Answers to P-Set # 03, (18.353/12.006/2.050)j
MIT (Fall 2023)

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1.1 Statement: Speed of a non-linear diffusion front

Consider some substance diffusing in an **isotropic** [1] **medium at rest.**
Then the following equation applies
$$\mathcal{C}_t = \operatorname{div}(\nu\,\nabla\mathcal{C}), \tag{1.1}$$
where

1. Fick's law of diffusion is assumed: the substance flux due to diffusion is along the gradient of the concentration, from higher to lower concentrations. For this it is important that the medium be isotropic — else the flux diffusion may occur along directions that depend both on the gradient of the concentration, as well as special directions in the media.
2. $\mathcal{C} = \mathcal{C}(\vec{x},\, t)$ is the substance concentration (mass per volume) — e.g.: grams per liter.
3. $\nu > 0$ is the *diffusion coefficient.*
4. $\nabla\mathcal{C}$ is the gradient of the concentration, and div denotes the divergence of a vector field.

When ν is a constant (1.1) reduces to the *linear diffusion equation*
$$\mathcal{C}_t = \nu\,\Delta\,\mathcal{C}, \tag{1.2}$$
where Δ is the *Laplace operator*, $\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2$.

However, there are situations where the diffusion coefficient is **not** constant, and depends (for example) on the concentration itself: $\nu = \nu(\mathcal{C})$. Here we will consider the particular case where $\nu \propto \mathcal{C}^2$ and the medium is homogeneous. [2]

[1] **Isotropic** means that the properties of the medium are invariant under rotation.
[2] **Homogeneous** means that the properties of the medium are the same everywhere.

Then (1.1) reduces to

$$\mathcal{C}_t = \mu \,\mathrm{div}\left(\mathcal{C}^2\, \nabla \mathcal{C}\right), \tag{1.3}$$

where $\boldsymbol{\mu > 0}$ **is a constant.**

Under conditions where (1.3) applies, imagine that at $t = 0$ there is a very tiny blob of substance somewhere in the media. Then, due to the diffusion, the size of the blob will increase with time — with a sharp edge between the region where $\mathcal{C} > 0$, and the region where there is no substance. *Note: In this the behavior of (1.3) differs from that of (1.2). In the case of the linear diffusion equation, the blob's edge ceases to be sharp for $t > 0$, even if the initial blob has a sharp edge.*

Problem Tasks. What you are expected to do.

Let $\boldsymbol{M}$ **be the total mass in the blob,** and make the approximation that, **at time $\boldsymbol{t = 0}$ the blob is just a point** — i.e.: all the substance's mass is concentrated in a blob so tiny that you can think of it as a point. Then perform the tasks below, ***using qualitative (but precise) physical arguments only*** — *do not solve the equation.*

q1. What dimensions does μ have?

q2. Argue that the shape of the blob is a sphere for $t > 0$.

q3. Find a formula for the radius of the blob $R = R(t)$ as a function of time — namely: a formula of the form $R = \alpha\, f(t)$, where α is a dimension-less constant (a number), and $f(t)$ is some function of time. You will not able to calculate α without solving the p.d.e. (1.3), which you are not expected to do. But you should be able to fully determine the function $f(t)$.

Assume now that $\boldsymbol{R(t) = 5}$ **mm** when $\boldsymbol{t = 3.75}$ **sec.**

q4. What value does R take for $t = 960\,\mathrm{sec} = 16\,\mathrm{min}$?

q5. For what value of t is $R = 5\,\mathrm{cm}$?

q6. Would your answers change if the nonlinear diffusion occurred in the plane, instead of in 3-D? In particular, what are the answers to **q1** and **q3** in 2-D?

Hint. The formula giving the radius as a function of time must involve physical constants that allow it to transform time into length. These physical constants must result from the physical constants in the problem, and only them — e.g.: if you write a formula that involves the speed of light in it, something went wrong with your reasoning!

1.2 Answer: Speed of a non-linear diffusion front

q1. All the terms in (1.3) must have the same units. Hence: **dimensions($\boldsymbol{\mu}$)** $= \dfrac{(\text{length})^{\mathbf{8}}}{(\text{mass})^{\mathbf{2}}(\text{time})}$.

q2. The situation is invariant under rotations. Hence the only shape that the blob is allowed to have is a sphere. *Note that, in practice, the initial blob is never a point. However, once the blob dimensions become much larger than its initial size, any irregularities produced by an initially asymmetric blob become negligible.*[3]

q3. We must find a relationship between R and t. There is no length scale provided by the initial conditions, and the only dimensional parameters in the problem are μ and M. Hence it must be that

$$R = \alpha\, \left(M^2\, \mu\, t\right)^{1/8}, \tag{1.4}$$

where α is a numerical constant (no dimensions).

q4. Let $t_1 = 3.75\,\mathrm{sec}$ and $t_2 = 960\,\mathrm{sec}$. Then $t_2 = t_1 \times 2^8$. Thus $R(t_2) = 2\,R(t_1) = 10\,\mathrm{mm}$.

q5. For the radius to increase by a factor of 10, time must increase by a factor of 10^8. Thus $R(t) = 5\,\mathrm{cm}$ when $t = 3.75 \times 10^8\,\mathrm{sec} = 6.25 \times 10^6\,\mathrm{min} \approx 1.042 \times 10^5\,\mathrm{hours} \approx 4.340 \times 10^3\,\mathrm{days} \approx 11.9\,\mathrm{years}$.

[3] You should be aware that **situations where asymmetries are amplified are possible, and (in fact) occur,** and give rise to very interesting physical questions (google "spontaneous symmetry breaking" and see what you get). But this is not the case here, though proving this is non-trivial, and beyond the scope of this class.

q6. These answers depend on the dimension where diffusion happens. Thus, in 2-D the answer to **q1** changes to **dimensions**$(\boldsymbol{\mu}) = \dfrac{(\text{length})^{\mathbf{6}}}{(\text{mass})^{\mathbf{2}}(\text{time})}$, while the answer to **q3** changes to
$$R = \alpha\,\left(M^2\,\mu\,t\right)^{1/6}. \qquad (1.5)$$
The answers to **q4** and **q5** are correspondingly affected.

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2 Overdamped pendulum with a torsional spring

2.1 Statement: Overdamped pendulum with a torsional spring

Suppose that the overdamped pendulum is connected to a torsional spring. As the pendulum rotates, the spring winds up and generates an opposing torque $-k\,\theta$, $k > 0$. The equation of motion is then $\boldsymbol{b\dot{\theta} + mgL\sin\theta = \Gamma - k\theta}$, where b is the damping coefficient.

a. Does this equation give a well-defined vector field on the circle?

b. Nondimensionalize the equation.

c. What does the pendulum do in the long run?

d. Show that many bifurcations occur as k is varied from 0 to ∞. What kind of bifurcations are they?

2.2 Answer: Overdamped pendulum with a torsional spring

a. Here $\boldsymbol{\dot{\theta} = f(\theta) = \frac{1}{b}(\Gamma - k\theta - mgL\sin\theta)}$, where $f(\theta)$ is not a periodic function for $k \neq 0$. Therefore this equation **does not** give a well-defined vector field on the circle.

b. We scale time so that the drag and gravity terms balance. Specifically, we define the dimensionless time $\tau = t/T$ where $T = b/(mgL)$. Then $\theta(\tau)$ evolves according to
$$\frac{\mathrm{d}\theta}{\mathrm{d}\tau} + \sin\theta = \gamma - \kappa\theta, \quad \text{where} \quad \gamma = \frac{\Gamma}{mgL}, \quad \kappa = \frac{k}{mgL}.$$

c. The pendulum approaches a stable fixed point for $\kappa \neq 0$, for all γ. If $|\gamma| > 1$ and $|\kappa|$ is small, then many loops of the pendulum may occur before the fixed point is reached.

d. See Figure 2.1

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3 Exponential to algebraic decay transition

3.1 Statement: Exponential to algebraic decay transition

Consider the equation[4] $\boldsymbol{\dot{x} = -r\,x - x^3}$, with initial data $\boldsymbol{x(0) = x_0}$. *In the limit* $\boldsymbol{r \downarrow 0}$*, the solution to this equation transitions from exponential decay as* $\boldsymbol{t \to \infty}$ *(when* $\boldsymbol{r > 0}$*) to algebraic decay when* $\boldsymbol{r = 0}$*.*

Why? Because when x becomes very small, the behavior of the solution is controlled by the term $-r\,x$ if $r > 0$. On the other hand, if $r = 0$, the exact solution $x = x_0/\sqrt{1 + 2\,x_0^2\,t}$ decays like $1/\sqrt{2\,t}$. However, how do we know that the solutions decay when $r > 0$? Because then $\dot{x}/x = -(r + x^2) < r$, so that the solution size is less than $|x_0|\,e^{-r\,t}$.

[4] Note that this equation is one of the two possible normal forms for pitchfork bifurcations.

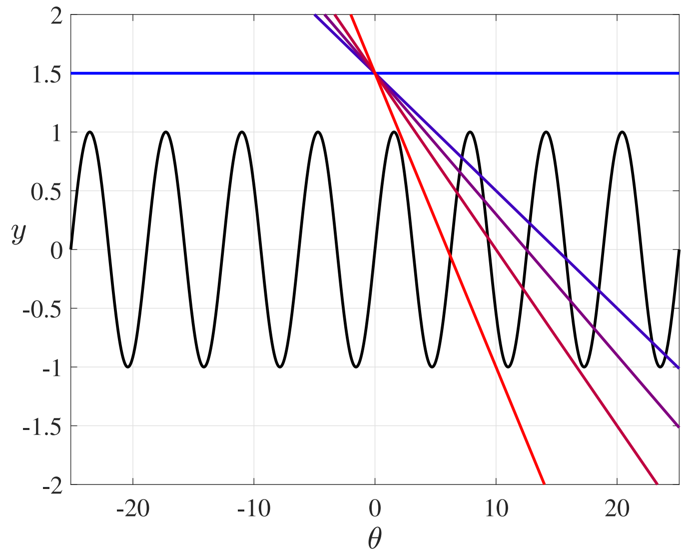

Figure 2.1: Example of the change in the number of solutions as κ is varied for fixed $\gamma = 1.5$. Going from blue to red, κ takes values 0, 0.10, 0.12, 0.15, and 0.25. For $\kappa = 0$, there are no fixed points. For infinitesimal $\kappa > 0$, there are infinitely solutions for $\theta > 0$. As κ is increased, the number of fixed points decreases in pairs, each via a saddle-node bifurcation. As $\kappa \to \infty$, the only fixed point is at $\theta = 0$. We note that the system is symmetric under the mapping $\theta \to -\theta$ and $\kappa \to -\kappa$, so the picture is reflected about $\theta = 0$ for negative κ.

Assume that $\mathbf{0 < r \ll 1}$. Then the solution should decay exponentially fast as $t \to \infty$, yet (because r is small) the solution should also be "close" to the solution to the problem when the term $-r\,x$ is neglected. That is: (3.1).

$$x_* = \frac{x_0}{\sqrt{1 + 2\,x_0^2\,t}} \tag{3.1}$$

How can these two things happen simultaneously?

- **(a)** Use separation of variables (or any other method) to solve $\dot{x} = -r\,x - x^3$, $x(0) = x_0$, $r > 0$, analytically. *Hint: what equation does $y = 1/x^2$ satisfy? Once you know y, finding the sign of x is easy.*
- **(b)** For the solution in item **a**, show that $x \sim C\,e^{-r\,t}$ for $t \to \infty$, where C is a constant that you should compute.
- **(c)** For the solution in item **a**, show that $x \sim x_*$ for $0 \le r\,t \ll 1$, where x_* is as in (3.1). Notice that, as $r \downarrow 0$, the time interval over which this is valid gets larger and larger.
- **(d)** To get some intuition on what is going on, plot the exact solution you obtained in item **a** versus x_* in (3.1) [plot #1]. In addition, plot the exact solution you obtained in item **a** versus the approximation in item **b** [plot #2]. Use the same parameter values for both plots, but different time ranges. *Suggestion: Use* $x_0 = 1$ *and* $r = 0.01$. *Then, for plot #1 use the range* $0 < t < 50$, *and for plot #2 the range* $0 < t < 100$.

3.2 Answer: Exponential to algebraic decay transition

- **(A)** It is easy to see that $y = 1/x^2$ solves $\dot{y} = 2\,r\,y + 2$. From this, or from separation of variables, we obtain

$$x = x_e(t) = \frac{x_0\,\sqrt{r}\,e^{-r\,t}}{\sqrt{r + x_0^2\,(1 - e^{-2\,r\,t})}}. \tag{3.2}$$

- **(B)** From (3.2) it follows that, as $t \to \infty$, $x_e \sim C\,e^{-r\,t}$, where

$$C = \frac{x_0\,\sqrt{r}}{\sqrt{r + x_0^2}}. \tag{3.3}$$

- **(C)** Let $0 \le rt \ll 1$, with $0 < r \ll 1$. Substitute $e^{-rt} \sim 1$ into the numerator of (3.2), and $e^{-2rt} \sim 1 - 2rt$ into the denominator. What follows is x_* in (3.1). Hence $x_s \sim x_*$ for $0 \le t \ll 1/r$.
- **(D)** See figure 3.1.

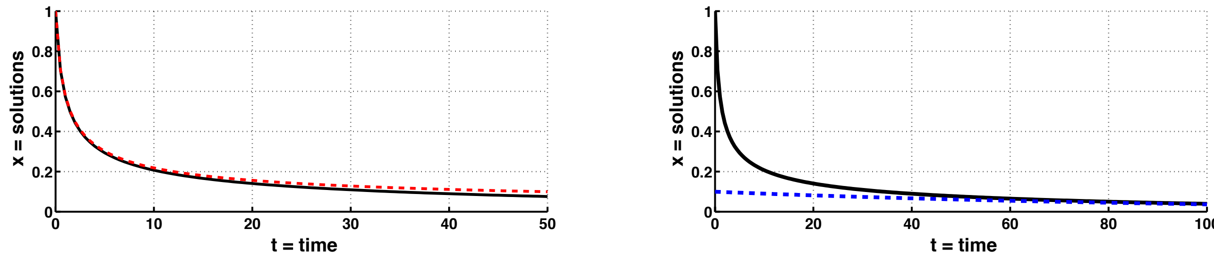

Figure 3.1: (Algebraic versus exponential decay). Plot of the solution to $\dot{x} = -0.01\,x - x^3$, $x(0) = 1$, (black, solid line), versus: (i) Left panel: x_* in (3.1) (red, dashed line); and (ii) Right panel: $C\,e^{-r\,t}$, (blue dashed line), where C is as in (3.3). Note that here $r = 0.01$ and $x_0 = 1$. The left panel illustrates the match between the exact solution and x_* for $0 < t \ll 1/r$, while the right panel illustrates the onset of exponential decay for $t \gg 1/r$.

4 Attracting and Liapunov stable

4.1 Statement: Attracting and Liapunov stable

Recall the *definitions for the various types of stability* that concern critical points:
Let $\mathbf{x}^*$ be a fixed point of the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$. Then:

1. $\mathbf{x}^*$ is **attracting** if there is a $\delta > 0$ such that $\lim_{t\to\infty} = \mathbf{x}^*$ whenever $\|\mathbf{x}(0) - \mathbf{x}^*\| < \delta$. That is: any trajectory that starts within δ of $\mathbf{x}^*$ *eventually* converges to $\mathbf{x}^*$. Note that trajectories that start nearby $\mathbf{x}^*$ *need not stay close in the short run*, but *must* *approach* $\mathbf{x}^*$ *in the long run.*
2. $\mathbf{x}^*$ is **Liapunov stable** if for each $\epsilon > 0$, there is a $\delta > 0$ such that $\|\mathbf{x}(t) - \mathbf{x}^*\| < \epsilon$ for $t > 0$, whenever $\|\mathbf{x}(0) - \mathbf{x}^*\| < \delta$. Thus, trajectories that start within δ of $\mathbf{x}^*$ stay within ϵ of $\mathbf{x}^*$ for all $t > 0$.
   In contrast with attracting, Liapunov stability requires nearby trajectories to remain close *for all* $t > 0$.
3. $\mathbf{x}^*$ **is asymptotically stable** if it is *both* attracting and Liapunov stable.
4. $\mathbf{x}^*$ is **repeller** if there exist $\epsilon > 0$ and $\delta > 0$ such that: if $0 < \|\mathbf{x}(0) - \mathbf{x}^*\| < \delta$, then (after some critical time) it will be $\|\mathbf{x}(t) - \mathbf{x}^*\| > \epsilon$ (i.e., for $t > t_c$). Repellers are a special kind of *unstable* critical points.

For each of the following systems, decide whether the origin is attracting but not Liapunov stable, Liapunov stable, asymptotically stable, repeller, or unstable but not a repeller.

a) $\dot{x} = 2\,y$ and $\dot{y} = -3\,x$.

b) $\dot{x} = y\,\cos(x^2 + y^2)$ and $\dot{y} = -x\,\cos(x^2 + y^2)$.

c) $\dot{x} = -x$ and $\dot{y} = -|y|\,y$.

d) $\dot{x} = 2\,x\,y$ and $\dot{y} = y^2 - x^2$. *Hint: what happens along* $x = 0$*?*

e) $\dot{x} = x - 2\,y\,x^2 - 4\,y^3$ and $\dot{y} = y + x^3 + 2\,x\,y^2$.

f) $\dot{x} = y$ and $\dot{y} = x$.

g) Finally, consider the critical point $(x,\,y) = (1,\,0)$, for the system

$$\dot{x} = (1 - r^2)\,x - (1 - \tfrac{x}{r})\,y \quad \text{and} \quad \dot{y} = (1 - r^2)\,y + (1 - \tfrac{x}{r})\,x, \tag{4.1}$$

defined in the "punctured" plane $r = \sqrt{x^2 + y^2} > 0$. *Hint: write the equations in polar coordinates.*

Additional hints. In some cases you can get the answer by finding a function $\mathcal{J} = \mathcal{J}(x,\,y)$ *with a local minimum at the origin such that* $\frac{d\mathcal{J}}{dt} > 0$ *along trajectories — or maybe one such* $\frac{d\mathcal{J}}{dt} < 0$*, or maybe one such* $\frac{d\mathcal{J}}{dt} = 0$*. In other cases look for special trajectories that either leave, or approach, the origin.*

4.2 Answer: Attracting and Liapunov stable

These are the answers, for each system:

a) It is easy to check that $\mathcal{J} = 3\,x^2 + 2\,y^2$ is constant along the trajectories.
The origin is a center, thus **Liapunov stable.**

b) It is easy to check that $\mathcal{J} = x^2 + y^2$ is constant along the trajectories.
The origin is a center, thus **Liapunov stable.** Unlike the system in (a), this is a nonlinear center.

c) In terms of the initial values, the solution for $t > 0$ is $x = x_0\,e^{-t}$ and $y = y_0/(1 + |y_0|\,t)$.
The origin is **asymptotically stable** — it is, in fact, a **nonlinear, stable, node.**

d) The first equation shows that, if x vanishes anywhere, then it vanishes everywhere — i.e., $x = 0$ is an invariant curve. The other equation then reduces to $\dot{y} = y^2$. Thus $x = 0$ and $y > 0$ is an orbit leaving the origin, and $x = 0$ and $y < 0$ is an orbit approaching the origin.
The origin is **unstable, but not a repeller.**
Note: for this system it can be shown that all the trajectories that have $x \neq 0$ somewhere (thus everywhere, why?) approach the origin as $t \to \infty$.

e) It is easy to check that $\mathcal{J} = x^2 + 2\,y^2$ satisfies $\frac{d\mathcal{J}}{dt} = 2\,\mathcal{J}$ along the trajectories.
The origin is a **repeller.**
Note: in fact, the origin is a nonlinear, unstable, spiral. This follows from the fact that the equations yield, for the polar angle θ, $\dot{\theta} = \mathcal{J}/r^2$ — where $r^2 = x^2 + y^2$.

f) It is easy to check that $\mathcal{J} = x^2 - y^2$ is constant along the trajectories.
The origin is a saddle (in fact, a linear saddle), thus **unstable, but not a repeller.**

g) In polar coordinates equation (4.1) takes the form

$$\dot{r} = (1 - r^2)\,r \quad \text{and} \quad \dot{\theta} = 1 - \cos\theta. \tag{4.2}$$

From this it should be clear that, for any initial data $0 < r_0$ and $0 < \theta < 2\,\pi$, the solution approaches the critical point as $t \to \infty$. Furthermore, the real positive axis (i.e., $\theta = 0$) is an invariant curve, and along it the system reduces to $\dot{x} = (1 - x^2)\,x$ — thus, again, the trajectories approach the critical point as $t \to \infty$. It follows that **the critical point is attracting.**

On the other hand, consider the trajectory $r \equiv 1$ and $0 < \theta < 2\,\pi$. This trajectory *starts* at the critical point at $t = -\infty$, goes around the unit circle counterclockwise, and arrives back at the critical point at $t = \infty$. In fact, any solution starting near the critical point with $y > 0$, initially moves away from the critical point (as far as a distance ≈ 2), before returning to the critical point as $t \to \infty$. Thus **the critical point is not Liapunov stable.**

THE END.