Answers to P-Set # 02, (18.353/12.006/2.050)j
MIT (Fall 2023)

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1 Blow up in finite time #01

1.1 Statement: Blow up in finite time #01

Consider the evolution of $x(t)$ according to $\dot{x} = rx + x^3$, where $r > 0$ is fixed.

a. Show that the origin is an unstable fixed point.

b. For $x(0) = x_0 \neq 0$, show that $|x(t)| \to \infty$ as $t \to t_0$, where

$$t_0 = \frac{1}{2r} \log\left(1 + \frac{r}{x_0^2}\right).$$

1.2 Answer: Blow up in finite time #01

a. Define $f(x) = r\,x + x^3$ and note that $f'(x) = r + 3\,x^2$. Hence $f'(0) = r > 0 \Rightarrow$ the origin is unstable according to linear stability analysis.

b. By partial fractions, we obtain

$$\left(\frac{1}{x} - \frac{x}{x^2 + r}\right) \frac{\mathrm{d}x}{\mathrm{d}t} = r.$$

Integrating with respect to t and applying the initial condition yields the following implicit solution for $x(t)$:

$$\log\left(\frac{|x|}{\sqrt{x^2 + r}}\right) = r\,t + \log\left(\frac{|x_0|}{\sqrt{x_0^2 + r}}\right).$$

Exponentiating and then squaring both sides sides gives (after some rearrangement)

$$[x(t)]^2 = \frac{r\,x_0^2\,e^{2\,r\,t}}{x_0^2\,(1 - e^{2\,r\,t}) + r}.$$

Blow-up occurs at the time $t = t_0$ at which the denominator vanishes, namely

$$x_0^2\,(e^{2\,r\,t_0} - 1) = r \quad \Longrightarrow \quad t_0 = \frac{1}{2\,r} \log\left(1 + \frac{r}{x_0^2}\right).$$

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2 Leaky Bucket by dimensional analysis

2.1 Statement: Leaky Bucket by dimensional analysis

Consider a cylindrical bucket of cross-sectional area A, with a small hole at the bottom (of cross-sectional area $a \ll A$), filled with water up to a depth $h = h(t)$ (h is a function of time because the bucket is leaking water). The **objective** is to write an equation for h, $\dot{\boldsymbol{h}} = \boldsymbol{f(h)}$, under the following assumptions:

1. The leaking is very slow, so that we can neglect any fluid motion within the bucket.
2. The small hole at the bottom is large enough that it is safe to neglect surface tension — thus the leaking occurs though a small continuous stream (jet) out of the hole, rather than a "drip-drip" involving drops.
3. We now argue that *the speed at which the bucket empties* (i.e.: $\dot{\boldsymbol{h}}$) *depends on A and a through their quotient a/A only, and it is proportional to it.* That is
$$\dot{\boldsymbol{h}} = (\boldsymbol{a}/\boldsymbol{A})\,\boldsymbol{F}(\boldsymbol{h}), \qquad (2.1)$$
where F depends on neither a nor A.
Argument: for a given water flux through the small hole, it is clear that $\dot{\boldsymbol{h}}$ is proportional to $1/A$. On the other hand, the flux through the small hole is proportional to the pressure difference across the hole, times a — where the pressure difference depends on h, g, etc., but **not** A. A somewhat simpler argument is: n equal buckets will empty at the same rate; but this is the same as multiplying both A and a by n (this works for integers only, though).

Given the assumptions above, $\boldsymbol{F}$ in (2.1) should **depend only** on the following dimensional quantities: $\boldsymbol{g}$ (the acceleration of gravity), $\boldsymbol{\rho}$ (water density), and (of course) $\boldsymbol{h}$.
Task: use dimensional analysis to *determine F up to a multiplicative constant.*[1] A bit of extra thinking will also tell you *what is the sign of F.*

2.2 Answer: Leaky Bucket by dimensional analysis

The density $\boldsymbol{\rho}$ involves mass, while neither of $\dot{h}$, h, or g, does. Hence $\boldsymbol{F}$ **does not depend on ρ; it can only depend on g and h.** It then follows that

$$\dot{\boldsymbol{h}} = -\gamma\,(\boldsymbol{a}/\boldsymbol{A})\,\sqrt{\boldsymbol{g}\,\boldsymbol{h}}, \qquad (2.2)$$

where $\boldsymbol{\gamma > 0}$ is some **numerical constant** (turns out $\gamma = \sqrt{2}$).
Note #1. The sign in (2.2) follows because h has to decrease.
Note #2. Doing a derivation of the equations using a bit of hydrodynamics, it can be shown that $\boldsymbol{\gamma = \sqrt{2}}$.
Note #3. It is a bit counter-intuitive that the equation does not involve ρ. For a heavier (lighter) fluid the pressure at the bottom should be larger (smaller). Should this not produce a larger (smaller) outflow? Actually, no, because the mass per unit time of the outgoing fluid will also be larger (smaller), requiring more (less) energy to be pushed out — and the two effects balance each other. It is, basically, the same reason heavy objects do not fall faster than light ones (in vacuum).

3 Get equation from phase line portrait problem #07

3.1 Statement: Get equation from phase line portrait problem #07

Consider the ode on the line

$$\frac{dx}{dt} = f(x), \qquad (3.1)$$

where f is some function which has a continuous derivative. Assume that (3.1) has at least two critical points, and let $x_1 < x_2$ be two consecutive critical points (there is no other critical point between them). Assume now that both x_1 and x_2 are stable. **Is this possible? Does a function $f = f(x)$ yielding this exist?**
If the answer is no, prove it.
If the answer is yes, prove it by giving an example.

3.2 Answer: Get equation from phase line portrait problem #07

The answer is no. Because f is continuous, it must either be: $f(x) > 0$ for $x_1 < x < x_2$ (in which case x_1 cannot be stable), or $f(x) < 0$ for $x_1 < x < x_2$ (in which case x_2 cannot be stable).

4 Get equation from phase line portrait problem #08

4.1 Statement: Get equation from phase line portrait problem #08

Consider the ode on the line

$$\frac{dx}{dt} = f(x), \qquad (4.1)$$

where f is some function which has a continuous derivative. Assume that (4.1) has infinitely many critical points $-\infty < \dots x_n < x_{n+1} < \dots \infty$. Assume that all the critical points are semi-stable.[2] **Is this possible? Does a function $f = f(x)$ yielding this exist?**

[1] Dimensional analysis alone cannot distinguish F from $2F$, or from πF, or from ...
[2] A critical point is semi-stable if the solutions diverge from the critical point on one side, and converge on the other.

If the answer is no, prove it.
If the answer is yes, prove it by giving an example.

4.2 Answer: Get equation from phase line portrait problem #08

The answer is yes. Example
$$\boldsymbol{f(x) = (\sin(\pi\, x))^2}. \tag{4.2}$$
In this case $x_n = n$, where n is an arbitrary integer.

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5 Implicit function problem #02

5.1 Statement: Implicit function problem #02

Consider the following equation
$$\boldsymbol{f(x,\,\lambda) = x + \lambda\,\cos(x) + \lambda^3 = 0}, \tag{5.1}$$
with the particular solution $\boldsymbol{(x,\,\lambda) = (0,\,0)}$. Since $f_x(0,\,0) = 1$, the implicit function theorem guarantees that: *there is a neighborhood of $\lambda = 0$ where (5.1) has a unique solution, $x = X(\lambda)$, such that $X(0) = 0$.* Furthermore, since f is an analytic function, X is an analytic function of λ. This means that X has a Taylor series
$$\boldsymbol{X = \sum_{n=0}^{\infty} x_n\,\lambda^n}, \tag{5.2}$$
which converges for $|\lambda|$ small enough. **Find $\boldsymbol{x_1}$, $\boldsymbol{x_3}$, $\boldsymbol{x_5}$, and $\boldsymbol{x_n}$ for all even $\boldsymbol{n}$.**

Hint #1: Since X is small, write the cosine in (5.1) as a power series. Next substitute (5.2) into (5.1) and reorganize the result in terms of powers of λ. Finally make the coefficient of each power of λ vanish.[3] *This will give you equations that will allow you to solve for x_0 first, then x_1, and so on.*
Hint #2: There is a special property of the equation that tells you what x_n is for n even. ***Figure this out first,*** *it will save you a lot of pointless algebra. This problem involves much less calculation than you might think.*

5.2 Answer: Implicit function problem #02

First, since $X(0) = 0$, $x_0 = 0$. Note that (5.1) is invariant under the transformation $x \to -x$ and $\lambda \to -\lambda$, which implies that $x = -X(-\lambda)$ is also a solution. From uniqueness it follows that **$\boldsymbol{X}$ is an odd function,** so that
$$\boldsymbol{x_n = 0} \quad \textbf{for all } \boldsymbol{n} \textbf{ even.} \tag{5.3}$$
Next, substitute (5.2) into (5.1), and use the Taylor series for the cosine. Upon collecting equal powers of λ, this yields
$$(1 + x_1)\,\lambda + (x_3 - \frac{1}{2}\,x_1^2 + 1)\,\lambda^3 + (x_5 - x_1\,x_3 + \frac{1}{24}\,x_1^4)\,\lambda^5 + \cdots = 0. \tag{5.4}$$
Equating to zero the coefficients of the powers of λ, we obtain
$$\boldsymbol{x_1 = -1, \quad x_3 = -\frac{1}{2}, \quad x_5 = \frac{11}{24}, \quad \ldots} \tag{5.5}$$
Further coefficients can be obtained by calculating the higher order terms in (5.4). *The implicit function theorem guarantees that the resulting equations will uniquely determine the coefficients x_n.*

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[3] You only need to carry this calculation up to powers λ^n with $n \le 5$.

6 Inverse function problem #01

6.1 Statement: Inverse function problem #01

Consider the following equation
$$\boldsymbol{y = x + \sin(x) = f(x)}, \tag{6.1}$$
where $\boldsymbol{f(0) = 0}$ and $\boldsymbol{f'(0) = 2 \neq 0}$. The inverse function theorem guarantees that *there is a neighborhood of $x = 0$ where f has a unique inverse, $x = X(y)$, such that $X(0) = 0$.* Furthermore, since f is an analytic function, X is an analytic function. This

means that X has a Taylor series
$$\boldsymbol{X = \sum_{n=0}^{\infty} x_n\, y^n}, \tag{6.2}$$
which converges for $|y|$ small enough. **Find x_1, x_3, x_5, and x_n for all even n.**

6.2 Answer: Inverse function problem #01

Since $X(0) = 0$, $x_0 = 0$. Further, equation (6.1) is invariant under the transformation $x \to -x$ and $y \to -y$, thus $x = -X(-y)$ is also a solution. From uniqueness it follows that **X is an odd function,** so that
$$\boldsymbol{x_n = 0} \quad \textbf{for all } \boldsymbol{n} \textbf{ even.} \tag{6.3}$$
Next, substitute (6.2) into (6.1), and use the Taylor series for the sine. Upon collecting equal powers of y, this yields
$$(2\,x_1 - 1)\,y + (2\,x_3 - \frac{1}{6}\,x_1^3)\,y^3 + (2\,x_5 - \frac{1}{2}\,x_1^2\,x_3 + \frac{1}{120}\,x_1^5)\,y^5 + \cdots = 0. \tag{6.4}$$
Equating to zero the coefficients of the powers of y, we obtain
$$\boldsymbol{x_1 = \frac{1}{2}, \quad x_3 = \frac{1}{96}, \quad x_5 = \frac{1}{1920}, \quad \ldots} \tag{6.5}$$
Further coefficients can be obtained by calculating the higher order terms in (6.4). *The implicit function theorem guarantees that the resulting equations will uniquely determine the coefficients x_n.*

7 Phase line portrait problem #02

7.1 Statement: Phase line portrait problem #02

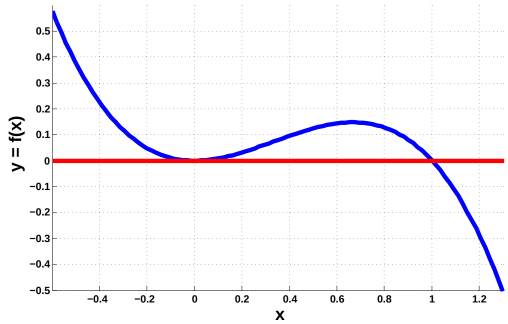
Consider the following ode on the line
$$\boldsymbol{\frac{dx}{dt} = f(x) = f(x) = x^2 - x^3}. \tag{7.1}$$
Draw its phase line portrait, indicating the critical points, and their stability properties.

In addition: describe *quantitatively* the behavior of the solutions near the critical points (i.e.: at what rate do they approach or leave them), as well as the behavior of the solutions when $|x|$ is large. **In particular:** *Are there solutions that cease to exist for some finite value of t, or are the solutions valid for all times?* If some solutions are defined for all times, and others are not: *state a condition that guarantees that a solution is defined for all times* — e.g.: if $x(t)$ is in some range/interval.

Important: *Do not attempt to do this problem by solving the equation analytically. First do a graphic analysis to obtain a qualitative picture of what happens. Then analize the behavior near critical point by solving the leading order approximation (first nonvanishing term in a Taylor expansion of $\boldsymbol{f}$). Finally, the behavior for $|\boldsymbol{x}|$ large can be similarly obtained by looking at the dominant behavior of $\boldsymbol{f}$.* By the way, note that here **all times means** $\boldsymbol{-\infty < t < \infty}$; so what happens in both limits $\boldsymbol{t \to \pm\infty}$ matter.

7.2 Answer: Phase line portrait problem #02

The zeros of f are at $x_1 = 0$ (a double zero) and $x_2 = 1$ (a simple zero), with f positive for $x < x_2$, and negative for $x > x_2$. See figure 7.1. It follows that x_1 is an semi-stable critical point (stable from the left and unstable from the right), while x_2 is a stable critical point. The phase line portrait is as in figure 7.1.



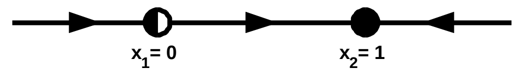


Figure 7.1: Problem 7, equation (7.1).
The zeros of f are at $x_1 = 0$ and $x_2 = 1$.
Left: plot of $\dot{x} = f(x)$ versus x. Red: x-axis; blue: $f(x)$.
Right: phase line portrait.

Because x_2 is a simple zero of f, the solutions approach it exponentially, with a rate constant [4] $r = -1$. On the other hand, for x small (near the double zero x_1) the solutions behave like the solutions to the ode $\dot{x} = x^2$ — that is:

$$\boldsymbol{x \sim \frac{1}{\tau - t}}, \tag{7.2}$$

where τ is some constant,[5] and t is far away from τ (so that x is small). For $t > \tau$, these are the solutions on the left (stable side) of x_1, which approach it algebraically as $t \to \infty$. For $t < \tau$, these are the solutions on the right (unstable side) of x_1, which approach it algebraically as $t \to -\infty$.

Since $f(x) \sim -x^3$ for $|x| \gg 1$, when $|x|$ becomes large on a solution, the solution acquires the behavior

$$\boldsymbol{x \sim \pm\frac{1}{\sqrt{t - \tau}}}, \tag{7.3}$$

where τ is some constant [6] and, of course, it must be $t > \tau$ (with t close to τ so $|x|$ is large). The negative sign in (7.3) corresponds to the solutions to the left of x_1. These solutions approach $x = -\infty$ as $t \downarrow \tau$, and $x = 0$ as $t \to \infty$ — **they do not exist for all times, only for $\tau < t < \infty$.** The positive sign in (7.3) corresponds to the solutions to the right of x_2. These solutions approach $x = \infty$ as $t \downarrow \tau$, and $x = 1$ as $t \to \infty$ — **they do not exist for all times, only for $\tau < t < \infty$.**
It follows that ***some solutions are NOT defined for all times.***

Note that ***the solutions in $0 < x < 1$ are defined for all times.*** They approach $x = 0$ (algebraically) as $t \to -\infty$, and $x = 1$ (exponentially) as $t \to \infty$. ***These are the only solutions defined for all times.***

8 Toy model for shell buckling

8.1 Statement: Toy model for shell buckling

Hold a ping-pong ball between your thumb and index fingers and squeeze it. If you do not apply enough force, the ball will deform slightly with a purely elastic response. But, if you push hard enough, the ball will buckle and you will make a (permanent) dent on it — and the ball will be ruined. This is the phenomena of (thin) shell buckling.

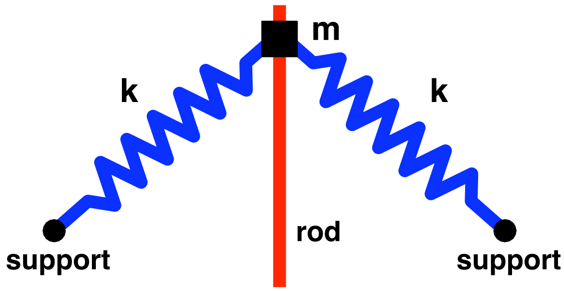
Shell buckling is a very rich phenomena,[7] way beyond the scope of this course. Here we will study an extremely simplified (1-D) version of this phenomena (the emphasis here being on "toy" model) where all the geometrical richness of the original setting is gone, and only the buckling bifurcation remains.

A sketch depicting the model is shown in figure 8.1. Further assumptions and notation are:

1. Idealize the bead as a point mass.

2. Let x be the vertical distance, along the rod, of the bead from the horizontal line joining the spring supports. Let $x > 0$ if the bead is above the supports and $x < 0$ if below.

[4] The rate constant is given by $f'(x_2)$. The solutions behave like $x = x_2 + \text{constant}\, e^{r\,t}$ near this critical point.
[5] Of course, τ is a solution dependent constant.
[6] Of course, τ is a solution dependent constant.
[7] Lots of interesting and important questions arise. For example: What is the shape of the dent that forms? The dent's edges have sharp corners: why these corners form, and how do they propagate as further pressure is applied?



A bead of mass m (black square) can slide along a rigid vertical rod (in red). The bead is connected by two equal springs (in blue), with spring constant k, to two supports placed symmetrically on each side of the rod. See the text for further details.

Figure 8.1: Toy model for shell buckling.

3. Let $\boldsymbol{h > 0}$ be the distance of the spring supports from the rod, and let $\boldsymbol{L > 0}$ be the springs equilibrium length. **Assume $\boldsymbol{L > h}$**, so that the springs are under compression for $x = 0$.

4. Hook's law applies to the springs. Thus they exert a force of magnitude $F = k\,(\ell - L)$, where ℓ is the spring length, along the spring axis, pushing if $\ell < L$, and pulling if $\ell > L$.

5. When the bead slides along the rod, the motion is opposed by a friction force of magnitude $b\,\dot{x}$, where $b > 0$ is a constant.

6. Because the rod is rigid, we need to consider only the vertical components of the various forces that act on the bead. These forces are: (i) Gravity, of magnitude $m\,g$, pointing down. (ii) The forces by the springs. (iii) Friction along the rod. *Note that here we assume that the force gravity is significant, so that there is no up-down symmetry in this problem.*

PROBLEM TASKS:

A. *Derive an ode for the bead position, and write it in appropriate a-dimensional variables.* [8]

B. *Assume that friction is large, so that inertia can be neglected. Exactly which a-dimensional number has to be small for friction to be "large"?*

C. *Analyze the bifurcations that occur for the equation resulting from item B, as the bead mass changes — in this toy model, increasing the bead mass plays the role of squeezing harder on the ping-pong ball. What type of bifurcation(s) occur?*

Hint: It is a bad idea to try to do this by attempting to solve for the critical points and bifurcation thresholds analytically. A qualitative, graphical, analysis is the best way to go.

D. *The picture in figure* 8.1 *corresponds, in this toy model, to the ping-pong ball in a more-or-less spherical shape. What is the "buckled" state?*

E. *What a-dimensional parameter controls when bifurcations happen?* This under the assumption:

$$\textbf{The ratio } \boldsymbol{\gamma = L/h > 1} \textbf{ is kept fixed.} \tag{8.1}$$

Thus γ is **not** the bifurcation parameter to use; something else is.

8.2 Answer: Toy model for shell buckling

Newton's law for the motion of the bead takes the form

$$m\,\ddot{x} + b\,\dot{x} = -m\,g + 2\,k\,\frac{x}{\sqrt{x^2+h^2}}\left(L - \sqrt{x^2+h^2}\right), \tag{8.2}$$

where the factor $2\,k$ arises because there are two springs, and the factor $x/\sqrt{x^2+h^2}$ is to compute the projection along the rod of the spring's forces. Note also that the signs are correct: when the springs are under compression ($\sqrt{x^2+h^2} < L$), and $x > 0$, the springs should be pushing x up — with the force sign switching if either $x < 0$ or $\sqrt{x^2+h^2} > L$.

[8] Suggestion: to a-dimensionalize use h for length and $b/(2\,k)$ for time.

Select a-dimensional variables via $\boldsymbol{x = h\,\tilde{x}}$ and $\boldsymbol{t = \dfrac{b}{2\,k}\,\tilde{t}}$. The equation then becomes

$$\epsilon\,\ddot{x} + \dot{x} = -r + \frac{x}{\sqrt{1+x^2}}\left(\gamma - \sqrt{1+x^2}\right), \tag{8.3}$$

where we have not written the tildes to simplify the notation,

$$\epsilon = \frac{2\,k\,m}{b^2}, \quad \text{and} \quad r = \frac{m\,g}{2\,k\,h}. \tag{8.4}$$

If $\epsilon \ll 1$, we can neglect inertia. Thus we arrive at the final equation (the toy model equation)

$$\dot{x} = -r + \underbrace{\frac{x}{\sqrt{1+x^2}}\left(\gamma - \sqrt{1+x^2}\right)}_{p=p(x)} = p(x) - r. \tag{8.5}$$

Since γ is kept fixed, **the bifurcation parameter is r.** To understand the critical point structure of this equation, in figure 8.2 we plot $y = p(x)$ and $y = r$ — this for some value of γ (there is no qualitative difference if γ is changed). **Let $r_c > 0$ be the value of p at the (single) local maximum for $x > 0$** — that is $r_c = p(x_c)$, **where x_c is the location at which the local maximum occurs.** Note that here we operate as if γ were a fixed constant, but (in fact) both r_c and x_c are functions of γ — which must be computed numerically, if needed.

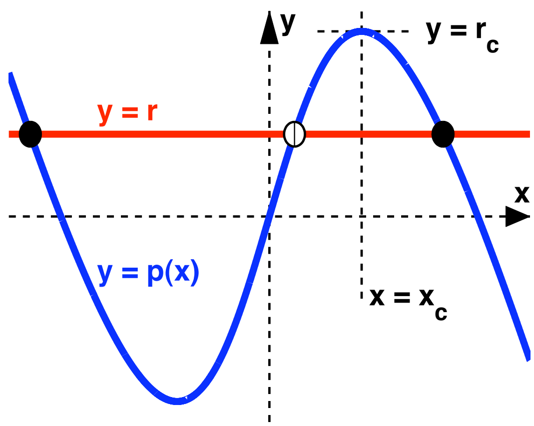


Figure 8.2: Critical points for the toy model for shell buckling. The critical points occur at the values of x where $y = r$ intersects $y = p(x)$ — where p is defined in equation (8.5).

Three cases arise:

c1. Case $0 < r < r_c$. Three critical points: x_1, x_2, and x_3 — which satisfy $x_1 < 0 < x_2 < x_c < x_3 < \sqrt{\gamma^2 - 1}$. Both x_1 and x_3 are stable, while x_2 is unstable.
- x_3 *corresponds to the configuration in figure* 8.1, with the bead being supported by the two (compressed) springs above the level $x = 0$.
- x_1 *corresponds to a configuration where the bead is hanging from the two (stretched) springs.* As follows from item c3, **this is the "buckled" state in this model.**

c2. Case $r_c < r$. Only one critical exists: the "buckled" state x_1 — which is stable.[9]

c3. Case $r_c = r$. Critical threshold value at which a saddle-node bifurcation occurs. As r increases through r_c, x_3 looses stability, and the system jumps to x_1 (if it was in x_3).

THE END.

[9] Because we are assume a situation where gravity matters, there is no truly "un-buckled" state — at best a slightly deformed one.