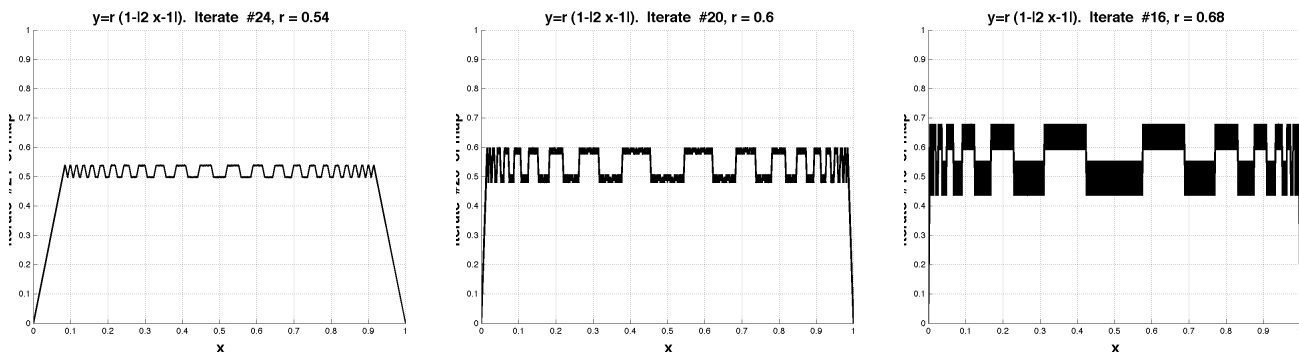
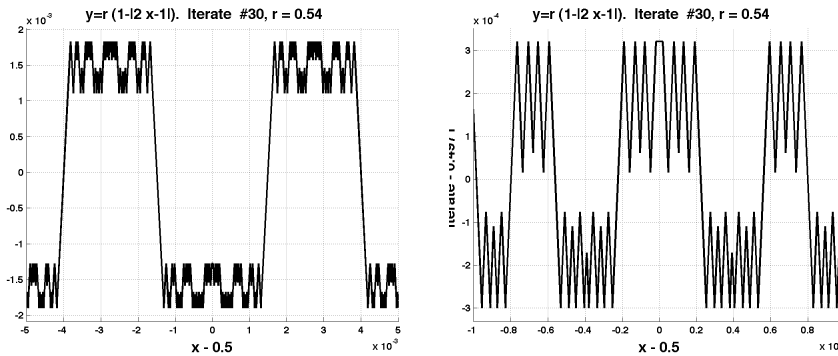


Figure 4.7: Iterates of the tent map for values of r corresponding to the main gap shown in figure 4.5. Specifically: **Left panel:** f_{24} for $r = 0.54$. **Middle panel:** f_{20} for $r = 0.60$. **Right panel:** f_{16} for $r = 0.68$.

figure 4.5. Specifically $r = 0.54, 0.60, 0.68$ (corresponding to the lower end, middle, and upper end of the gap, respectively). The take-away message is that the iterates develop a vaguely square wave shape, with sharp transitions between one set of “lower” values, and another set of “upper” values, leaving a gap between these two sets. This means that the values between the sets are extremely unstable (very large derivative) and will not show-up in the attractor. You can check that the gaps in figure 4.7 correspond to the gaps in figure 4.5.

Next, figure 4.8 shows the structure of one of the “oscillatory” regions in figure 4.7 (where details are beyond



In these plots the axis scales are shifted, as follows:
 Horizontal by 0.5.
 Left panel vertical by 0.4987.
 Middle panel vertical by 0.4971.
 See remark 4a.

Figure 4.8: Detail of f_{30} for $r = 0.54$. Left panel: $0.4950 < x < 0.5050$. Middle panel: $0.4990 < x < 0.5010$.

the resolution of the png file used). Notice how the same pattern seen in figure 4.7 repeats here, leading to the smaller gaps seen in figure 4.6.

Remark 4a: In figure 4.8 we have applied a shift to the vertical and horizontal scales. This is to avoid the need to have axis labels involve 4 decimals (which would make them unreadable). If you blow up the pdf image, you will see that they are now readable. ♣

THE END.