

Answers to P-Set # 04, (18.353/12.006/2.050)j MIT (Fall 2023)

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1 Volume evolution

1.1 Statement: Volume evolution

Consider some arbitrary orbit, Γ , for the system

$$\frac{d\vec{r}}{dt} = \vec{F}(\vec{r}), \quad \text{where } \vec{r} \text{ and } \vec{F} \text{ are vectors in } \mathcal{R}^n, \quad (1.1)$$

and \vec{F} has continuous partial derivatives up to (at least) second order. That is: Γ is the curve in \mathcal{R}^n given by some solution $\vec{r} = \vec{r}_\gamma(\mathbf{t})$ to (1.1). Then

A. Let $\Omega = \Omega(t)$ be an “infinitesimal” region that is being advected, along Γ , by the flow given by (1.1). *For example:*

A1. Let $\Omega(0)$ be a ball of “infinitesimal” radius dr , centered at $\vec{r}_\gamma(0)$.

A2. For every point $\vec{r}_p^0 \in \Omega(0)$, let $\vec{r} = \vec{r}_p(t)$ be the solution to (1.1) defined by the initial data $\vec{r}_p(0) = \vec{r}_p^0$.

A3. At any time t_* , the set $\Omega(t_*)$ is given by all the points $\vec{r}_p(t_*)$, where \vec{r}_p^0 runs over all the points in $\Omega(0)$.

Note that $\Omega(0)$ need not be a ball. Any infinitesimal region containing $\vec{r}_\gamma(0)$ will do. All we need is that the notion of hypervolume applies to it — see item **B**. In particular: *you do not need to use/know the formula for the hypervolume of a ball in n dimensions to do this problem!*

B. Let $\mathcal{A} = \mathcal{A}(t)$ be the hypervolume of $\Omega(t)$. Note: (i) if $n = 1$ the hypervolume is the length; (ii) if $n = 2$ the hypervolume is the area; (iii) if $n = 3$ the hypervolume is the volume; etc.

TASK: Find a differential equation for the time evolution of \mathcal{A} .

Optional: use the differential equation to show that $\det(e^{Bt}) = e^{\text{tr}(B)t}$ for any square matrix B — where $\text{tr}(B)$ denotes the trace of B . Note: what you are asked to do here is to do the proof using the differential equation, *specifically, not* by some other technique, like (say) linear algebra.

Hints.

h1. Introduce the vector $\delta\vec{r}_p = \delta\vec{r}_p(t) = \vec{r}_p - \vec{r}_\gamma$ for every point in $\Omega(t)$. This vector characterizes the evolution of the “shape” of Ω as the set moves along Γ . In order to calculate how $\mathcal{A}(t)$ evolves, you only need to know how the $\delta\vec{r}_p$ vectors evolve.

h2. For every vector $\delta\vec{r}_p$, write an equation giving $\delta\vec{r}_p(t + dt)$ in terms of $\delta\vec{r}_p(t)$ and the partial derivatives of \vec{F} along Γ . Since you are dealing with infinitesimal terms, you can neglect higher order terms, so as to obtain a relationship from $\delta\vec{r}_p(t)$ to $\delta\vec{r}_p(t + dt)$ given by a linear transformation. Make sure that this linear transformation correctly includes the $O(dt)$ terms, which you will need to calculate time derivatives.

h3. From the transformation in item **h2** derive a relationship between $\mathcal{A}(t + dt)$ and $\mathcal{A}(t)$. Note that:

- For linear transformations, hypervolumes are related by the absolute value of the determinant. †
- You need to calculate the determinant only up to $O(dt)$ terms (neglect higher orders).
- For any square matrix M , $\det(\mathbf{1} + \epsilon M) = \mathbf{1} + \epsilon \text{tr}(M) + O(\epsilon^2)$.

Optional: prove the formula in (c).

h4. Item **h3** will yield a formula of the form $\mathcal{A}(t + dt) = \mathcal{A}(t) + (\text{something}) dt$. What differential equation is this?

† Multi-variable calculus: For a transformation $\vec{x} \rightarrow \vec{y}$: $\int f(\vec{y}) d\vec{y} = \int f(\vec{y}(\vec{x})) |\det(\mathbf{J})| d\vec{x}$, where \mathbf{J} = matrix of partial derivatives $\frac{\partial y_m}{\partial x_n}$. Thus for an infinitesimal hypervolume $\delta\mathbf{V} \rightarrow |\det \mathbf{J}| \delta\mathbf{V}$. If $\vec{y} = M \vec{x}$ (linear transformation), $\mathbf{J} = M$.

1.2 Answer: Volume evolution

At any time t , we can write

$$\vec{r}_\gamma(t + dt) = \vec{r}_\gamma(t) + \vec{F}_\gamma(t) dt, \quad (1.2)$$

$$\vec{r}_p(t + dt) = \vec{r}_p(t) + \vec{F}_p(t) dt, \quad (1.3)$$

where we have neglected $O((dt)^2)$ contributions, $\vec{F}_\gamma = F(\vec{r}_\gamma)$, $\vec{F}_p = F(\vec{r}_p)$, and $\vec{r}_p = \vec{r}_p(t)$ tracks an arbitrary point in $\Omega(t)$ — as in item **A2**. Hence we can write

$$\begin{aligned}\delta\vec{r}_p(t+dt) &= \delta\vec{r}_p(t) + \left(\vec{F}_p(t) - \vec{F}_\gamma(t)\right) dt \\ &= (I + M_\gamma(t) dt) \delta\vec{r}_p(t),\end{aligned}\tag{1.4}$$

where: **(a)** $\delta\vec{r}_p = \vec{r}_p - \vec{r}_\gamma$, **(b)** I is the identity matrix, **(c)** $M = \{\partial F_i / \partial x_j\}$ is the matrix of partial derivatives of \vec{F} , **(d)** $M_\gamma = M(\vec{r}_\gamma)$, and **(d)** we have neglected $O((dr)^2 dt)$ terms to arrive at the second line in (1.4). Therefore

$$\mathcal{A}(t+dt) = \det\left(I + M_\gamma(t) dt\right) \mathcal{A}(t) = \left(1 + \text{tr}(M_\gamma(t) dt)\right) \mathcal{A}(t),\tag{1.5}$$

where we have neglected $O((dt)^2)$ terms when computing the determinant.¹ From this last equation we obtain

$$\frac{d}{dt} \mathcal{A} = \text{div}(\vec{F}) \mathcal{A},\tag{1.6}$$

since $\text{tr}(M_\gamma(t)) = \text{div}(\vec{F})$, **with the divergence evaluated along Γ** .

The optional tasks

Consider the equation $\frac{d\vec{r}}{dt} = B\vec{r}$, for which (1.6) yields $\mathcal{A} = e^{\text{tr}(B)t}$ **[#1]**. On the other hand $\vec{r}_p(t) = e^{Bt}\vec{r}_p^0$, so that $\mathcal{A}(t) = |\det(e^{Bt})|$. But $\det(e^{Bt})$ is never zero and starts at one, hence $\det(e^{Bt}) > 0$ and the absolute value is not needed. Thus $\mathcal{A}(t) = \det(e^{Bt})$ **[#2]**.

From **[#1]** and **[#2]**,

$$\det(e^{Bt}) = e^{\text{tr}(B)t}.$$

For the second optional task, consider first the case where all the eigenvalues of M , $\{\lambda_n\}$, are distinct, with $M\vec{v}_n = \lambda_n\vec{v}_n$ the eigenvectors. It is then easy to see that the eigenvalues of $1 + \epsilon M$ are $\{1 + \epsilon\lambda_n\}$.

Hence: $\det(1 + \epsilon M) = \prod(1 + \epsilon\lambda_n) = 1 + \epsilon \sum \lambda_n + O(\epsilon^2) = 1 + \text{tr}(M) + O(\epsilon^2)$.

The case of repeated eigenvalues follows by adding a small perturbation to M , which splits the eigenvalues, and then taking the limit.

Note: **there are many other ways to prove this. For example:**

(a) You could expand the determinant using the first row, and then noticing that this yields $\det(1 + \epsilon M) = (1 + \epsilon m_{11}) \det(1 + M_{11}) + O(\epsilon^2)$,[‡] where m_{ij} are the entries of M , and M_{11} is the matrix that results from eliminating the first row and column of M . Then use induction.

[‡] The tricky part here is showing that the $O(\epsilon^2)$ is really $O(\epsilon^2)$.

(b) You could use that, for any square matrix, $\det(A) = \sum \text{sign}(\sigma_s) \Pi_s$, where the sum is over all the products $\Pi_s = \prod_i a_i \sigma_s(i)$, where σ_s is a reordering of $1, 2, \dots, n$. Then notice that, for $A = 1 + \epsilon M$, the only σ_s for which Π_s is not $O(\epsilon^2)$ (or higher) is the identity.

(c) You could use that $\ln \det(1 + \epsilon M) = \text{tr}(\ln(1 + \epsilon M)) = \text{tr}(\epsilon M - \frac{1}{2}\epsilon^2 M^2 + \dots)$.

2 Neglect terms in equations (inertia in a forced-damped oscillator)

2.1 Statement: Neglect terms in equations

When modeling physical systems it is useful to be able to estimate how important the various physical effects that bear on the problem are, so that only the physics that matters is included in the model. The “kitchen sink” approach leads to models with too many unknown “free” parameters, and complicated system of equations that are very hard

¹ No absolute value is needed because $I + M_\gamma(t) dt$ is infinitesimally close to the identity, so its determinant is positive $= 1 + O(dt)$.

to solve (even with a computer). Not to mention the fact that “numerically” solving a system of equations whose behavior you do not understand can easily lead to trouble.

There is no such thing as fool-proof software that can reliably solve problems the user does not understand. This is particularly true for problems that involve solving pde.

Dimensional analysis is a tool that can help to identify effects that (maybe) can be neglected. I say “maybe” because the fact that some effect appears to be small does not mean that it can be neglected — for example: boundary layers and shocks are related to physical terms that a naive analysis classifies as “small”. But a necessary condition to be able to neglect an effect is that it be small. Beyond this, hard thinking is needed. There is no magic bullet.

Here we will consider a *very simple example, basically: a toy model for a shock absorber*. Thus, imagine a mass m , attached both to a spring (with spring constant $k > 0$) and a damper (with damping constant $\nu > 0$). We assume that the damper produces a force opposing the motion and proportional to the velocity. We also assume motion in 1-D, and a Hook law spring. In addition, we will assume that an external, time varying force, is acting on the mass.

Let x be the deviation of the mass from its equilibrium position (where the spring force vanishes). Then Newton’s laws of motion involve four terms: (1) *inertia*, $m \ddot{x}$, (2) *spring force*, $-k x$, and (3) *damping force*, $-\nu \dot{x}$. (4) *applied force*, $f = f(t)$. Now **answer/perform the following questions/tasks**:

- q1. What dimensions do k and ν have?
- q2. The balance between damping and the spring force produces a characteristic time scale, τ_d . Write a formula for τ_d — **without solving any equations**.
Assume that the external force has the form $f = f_0 F(t/\tau_d)$, where f_0 is a typical force, and F is an a-dimensional function, with both the function and derivatives of size $O(1)$.
- q3. The balance between inertia and the spring also produces a characteristic time scale, τ_s . Write a formula for τ_s — **without solving any equations**.
- q4. The balance between the spring force and the applied force produces a characteristic length scale, L . Write a formula for L — **without solving any equations**.
- q5. Write the equation for the mass spring system using a-dimensional variables. Specifically: write the equation in terms of the variables $\tilde{x} = x/L$ and $\tilde{t} = t/\tau_d$. The equation can be written so that it involves a single a-dimensional number, ϵ , multiplying the second derivative. Write ϵ in terms of τ_d and τ_s .

Now **answer the questions**: What condition on ϵ is needed to be able to neglect the effects of inertia (i.e.: the term $m \ddot{x}$ in the equations) on the behavior? What does the condition mean in terms of the times τ_d and τ_s ? Can you interpret the condition in terms of something being at, or almost, at equilibrium? Why is this a regime at which you would like a car shock-absorber to operate at?

- q6. Note that τ_s is related to the period of oscillation, $T = \alpha \tau_s$, via some numerical constant α . **Find α** — you need to solve an equation to find α .

2.2 Answer: Neglect terms in equations

- q1. The three terms $m \ddot{x}$, $\nu \dot{x}$, and $k x$ are all forces, and must have the same dimensions. Hence

$$\text{dimension}(k) = \frac{(\text{mass})}{(\text{time})^2} \quad \text{and} \quad \text{dimension}(\nu) = \frac{(\text{mass})}{(\text{time})}.$$

- q2. A balance between damping and the spring involves ν and k only. Hence $\tau_d = \nu/k$.
- q3. A balance between inertia and the spring involves m and k only. Hence $\tau_s = \sqrt{m/k}$.
- q4. A balance between the applied force and the spring involves f_0 and k only. Hence $L = f_0/k$.
- q5. In terms of the variables \tilde{x} and \tilde{t} , the equation $m \ddot{x} + \nu \dot{x} + k x = f$ takes the form

$$\epsilon \ddot{\tilde{x}} + \dot{\tilde{x}} + \tilde{x} = F(\tilde{t}), \quad \text{where} \quad \epsilon = \frac{\tau_s^2}{\tau_d^2}. \quad (2.1)$$

From this we see that *in order to be able to neglect the effects of inertia*, $\epsilon \ll 1$ is needed. This means that $\tau_s \ll \tau_d$, so that (over time scales of size τ_d) the inertial effects have time to reach equilibrium — with the time evolution of the system then being controlled by the damping.

If this were a car shock-absorber, what this behavior means is that the car height changes on the (relatively) slow time scale τ_d , and avoids the rattling that the car springs alone would cause. If you were actually designing a car, † you would start from the applied force that you expect, and estimate its time scale τ_f (this depends on the car velocity and road type), as well as its strength. Then you would select the damper parameters ‡ k and ν so that $\tau_d \approx \tau_f$, $\tau_s \ll \tau_d$, and so that L is kept reasonably small — from the formulas in item **q2–q4** it should be clear that this is possible. Unfortunately, the resulting shock-absorber would be pretty minimal — then again, actual shock absorbers are more complicated than this, and incorporate other parameters that can be adjusted.

† Bear in mind that this is an **extremely** simplified form of the real problem!

‡ The car mass m is not a parameter that you can choose here.

- q6.** The relevant equation is $m\ddot{x} + kx = 0$, whose solutions are $x = a \cos\left(\frac{t}{\tau_s} + \theta_0\right)$, where a and θ_0 are constants. It follows that $T = 2\pi\tau_s$ — i.e.: $\alpha = 2\pi$.

3 Dipole system

3.1 Statement: Dipole system

Task #1. Plot a computer generated phase plane portrait for the “Dipole system”

$$\dot{x} = 2xy \quad \text{and} \quad \dot{y} = y^2 - x^2. \quad (3.1)$$

I strongly suggest that you use the PHPLdemoB MatLab script provided to you in the class website [MatLab toolkit].

Task #2. Find the critical points for this system, and linearize near them. What do the linearized equations tell you about the behavior near the critical points?

Task #3. Use the generated phase plane portrait to compute the index of the critical points.

Task #4. The generated phase plane portrait should suggest that the orbits for this system are circles. † In fact *any circle tangent to the y-axis at the origin would seem to be an orbit*. **Show that this is correct.**

† In MatLab, use “axis square” when plotting, so there are no distortions.

Hint for #3. Write a function E whose level curves are **all** the circles tangent to the y -axis at the origin, and show that E is conserved. You will not be able to obtain a function E without some singularity on the y -axis. ‡ The best you can do is have E singular at the origin only. This is related to the fact that the y -axis is the circle tangent to the y -axis at the origin, whose radius is infinity; while the origin itself corresponds to a zero radius.

‡ This is not a problem, since the y -axis can be easily analyzed separately.

3.2 Answer: Dipole system

Task #1. Figure 3.1 shows a computed generated phase plane portrait. Because (3.1) is invariant under $x \mapsto ax$, $y \mapsto ay$, and $t \mapsto t/a$ ($a > 0$ any constant), **the phase plane portrait is invariant under stretching**. Once we know what happens in a neighborhood of the origin, we know what happens everywhere.

Task #2. The system has only one critical point, the origin. The linearized equations there are $\dot{\vec{x}} = \mathbf{0}$, which tells nothing about the behavior of the system near the critical point.

Task #3. **The index of the origin in the figure is $I = 2$.** As we go around the origin counterclockwise, starting on the positive real axis, in each quarter turn of the path the flow vector rotates (also counterclockwise) a half turn.

Task #4. The radius $|r|$ circle, † tangent to the y -axis at the critical point, is given by $(x - r)^2 + y^2 = r^2$. Solving

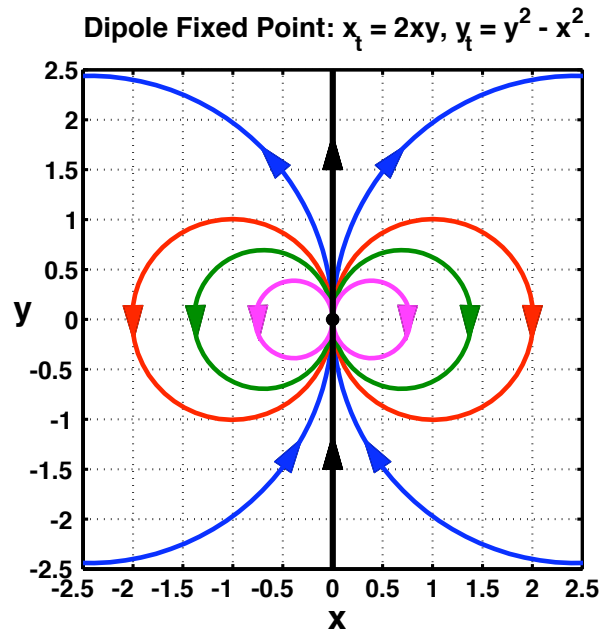


Figure 3.1: Phase plane portrait for the Dipole system.

for r yields

$$r = \frac{x^2 + y^2}{2x}, \quad \text{or} \quad \frac{1}{r} = \frac{2x}{x^2 + y^2}. \tag{3.2}$$

To show that these are conserved quantities,[‡] we use (3.1) to obtain: $\dot{\rho} = 2y\rho$, where $\rho = x^2 + y^2$. Hence

$$\dot{r} = \frac{\dot{\rho}}{2x} - \frac{\rho \dot{x}}{2x^2} = 0. \tag{3.3}$$

This proves that both $E_1 = r$ and $E_2 = 1/r$ are conserved quantities. Hence **the circles (with the origin missing) $r = \text{constant}$, $-\infty < r \neq 0 < \infty$, are orbits** — these circles are contained within either $x > 0$ or $x < 0$. The y -axis requires a separate argument, which is trivial because there the equations reduce to $\dot{y} = y^2$.

[†] Here $r > 0$ corresponds to a circle to the right of the y -axis, while $r < 0$ is a circle to the left.

[‡] Note that $E_1 = r$ is singular along the whole y -axis, while $E_2 = 1/r$ is singular at the origin only.

Task #4 (alternative). Seek for solutions of the form $x = r(1 + \cos \phi)$ and $y = r \sin \phi$, (3.4)

where r is a constant and $\phi = \phi(t)$ — i.e.: circles of radius $|r|$, centered at $(x, y) = (r, 0)$. Then

$$\dot{\phi} = -2r(1 + \cos \phi), \tag{3.5}$$

which has a semi-stable critical point at $\phi = \pm\pi$. Clearly:

(3.4–3.5) yield the phase portrait in figure 3.1, except for the y -axis, which requires a separate argument.

Remark (on global attractors that are not Liapunov stable). Figure 3.1 shows that there is exactly one orbit that never returns to the critical point (the positive y -axis). Were it not for this orbit, this critical point would be an example of a global attractor that is not Liapunov stable. However, it is easy to construct an *example of a global attractor that is not Liapunov stable* by “projecting” the phase portrait in figure 3.1 onto a sphere, as follows:

1. Pick a point on the sphere and call it “the origin” (this will be the critical point).
2. Draw a straight line *tangent to the sphere* at the critical point, and select a direction along the line.
3. Let the orbits be the intersections of the sphere with any plane that contains the line selected in item 2.
4. Let the flow direction along any of the orbits in item 3 be the same as that selected for the line in item 2.

This second system is an example of a global attractor that is not Liapunov stable. In this picture the y -axis gets mapped into a geodesic circle going through the origin and its antipode. ♣

4 Phase Plane Center Question #01

4.1 Statement: Phase Plane Center Question #01

Consider the equation

$$\ddot{x} + (1 - \cos(\dot{x})) + x = 0. \quad (4.1)$$

This equation has a critical point at $x = \dot{x} = 0$, which is a center for linearized analysis ([show this](#)). Is it a center for the full nonlinear equation as well? **Justify your answer** with an analytical argument. Finally: **plot a computer generated phase plane portrait for the system** (use the PHPLdemoB MatLab script provided with the MatLab toolkit in the course web page). Examine the phase plane in a reasonably large region enclosing the critical point; specifically: $-4 < x < 1$ and $-2.5 < \dot{x} < 2.5$. *What do you see? Can you guess what the complete phase plane looks like from this picture? How can you check if your guess is correct?*

4.2 Answer: Phase Plane Center Question #01

The linearization for equation (4.1) near the critical point $x = \dot{x} = 0$ is

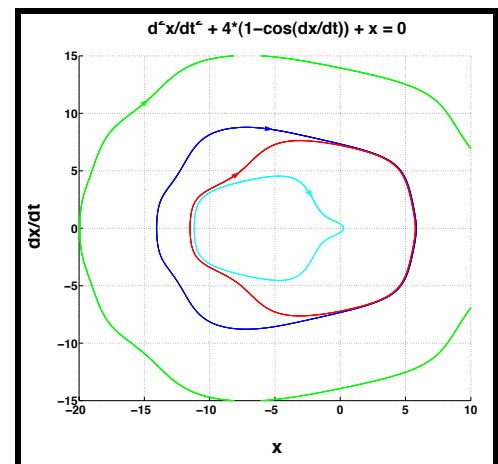
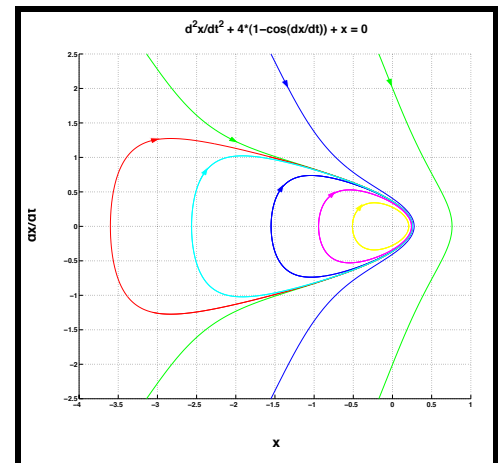
$$\ddot{x} + x = 0. \quad (4.2)$$

This is the harmonic oscillator equation, thus the critical point is a center in the linearized analysis. In addition, equation (4.1) is invariant under the change $t \rightarrow -t$; i.e.: (4.1) is a **reversible system**. It follows that $x = \dot{x} = 0$ is a **center for the non-linear problem as well**.

The picture on the right shows a computed generated phase plane portrait for $-4 < x < 1$ (horizontal axis) and $-2.5 < \dot{x} < 2.5$ (vertical axis). Note that, while near the center the orbits become nearly circular, far away they become more and more elongated. In fact, the periodic orbits seem to be restricted to be within a vaguely paraboloidal region. However, this is not true, as the phase plane picture below, on a much larger region, shows. This illustrates how easy is to draw the wrong conclusion from limited information.

The picture on the right indicated that all the orbits for equation (4.1) are closed curves, with the very large orbits transitioning back to circles — this last is not surprising, since when \ddot{x} and x are large they dominate over the $(1 - \cos(\dot{x}))$ term, which is constrained to be between 0 and 2.

Can we prove that all the orbits are closed? Yes: Introduce polar coordinates $x = r \cos \theta$ and $\dot{x} = r \sin \theta$. Then $\dot{\theta} = \frac{1}{r^2} (x\ddot{x} - \dot{x}^2) = -\frac{1}{r^2} (x^2 + \dot{x}^2 + (1 - \cos \dot{x})) < 0$. Thus an orbit that crosses the x -axis at some $x_1 < 0$, will cross it again at some $x_2 > 0$, and then again at some $x_3 < 0$. But the system is reversible, so it must be $x_1 = x_3$. **QED**



5 Reversible system #01 [\(show reversible and sketch phase portrait\)](#)

5.1 Statement: Reversible system #01

Show that the system $\ddot{x} + x\dot{x} + x = 0$ is reversible and sketch the phase portrait. You can use a computer to do the picture, but you must justify the plot; in particular: include an analysis of any fixed point that occurs.

5.2 Answer: Reversible system #01

Start by writing the system in the 2-D form

$$\dot{x} = v \quad \text{and} \quad \dot{v} = -x - xv.$$

It should be clear that this is **invariant under**

$$x \mapsto -x, \quad v \mapsto v, \quad t \mapsto -t.$$

Hence **the system is reversible (relative to the v -axis)**.

There is only one fixed point, at $(0, 0)$, and the linearization

$\dot{x} = v$, $\dot{v} = -x$ shows that it is a linear center. Because the system is reversible, it is also **a nonlinear center**.

There are three nullclines:

1. $x = 0$, where $\dot{v} = 0$. The orbits cross this nullcline left to right for $v > 0$, and right to left for $v < 0$.
2. $v = 0$, where $\dot{x} = 0$. The orbits cross this nullcline going down for $x > 0$, and going up for $x < 0$.
3. $v = -1$, where $\dot{v} = 0$. *This nullcline is also a trajectory; along it $\dot{x} = -1$.*

Looking at the direction field, it is easy to see that the trajectories swirl around the critical point above $v = -1$. Further, because of the reversibility, they actually close (instead of spiral in or out). Similarly, below $v = -1$, the trajectories diverge to infinity. From this the phase portrait is shown in Fig. 5.1 follows.

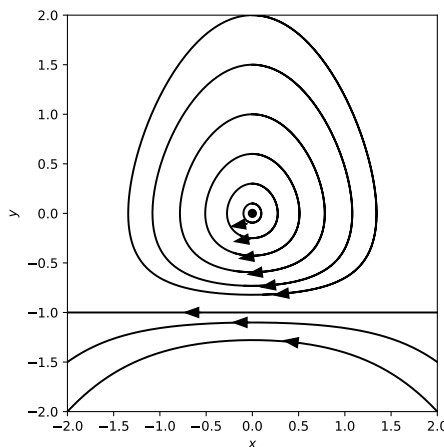


Figure 5.1: Phase portrait for the “reversible system #01”. Because of reversibility, the phase portrait is symmetric about the v -axis

6 A system both gradient and Hamiltonian

6.1 Statement: A system both gradient and Hamiltonian

Consider the system $\dot{x} = \cos x \cosh y = f$ and $\dot{y} = \sin x \sinh y = g$.

- a. Then $f_y = g_x$, hence this is a gradient system.² **Find the potential $V = V(x, y)$.**
- b. **Show that the system is also Hamiltonian, and find the Hamiltonian $H = H(x, y)$.**

² Why? Can you show this?

6.2 Answer: A system both gradient and Hamiltonian

- We have $f_y = \cos x \sinh y = g_x$. We can write $\mathbf{f} = -\mathbf{V}_x$ and $\mathbf{g} = -\mathbf{V}_y$, where $\mathbf{V} = -\sin x \cosh y$. A necessary condition for a gradient system is $\mathbf{f}_y = -(\mathbf{V}_x)_y = -(\mathbf{V}_y)_x = \mathbf{g}_x$. It is sufficient on simply connected sets.
- Since $f_x = -\sin x \cosh y = -g_y$, the system is Hamiltonian (*why?*), and we can write $\mathbf{f} = -\mathbf{H}_y$ and $\mathbf{g} = \mathbf{H}_x$, where $\mathbf{H} = -\cos x \sinh y$.
- Finally, note that we can write the system in the complex form $\dot{z} = \cos \bar{z}$, where $z = x + iy$ and $\bar{z} = x - iy$. This follows from the trigonometric identity $\cos(x - iy) = \cos x \cosh y + i \sin x \sinh y$.

Further “fun” facts.

- A Hamiltonian system has centers and saddles only, while a gradient system can have sinks and sources. *How can a system be both gradient and Hamiltonian?* Answer: by having saddles only; which are also allowed by gradient systems.

In this case, at a critical point: $\cos x = 0$ (since \cosh never vanishes), hence $\sin x = \pm 1$. Thus $\mathbf{y} = \mathbf{0}$, hence $\cosh y = 1$ and $\sinh y = 0$. You can then check that the critical point is a saddle, with eigenvalues $\lambda = \pm 1$.

- Let $\dot{\mathbf{x}} = -\mathbf{H}_y = -\mathbf{V}_x$ and $\dot{\mathbf{y}} = \mathbf{H}_x = -\mathbf{V}_y$ be a system that is both Hamiltonian and Lagrangian. Then you can check that both \mathbf{H} and \mathbf{V} satisfy the Laplace equation $\Delta \mathbf{H} = \Delta \mathbf{V} = 0$.

Viceversa, **suppose that H satisfies the Laplace equation.** Then you can construct a \mathbf{V} such that $\mathbf{V}_x = \mathbf{H}_y$ and $\mathbf{V}_y = -\mathbf{H}_x$. That is: $\dot{\mathbf{x}} = -\mathbf{H}_y$ and $\dot{\mathbf{y}} = \mathbf{H}_x$ is both gradient and Hamiltonian, with **saddles only**.

Proof. For any $\vec{r} = (x, y)$, consider a curve Γ from the origin $\vec{0}$ to \vec{r} , and define $V(\vec{r}) = \int_{\Gamma} (\mathbf{H}_y dx - \mathbf{H}_x dy)$. From Green's theorem the answer does not depend on Γ , so V is well defined. Obviously $\mathbf{V}_x = \mathbf{H}_y$ and $\mathbf{V}_y = -\mathbf{H}_x$ is satisfied.

- Harmonic functions min-max theorem:* if $\Delta \mathbf{H} = 0$, then \mathbf{H} has no isolated local maximums (nor minimums).

Proof. Such maximums or minimums would be centers for the Hamiltonian system $\dot{\mathbf{x}} = -\mathbf{H}_y$ and $\dot{\mathbf{y}} = \mathbf{H}_x$. Now use f1–f2.

Note: this is a “weak” form of the min-max theorem. The full theorem applies in any dimension, and does not require “isolated”. It also applies on “generic” regions, where it says that the min-max of an harmonic function has to be on the boundary. However, it is interesting that a Dynamical Systems proof of the theorem is possible, even if limited in scope.

7 Two closed orbits enclosed by a third

7.1 Statement: Two closed orbits enclosed by a third

Consider a phase plane system, $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{y})$ and $\dot{\mathbf{y}} = \mathbf{g}(\mathbf{x}, \mathbf{y})$, where \mathbf{f} and \mathbf{g} are nice and smooth. Imagine that the system has two disjoint closed orbits (say: Γ_1 and Γ_2), and a third one (say: Γ_3) that encloses both. **† What is the minimal number of critical points that the system can have, and what are their indexes?**

† Example: $\Gamma_1 =$ radius 1 circle centered at $(2, 0)$, $\Gamma_2 =$ radius 1 circle centered at $(-2, 0)$, $\Gamma_3 =$ radius 4 circle centered at $(0, 0)$.

7.2 Two closed orbits enclosed by a third

Since Γ_1 and Γ_2 each has index 1, each must enclose at least one critical point with index 1. Then, because Γ_3 also has index 1, a third critical point (outside Γ_1 and Γ_2 , but inside Γ_3) is needed, with index -1. Thus the answer is: **minimum needed is three critical points, with indexes $I_1 = 1$, $I_2 = 1$, and $I_3 = -1$.** The equation $\ddot{\mathbf{x}} + \mathbf{V}'(\mathbf{x}) = \mathbf{0}$, with $\mathbf{V} = -\frac{1}{2} \mathbf{x}^2 + \frac{1}{4} \mathbf{x}^4$, provides an example — with many possible choices for the Γ_j .

8 Find a conserved quantity #01 (and sketch phase portrait)

8.1 Statement: Find a conserved quantity #01 (and sketch phase portrait)

Find a conserved quantity for the system $\ddot{x} = a - e^x$, and sketch phase portraits characteristic of the cases $a < 0$, $a = 0$, $a > 0$. Include an analysis of any fixed point that occurs.

8.2 Answer: Find a conserved quantity #01 (and sketch phase portrait)

Write the equation in the form $\ddot{x} = -dV/dx$ with $V(x) = -ax + e^x$.

Then, multiply by \dot{x} and note that the result is

Thus, **a conserved quantity** is

$$E(x, \dot{x}) = \frac{1}{2} \dot{x}^2 + V(x).$$

$$\frac{d}{dt} \left(\frac{1}{2} \dot{x}^2 + V(x) \right) = 0.$$

In 2-D form, the system is $\dot{x} = v \quad \dot{v} = a - e^x$,

with **a fixed point at $v = 0$ and $x = \ln a$** — only for $a > 0$;

there is **no fixed point if $a \leq 0$** .

For $a > 0$, expand the conserved quantity around the fixed point by setting $x = \log a + u$ (u small). This yields

$$2E(u, v) \approx a(1 - \log a) + v^2 + \frac{a}{2} u^2,$$

which shows that **the conserved quantity has a local**

minimum at the fixed point. Therefore **the fixed point is a nonlinear center**.

For $a \leq 0$, $\dot{v} < 0$ everywhere. Hence eventually $\dot{x} < 0$ along any trajectory. It follows that all the trajectories diverge to $(-\infty, -\infty)$ as $t \rightarrow \infty$. Further: for $|v|$ large (at a fixed x) the trajectories become parallel to the x -axis.

The phase portraits are shown in Fig. 8.1. Note that **this system is reversible** (it is invariant under the transformation $x \mapsto x, v \mapsto -v, t \mapsto -t$); hence the phase portraits are symmetric relative to the x -axis.

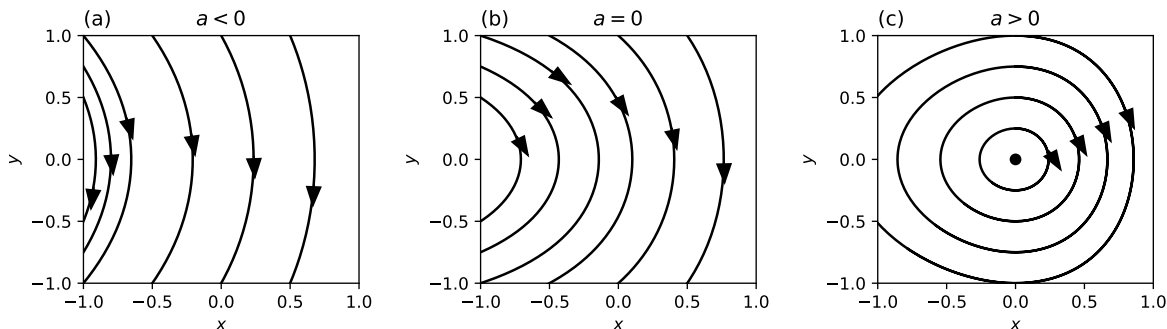


Figure 8.1: Phase portraits for “Find a conserved quantity #01 (and sketch phase portrait)”.

THE END.