Answers to P-Set # 05, (18.353/12.006/2.050)j
MIT (Fall 2023)

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1.1 Statement: Computer generated phase portrait: The Eyes

First, **plot the phase plane portrait** for

$$\dot{x} = y + y^2 \quad \text{and} \quad \dot{y} = -\frac{1}{2}\,x + \frac{1}{5}\,y - x\,y + \frac{6}{5}\,y^2, \qquad (1.1)$$

(the eyes system) using a computer. I encourage you to *use the* PHPLdemoB *MatLab script* provided with the class Toolkit. **Do two plots, say: $-7 < x,\ y < 7$ ("large scale") and $(-6 < x < 4;\quad -4 < y < 3)$ for a bit more detail.**

Next, **find the critical points and classify them — does what you observe in the plot match what the theory predicts? Explain any discrepancies.** *Be careful with this part!* Explore (numerically) what happens close to the critical points; *what do you see there and how does it make the portrait consistent with the theory?*

Optional tasks.

(1) Look at the phase portrait. There is an "obvious" visual symmetry that you should spot. **Explicitly write the symmetry of the system** that explains what you see (a change of variables that leaves the system invariant).

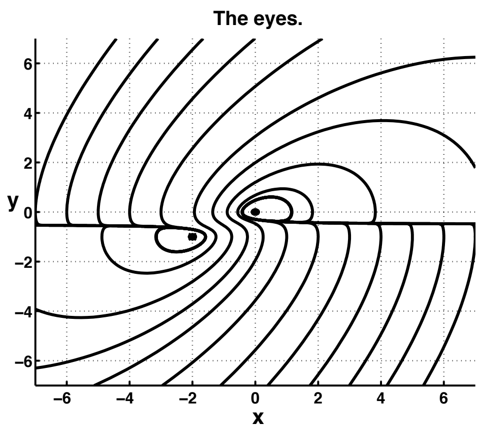
Hint. Look at the critical points. What transformation (consistent with the visual symmetry) maps one to the other?

(2) In your "large scale" portrait you should be able to see that the orbits, for $|x|$ large, approach a curve with $y \approx -0.5$ (for $x > 0$) or leave it for $x < 0$. **Make a theoretical argument explaining why this happens.**

Hint. The same type of approach used to analyze relaxation limit cycles works here, because something is large.

1.2 Answer: Computer generated phase portrait: The Eyes

Figure 1.1 (left) shows a computer generated "large scale" phase portrait for the solutions of (1.1) — to avoid crowding the orbit's flow directions are not shown. Figure 1.1 (right) shows a smaller region of the phase plane. Note



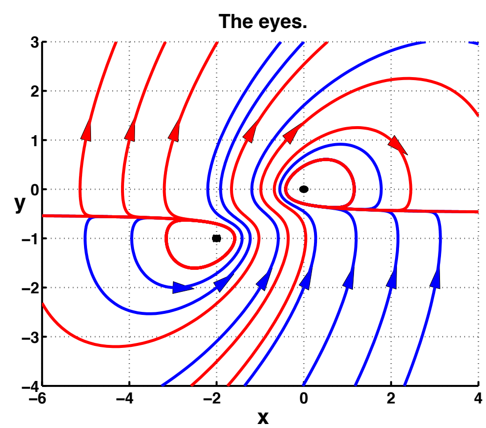


Figure 1.1: Phase plane portrait for "The Eyes" (global picture). The critical points are the eyes' pupils.

that something "funny" is going on near the critical points, as explained below.

The critical points for equation (1.1) are $\boldsymbol{P_l = (-2,\ -1)}$ and $\boldsymbol{P_r = (0,\ 0)}$. Linear analysis shows that

A. $\boldsymbol{P_l}$ **is a stable spiral, with eigenvalues** $\boldsymbol{\lambda = -0.1 \pm i\,\sqrt{0.49}}$.

B. $\boldsymbol{P_r}$ **is an un-stable spiral, with eigenvalues** $\boldsymbol{\lambda = +0.1 \pm i\,\sqrt{0.49}}$.

On the other hand, the picture in figure 1.1 (right) gives the impression that the orbits are leaving P_l and approaching P_r, which certainly contradicts **A** and **B** above. **How do we explain this?**

The answer to the question in the prior paragraph is provided by figure 1.2, which shows a detail near P_r. The picture shows that $\boldsymbol{P_r}$ **is an un-stable spiral, but it is enclosed by a stable limit cycle.** Thus, *all the orbits that seem to be approaching* P_r *in figure* 1.1 *(right), in fact are approaching this limit cycle.* Similarly, **there is an un-stable limit cycle near** $\boldsymbol{P_l}$**,** so that the orbits that seem to be leaving P_l, are in fact leaving the unstable limit cycle near P_l (the

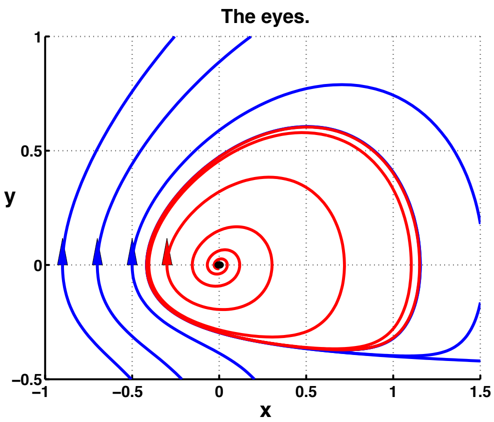


Figure 1.2: Phase plane portrait for "The Eyes"
Detail near the unstable spiral point P_r, where the dot at the eye's pupil is the critical point. Note the orbits spiraling away from the critical point, and approaching a (stable) limit cycle — the edge of the eye.

phase plane portrait near P_l looks quite similar to the picture shown in figure 1.2, except that the orbits move away from the limit cycle, rather than towards it).

In summary, **this is the global behavior for the orbits:**

1. Orbits starting within the limit cycle enclosing P_l, approach P_l as $t \to \infty$ and the limit cycle as $t \to -\infty$.
2. Orbits starting within the limit cycle enclosing P_r, approach P_r as $t \to -\infty$ and the limit cycle as $t \to \infty$.
3. All other orbits approach the limit cycle enclosing P_r (resp. P_l) as $t \to \infty$ (resp. $t \to -\infty$).

Optional task #1. Figure 1.1 has an obvious mirror symmetry roughly across the line $x = y$, but not quite. The critical points give the clue: re-write the system in terms of the variables $\boldsymbol{X = x + 1}$ and $\boldsymbol{Y = y + 0.5}$, so that the critical points are $\boldsymbol{(X,\, Y) = \pm(1,\, 0.5)}$, and have the desired symmetry. Then the system becomes

$$\dot{\boldsymbol{X}} = \boldsymbol{Y^2 - \tfrac{1}{4}} \quad \text{and} \quad \dot{\boldsymbol{Y}} = \boldsymbol{\tfrac{1}{5} - X\,Y + \tfrac{6}{5}\,Y^2}. \tag{1.2}$$

This is clearly invariant under $\boldsymbol{X \to -X}$, $\boldsymbol{Y \to -Y}$, and $\boldsymbol{t \to -t}$; an exact mirror symmetry across $\boldsymbol{X = Y}$, with time reversal (flip the arrows).

Optional task #2. We will make the argument for $\boldsymbol{x \gg 1}$. The case $\boldsymbol{-x \gg 1}$ follows from the symmetry discussed earlier. Let $\boldsymbol{x = \frac{1}{\epsilon}\,\chi}$, with $0 < \epsilon \ll 1$ and $\chi = O(1)$. Then the equations become

$$\dot{\boldsymbol{\chi}} = \boldsymbol{\epsilon\,(y + y^2)} \quad \text{and} \quad \dot{\boldsymbol{y}} = \boldsymbol{-\tfrac{1}{\epsilon}\,\left(y + \tfrac{1}{2}\right)\chi + \tfrac{1}{5}\,y + \tfrac{6}{5}\,y^2}. \tag{1.3}$$

Thus, for $y > -\frac{1}{2}$, $\dot{y}$ is large and negative, while $\dot{\chi}$ is small. Similarly, for $y < -\frac{1}{2}$, $\dot{y}$ is large and positive, while $\dot{\chi}$ is small. **It follows that y gets pushed into a narrow band near $y = -\frac{1}{2}$.** Note that, once an orbit ends up in this band, it gets funneled into the limit cycle. because it cannot cross. The end product is the "line" in figure 1.1 going into the right limit cycle.

2 Computer generated phase portrait: van der Pol #01

2.1 Statement: Computer generated phase portrait: van der Pol #01

Task #1. Plot a computer generated phase plane portrait for the *van der Pol oscillator:* $\ddot{x} - 2\,(1 - x^2)\,\dot{x} + 4\,x = 0.$ (2.1)

I strongly suggest that you use the PHPLdemoB *MatLab script provided to you in the class website [MatLab toolkit].* Note that, in order to use the script, you have to reduce the equation to a system written in terms of u and v; I suggest that you use $u = x$ and $v = \frac{1}{2}\,\dot{x}$, and plot in the square $-5 < u, v < 5$ [this will give you a nice plot including the "main features" in the phase portrait].

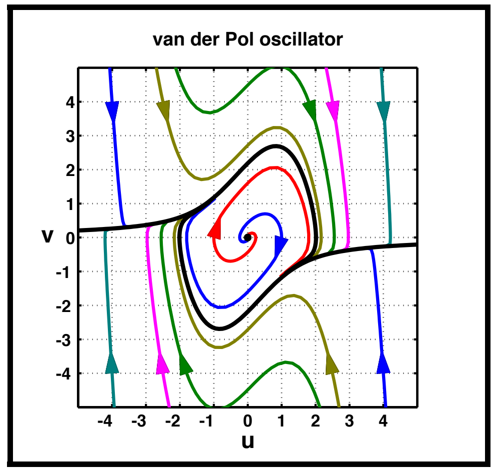
Task #2. What kind of critical point is $u = v = 0$? Find the eigenvalues of the linearized problem. Does your phase plane portrait agree with your analysis?

2.2 Answer: Computer generated phase portrait: van der Pol #01

With $u = x$ and $v = \frac{1}{2}\dot{x}$ the equation in (2.1) becomes the system

$$\dot{u} = 2\,v \quad \text{and} \quad \dot{v} = -2\,u + 2\,(1 - u^2)\,v.$$

A computer generated phase portrait for this system is shown by the picture on the right. Notice the **attracting limit cycle**, as well as the **unstable spiral point** at the origin, with linear eigenvalues $\lambda = 1 \pm i\,\sqrt{3}$.



3 Computer generated phase portrait: van der Pol #03

3.1 Statement: Computer generated phase portrait: van der Pol #03

Task #1. Plot a computer generated phase plane portrait for the *van der Pol oscillator:*

$$\ddot{x} - 4\,(1 - x^2)\,\dot{x} + x = 0. \qquad (3.1)$$

I strongly suggest that you use the PHPLdemoB *MatLab script provided to you in the class website [MatLab toolkit].* Note that, in order to use the script, you have to reduce the equation to a system written in terms of u and v; I suggest that you use $u = x$ and $v = \dot{x}$, and plot in the square $-6.5 < u, v < 6.5$ [this will give you a nice plot including the "main features" in the phase portrait].

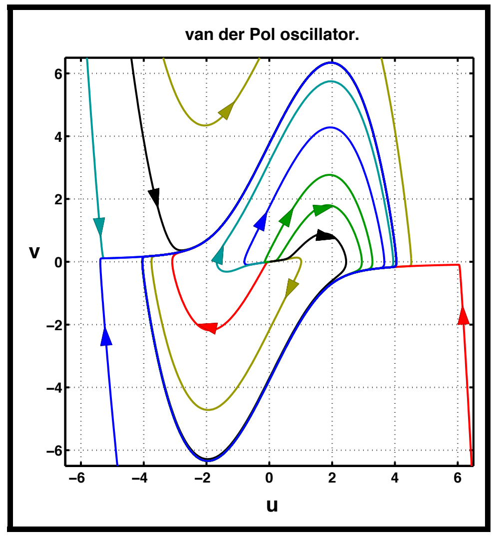
Task #2. What kind of critical point is $u = v = 0$? Find the eigenvalues of the linearized problem. Does your phase plane portrait agree with your analysis?

3.2 Answer: Computer generated phase portrait: van der Pol #03

With $u = x$ and $v = \dot{x}$ the equation in (3.1) becomes the system

$$\dot{u} = v \ \text{ and } \ \dot{v} = -u + 4\,(1 - u^2)\,v.$$

A computer generated phase portrait for this system is shown by the picture on the right. Notice the **attracting limit cycle**. The origin is a **node (unstable)**, with eigenvalues and eigenvectors $\lambda_{\pm} = 2 \pm \sqrt{3}$ and $\vec{e}_{\pm} = (1,\, \lambda_{\pm})$. The direction of the eigenvectors are evident in the plot, as the direction along which orbits leave the origin — all of them, but two, along $\vec{e}_{-}$ (as $t \to -\infty$ the eigenvector corresponding to the smallest eigenvalue dominates for a generic orbit).



4 Find the potential for a gradient system

4.1 Statement: Find the potential for a gradient system

Suppose that you are given a system $\dot{x} = f(x,\, y)$ and $\dot{x} = g(x,\, y)$. (4.1)

How can you to tell if (4.1) is a gradient system? Answer: **A necessary condition is $f_y = g_x$.** (4.2)

Why? Because if (4.1) is a gradient system, for some potential $V = V(x,\, y)$, $f = -V_x$ and $g = -V_y$.

In N dimensions, $\dot{\vec{x}} = \vec{f}(\vec{x})$, the condition (4.2) becomes $(f_n)_{x_m} = (f_m)_{x_n}$ for all n and m.

It turns out that, if the system is defined **in a simply connected region, (4.2) is also sufficient.** (4.3)

Then we can write $V(\vec{r}) = -\int_\Gamma (f\,\mathrm{d}x + g\,\mathrm{d}y)$, (4.4)

where Γ is a curve from some fixed point $\vec{p}$ in the region, to $\vec{r} = (x,\, y)$.

Because of the Green-Stokes theorem, the integral in (4.4) does not depend on the choice of Γ. *Make sure that you understand why this is so!*

If the region in (4.3) is a rectangle with sides parallel to the axes, this simple process works: (4.5)

First, integrate $V_x = -f$ for each value of y. This yields V up to some unknown, additive, function $h = h(y)$ (the integration constant). Then find h by using the other equation $V_y = -g$; which determines h up to a constant. *The reason that this works* is because a V exists, as shown by (4.4).

Your task: for the systems below, determine which ones are gradient systems and which ones are not, and find V for the ones that are gradient. The systems are defined everywhere in the plane.

- **(a)** $\dot{x} = 2\,x\,y - y\,\cosh(x)$ and $\dot{y} = x^2 - \sinh(x)$.
- **(b)** $\dot{x} = 3\,x^2 - e^y\,\cos(x)$ and $\dot{y} = -1 - e^y\,\sin(x)$.
- **(c)** $\dot{x} = 3\,x^2 - e^y\,\sin(x)$ and $\dot{y} = -1 - e^y\,\cos(x)$.

4.2 Answer: Find the potential for a gradient system

Using the process in (4.5) yields (below C is always a constant)

- **(a)** $f = 2\,x\,y - y\,\cosh(x)$ and $g = x^2 - \sinh(x)$.
  Then $f_y = g_x = 2\,x - \cosh(x)$, hence **the system is gradient,** with $V = -x^2\,y + y\,\sinh(x) + C$.
  $V_x = -f \;\Rightarrow\; V = -x^2\,y + y\,\sinh(x) + h(y)$. Then $V_y = -g \;\Rightarrow\; h =$ constant.
- **(b)** $f = 3\,x^2 - e^y\,\cos(x)$ and $g = -1 - e^y\,\sin(x)$.
  Then $f_y = g_x = -e^y\,\cos(x)$, hence **the system is gradient,** with $V = -x^3 + y + e^y\,\sin(x) + C$.
  $V_x = -f \;\Rightarrow\; V = -x^3 + e^y\,\sin(x) + h(y)$. Then $V_y = -g \;\Rightarrow\; h = y +$ constant.
- **(c)** $f = 3\,x^2 - e^y\,\sin(x)$ and $g = -1 - e^y\,\cos(x)$.
  Then $f_y = -e^y\,\sin(x) \neq e^y\,\sin(x) = g_x$, hence **the system is not gradient.**

5 Planetary orbits in General Relativity

5.1 Statement: Planetary orbits in General Relativity

The relativistic equation for the orbit of a planet around a star is

$$\frac{d^2 u}{d\,\theta^2} + u = \alpha + \epsilon\, u^2, \qquad (5.1)$$

where $(r,\, \theta)$ are the polar coordinates for the planet's position in the plane of motion, and $u = 1/r$. The parameter $\alpha > 0$ is related to the angular momentum of the orbit (it is the same as in classical Newtonian mechanics). Finally, the term ϵu^2 is the relativistic correction to Newtonian mechanics, where $0 < \epsilon \ll 1$. Note: we are **only interested in solutions with $u \geq 0$**. Note that $u = 0$ somewhere corresponds to the planet escaping the star's gravitational field.

- **(a)** Rewrite the equation as a system in the $(u,\, v)$ plane, where $v = \dfrac{du}{d\theta}$.
- **(b)** Find all the equilibrium points of the system.
- **(c)** Show that one of the equilibria is a center in the $(u,\, v)$ phase plane, according to the linearization. *Is it a nonlinear center?*
- **(d)** Show that the equilibrium point found in **(c)** corresponds to a circular planetary orbit.
- **(e)** **Optional.** The equation has solutions where u is a periodic function of θ.
  e1. *Do these solutions correspond to periodic orbits around the star?*
  e2. *If not, what do they correspond to?*
  e3. *What happens when $\epsilon = 0$ (Newtonian mechanics)?*
  Hint. Examine first **e3.** *Where are the elliptical orbits?*

5.2 Answer: Planetary orbits in General Relativity

- **(a)** With $v = \frac{du}{d\theta}$, the equation yields the system:
$$\frac{du}{d\theta} = v \quad \text{and} \quad \frac{dv}{d\theta} = -u + \alpha + \epsilon\, u^2. \qquad (5.2)$$
- **(b)** The equilibrium points for (5.2) are:
$$v = 0 \quad \text{and} \quad u_{\pm} = \frac{1}{2\epsilon}\left\{1 \pm \sqrt{1 - 4\,\epsilon\,\alpha}\right\}. \qquad (5.3)$$
  Thus: $u_+ \approx \frac{1}{\epsilon} - \alpha + \ldots$
  and $u_- \approx \alpha + \epsilon\,\alpha^2 \ldots$
- **(c)** Equation in (5.1) is a conservative system, with energy
$$E = \frac{1}{2}v^2 + V(u), \qquad (5.4)$$
  where $V = \frac{1}{2}u^2 - \alpha\, u - \frac{1}{3}\,\epsilon\, u^3$. Further, **$u_-$ is a local minimum** of V, while **u_+ is a local maximum.** Hence
$$(u,\, v) = (u_+,\, 0) \text{ is a saddle, and } (u,\, v) = (u_-,\, 0) \text{ is a center.} \qquad (5.5)$$
  Note: $(u_+,\, 0)$ corresponds to an unstable circular orbit with radius $r_+ < 2\,\epsilon$. This solution has no analog in Newtonian mechanics; its presence is related to the gravitational collapse ("black holes") predicted by General Relativity.
- **(d)** The equilibrium points have $u = 1/r$ constant, thus they correspond to circular orbits.
- **(e)** Near the critical point $(u,\, v) = (u_-,\, 0)$, the solutions of (5.2) are all periodic functions of θ. *Does this then mean that the planet orbit around the star follows a closed periodic orbit?* **The answer to this is no, unless the period in θ of the solution is an integer fraction of 2π** (and u does not vanish on the solution). Otherwise, as the planet goes around the star one turn, it does not return to its prior position (but some other), and the orbit does not close.

  e3. For $\epsilon = 0$ (Newtonian Mechanics) the solutions to (5.1) are
$$u = \alpha + c\,\sin(\theta + \theta_0), \qquad (5.6)$$
  where c is a constant. Solutions with $0 < |c| < \alpha$ are periodic of period $2\,\pi$ and stay positive $u > 0$. These solutions yield closed, periodic planetary orbits (Kepler's ellipses).[1] However, in the Relativistic case the period is a function of the amplitude of the deviation from the center, and the orbits are then not closed. **For small enough deviations, they look like ellipses whose principal axes rotate slowly in space — orbital precession.**

Figure 5.1 **shows a computer generated phase plane portrait for the system in (5.2).**

[1] The case $|c| = \alpha$ yields a parabolic orbit, while $|c| > \alpha$ yields hyperbolic orbits.

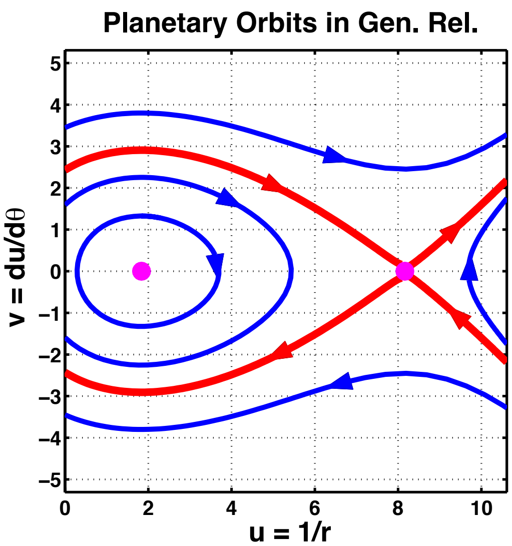


Figure 5.1: Planetary orbits in General Relativity

The picture on the left shows a computer generated phase portrait for the system in (5.2), with

$$\epsilon = 0.1 \quad \text{and} \quad \alpha = 1.5$$

Close to the critical point $(u,\, v) = (u_-,\, 0)$ periodic solutions fully within $u > 0$ arise. These give rise to bounded orbits (though not necessarily closed) around the star. Further away these "periodic" solutions reach $u = 0$ (i.e.: $r = \infty$), which corresponds to escape from the star system. The unstable circular orbit at the critical (saddle) point $(u,\, v) = (u_+,\, 0)$ is also shown.
Warning: these are not plots of the "actual" orbits in physical space, but of $u = 1/r$ and $v = \frac{du}{d\theta}$ as functions of θ. The arrows indicate the direction in which θ increases.

6 Saddle connections

6.1 Statement: Saddle connections

Consider a phase plane system with exactly two fixed points, both of which are saddles. Consider now the following situations:

(a) There is no orbit that connects the saddles.

(b) There is exactly one orbit that connects the saddles.

(b) There are exactly two orbits that connect the saddles.

In each case, **if possible, sketch a phase portrait for such a system, else give a reason for why it is not possible. The sketches MUST include what the stable and unstable manifolds of the two saddles do.**

6.2 Answer: Saddle connections

We begin by noticing that

1. The *only way that the two saddles can have a trajectory connecting them is if one of the saddle's unstable manifolds is a stable manifold for the other saddle.* Furthermore:

1a. *There cannot be any homoclinic connections.* Why? Because (index theory) this would require fixed points inside the "loop" formed by the homoclinic connection, with indexes adding up to 1, and we only have saddles.

Thus *if there is only one connection, the remaining stable/unstable manifolds must all be distinct* (this means 7 separate curves: 6 semi-infinite ones starting/ending at one of the saddles, and one connecting them). Modulo deformations this yields the picture on the right panel of figure 6.1.

Thus, **the answer to item b is: possible, see the right panel of figure 6.1.**

Note that, up to deformations (topology) this picture is the unique answer to item **b**. Furthermore:

1b. *No further connections can be added (i.e.: one is the maximum allowed).* The reason is, again, index theory. Two connections would create a "loop" (cycle graph), requiring fixed points we do not have.

Thus, **the answer to item c is: not possible.**

2. On the other hand, *if there is no connections between the saddles, all the 8 stable/unstable manifolds must be distinct curves. This is, in fact, enough to guarantee no connections.* The picture on the left panel of figure 6.1 gives an example of this.

Thus, **the answer to item a is: possible, see the left panel of figure 6.1.**

However, the allowed topology in the answer to item **a** is not unique — *can you think of other alternatives?*

From the answers above *it should be obvious why the problem statement requires that the answer includes what the stable and unstable manifolds of the saddles do.* They are the key to everything.

Extra stuff. *The problem does not request that you write an explicit system that produces the desired phase plane.* However, next I explain how I did this.

The easiest way to do it is to use conservative **Hamiltonian** systems:

$$\dot{\boldsymbol{x}} = -\boldsymbol{H_y} \text{ and } \dot{\boldsymbol{y}} = \boldsymbol{H_x}. \quad (6.1)$$

Then we only need to specify $H = H(x, y)$, with the orbits given by the level curves for the surface $z = H(x, y)$. When considering critical points for the system in (6.1), it is helpful to think of the surface $z = H(x, y)$ as a mountain range. Then saddles occurs at low points of ridges, or high points of valleys, while centers occur at high points of ridges, or low points of valleys. Thus, in order to get the phase plane portraits for this problem, the trick is to put valleys and ridges appropriately. Note that centers can only be avoided by having the ridges highest points (and valley lowest points) at ∞.

For item b of this problem, a simple way to achieve the desired result is to: **(B1)** Choose the stable and unstable manifolds for the saddles (i.e.: the level curves at the same level as the saddles, in the the surface $z = H(x, y)$). **(B2)** Choose H so it switches sign across the curves selected in **(B1)**, and grows in size away from them (this second condition is so H has no maximums or minimums, so that no centers occur). We implement this idea by selecting the saddles to be at $(x, y) = (\pm 1, 0)$, and the curves by $x \equiv \pm 1$ and $y \equiv 0$. Then take $\boldsymbol{H = y(x^2 - 1)}$. The right panel in figure 6.2 shows the results of this choice.

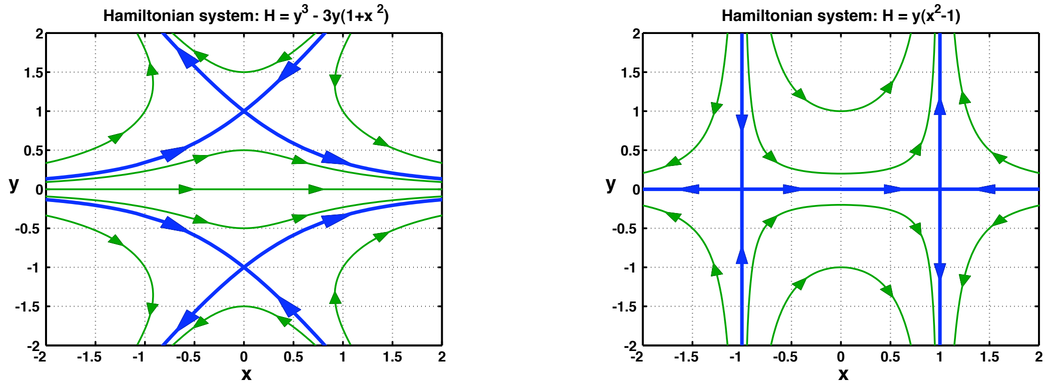


Figure 6.1: Saddle connections. The **panel on the left corresponds to case (a),** with no orbits that connect the saddles. Specifically, the phase portrait here corresponds to the Hamiltonian system with Hamiltonian $H = y^3 - 3\,y\,(1 + x^2)$. The **panel on the right corresponds to case (b),** with one orbit that connect the saddles. Specifically, the phase portrait here corresponds to the Hamiltonian system with Hamiltonian $H = y\,(x^2 - 1)$.

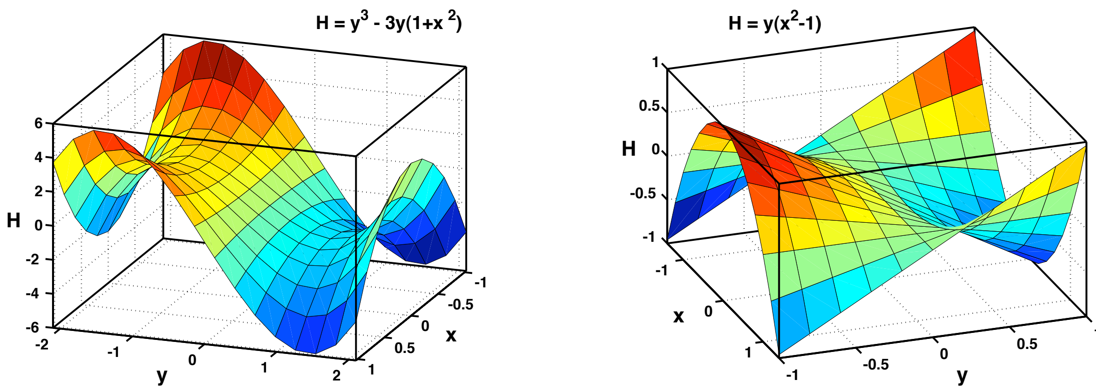


Figure 6.2: Saddle connections. **Left panel (item a),** surface determined by the Hamiltonian $H = y^3 - 3\,y\,(1 + x^2)$. **Right panel (item b),** surface determined by the Hamiltonian $H = y\,(x^2 - 1)$.

Constructing an appropriate surface for item a in the problem is a bit trickier. We need two saddles that occur at *different* levels in the mountain range, and nothing else. To get the first saddle, place a long ridge (from $x = -\infty$ to $x = \infty$), with a low point at $x = 0$ (this will create a saddle at the position of the low point). Right next to this ridge, place a long valley (notice that there is a duality between valleys and ridges), with a high point [2] at $x = 0$ (this will create a saddle at the position of the high point). If done carefully, the surface constructed in this fashion will have the right properties; since it will have a shape like the one shown on the left in figure 6.2. How do we get a formula that gives rise to such a surface? Well, notice that each of the $x =$ constant cross-sections of the surface looks like a cubic, while each of the $y =$ constant cross-sections looks like a parabola. Thus, take a cubic in y, with coefficients that depend quadratically on x. For example: $\boldsymbol{H = y^3 - 3\,(1 + x^2)\,y}$ does the trick.

[2] The valley high point should be below the ridge low point!

7 Two nested limit cycles and a single critical point

7.1 Statement: Two nested limit cycles and a single critical point

(1) Give an example of a (smooth) phase plane system with exactly two periodic orbits and a single critical point. Note: the two periodic orbits will necessarily by isolated, hence they will be limit cycles.
Hint: easy to do in polar coordinates (see remark below).

(2) **True or false:** In the situation of item **1**, both limit cycles can be repellers (i.e.: trajectories near the limit cycles diverge from them). **If true**, sketch an appropriate phase plane portrait. **If false:** explain why.
Hint: recall the Poincaré Bendixon theorem.

Remark. *How can you be sure that a system written in polar coordinates is smooth?* Answer: write the system in the form

$$\dot{r} = a\,r \quad \text{and} \quad \dot{\theta} = b, \tag{7.1}$$

where a and b are some functions of $(r,\,\theta)$ — equivalently, of $(x,\,y)$. In cartesian coordinates this corresponds to

$$\dot{x} = a\,x - b\,y = f, \quad \dot{y} = b\,x + a\,y = g. \tag{7.2}$$

Then $a = a(x,\,y)$ and $b = b(x,\,y)$ should be such that f and g are smooth. **For example, this happens if $a = a(r^2)$ and $b = b(r^2)$ are smooth functions of r^2.** However, it generally fails if they are functions of $r = \sqrt{x^2+y^2}$ only, because r is not smooth at the origin, which would render (7.2) not smooth there. ♣

Question: *why do we transform (7.1) to cartesian coordinates in order to determine if the system is smooth?* Answer: to remove the coordinate singularity at the origin, which interferes with the task.

7.2 Answer: Two nested limit cycles and a single critical point

(1) An example is given by the system

$$\dot{r} = (1-r^2)\,(r^2-9)\,r \quad \text{and} \quad \dot{\theta} = 1. \tag{7.3}$$

For this system the origin is the only critical point;
an unstable spiral point (there are no critical points for $r > 0$, because there $\dot{\theta} \neq 0$). Furthermore, $r \equiv 1$ is an attracting limit cycle (it attracts everything in $0 < r < 3$), while $r \equiv 3$ is a repeller limit cycle (backwards in time it attracts everything in $1 < r < \infty$).

In general, because there is a single critical point: **The two limit cycles must be nested, with the critical point inside the inner limit cycle, and no critical points in the annular region between the limit cycles.**

(2) **False.** Consider a trajectory in the annular region between the limit cycles. From the Poincaré Bendixon theorem, this trajectory must approach one of the limit cycles as $t \to \infty$, hence at least one of the limit cycles is not a repeller. The same argument, with $t \to \infty$, shows that: **both limit cycles cannot be attractors.**

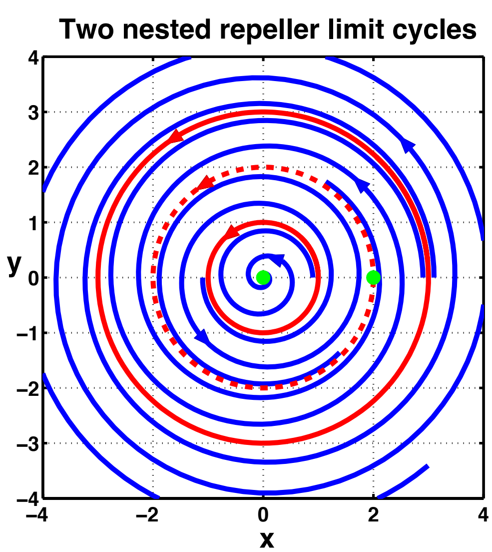


Figure 7.1: Two nested repeller limit cycles

The picture on the left shows a computer generated phase portrait for the system as described in (7.4), with

$a = +0.09/r$ for $r \geq 3.5$.
$a = -0.09\,\sin(\pi\,r)/r$ for $r \leq 3.5$.
$b = 1 - e^{-\rho/0.05}$, where $\rho = (x-2)^2 + y^2$.

These forms were selected to get a plot with a clearly visible structure. Note that a is an even, smooth, function of r for $r < 3.5$, and has a continuous derivative at $r = 3.5$. This is good enough for the purpose of illustrating the result of (7.4).
The two limit cycles, the cycle graph, the critical points, as well as some orbits are shown. The arrows indicate the flow direction. Clearly items **s1–s6** below (7.4) apply.

However, in regards to the question in item **(2)**, **if an extra critical point is allowed,** it can be placed between the limit cycles, to create an attracting cycle-graph in the annular region. **Then both limit cycles can be repellers.** One way to do this is as follows:

Take three radii $0 < r_1 < r_2 < r_3$. Then: **(i)** Let $a(r^2)$ have simple zeros at these values, with $a < 0$ for $r < r_1$; for example: $a = (r^2 - r_1^2)\,(r^2 - r_2^2)\,(r^2 - r_3^2)$. **(ii)** Let b be positive everywhere, except for a zero on the circle $r = r_2$; for example: $b = (x - r_2)^2 + y^2$. (7.4)

A system like this will have:

- **s1.** A stable spiral point at the origin, attracting all the trajectories in $0 < r < r_1$.
- **s2.** A repeller limit cycle with $r \equiv r_1$.
- **s3.** An attracting cycle graph with $r \equiv r_2$, attracting all the trajectories in $r_1 < r < r_2$ and $r_2 < r < r_3$. The cycle graph is made up by the critical point at the position of the zero of b, and an homoclinic connection from this critical point to itself (along $r \equiv r_2$).
- **s4.** A repeller limit cycle with $r \equiv r_3$.
- **s5.** All trajectories beyond $r = r_3$ spiral out to infinity.
- **s6.** Note that index theory implies that the critical point at the zero of b has index $= 0$.

An **example** of this is shown in figure 7.1.

THE END.