1 Bifurcations of a Critical Point for a 1-D map

1.1 Statement: Bifurcations of a Critical Point for a 1D map

When studying bifurcations of critical points in the lectures, we argued that we could understand most of the relevant possibilities simply by looking at 1-D flows. The argument was based on the idea that, generally, a stable critical point will become unstable in only one direction at a bifurcation — so that the flow will be trivial in all the other directions, and we need to concentrate only on what occurs in the unstable direction. The only important situation that is missed by this argument, is the case when two complex conjugate eigenvalues become unstable (Hopf bifurcation). Because in real valued systems eigenvalues arise either in complex conjugate pairs or as single real ones, the cases where only one (or a complex pair) go unstable are the the only ones likely to occur (barring situations with special symmetries that “lock” eigenvalues into synchronous behavior).

The same argument can be made when studying bifurcations of limit cycles in any number of dimensions. In this case one considers the Poincaré map near the limit cycle,1 with the role of the eigenvalues taken over by the Floquet multipliers. Again, we argue that we can understand a good deal of what happens by replacing the (multi-dimensional) Poincaré map by a one dimensional map with a stable fixed point, and asking what can happen if the fixed point changes stability as some parameter is varied. In this problem you will be asked to do this.

Remark 1.1 Some important cases are missed by this approach: the case where a pair of complex Floquet multipliers becomes unstable (Hopf bifurcation of a limit cycle), and the cases where a bifurcation occurs because of an interaction of the limit cycle with some other object (e.g., a critical point). Several examples of these situations can be found in section 8.4 of Strogatz’ book (Global Bifurcations of Cycles).

Consider a one dimensional (smooth) map from the real line to itself

\[ x \rightarrow y = f(x, \mu) \quad (1.1) \]

\[ ^1 \text{The limit cycle is a fixed point for this map.} \]
that depends on some (real valued) parameter $\mu$. **Assume that $x = 0$ is a fixed point for all values of $\mu$ — that is, $f(0, \mu) \equiv 0$.** Furthermore, assume that $x = 0$ is stable for $\mu < 0$ and unstable for $\mu > 0$. That is:

$$\left| \frac{\partial f}{\partial x}(0, \mu) \right| < 1 \quad \text{for } \mu < 0, \quad \text{and}$$

$$\left| \frac{\partial f}{\partial x}(0, \mu) \right| > 1 \quad \text{for } \mu > 0. \quad \text{(1.2)}$$

A further assumption, that involves no loss of generality (since the parameter $\mu$ can always be re-defined to make it true) is that

$$\frac{\partial^2 f}{\partial x \partial \mu}(0, 0) \neq 0. \quad \text{(1.4)}$$

This guarantees that the loss of stability is linear in $\mu$, as $\mu$ crosses zero. This is what is called a transversality condition. It means this:

Graph of the Floquet multiplier $\frac{\partial f}{\partial x}(0, \mu)$ as a function of $\mu$. Then the resulting curve crosses one of the lines

$y = \pm 1$ transversally (curves not tangent at the common point) for $\mu = 0$.

By doing an appropriate expansion of the map $f$ for $x$ and $\mu$ small (or by any other means), **show that** (generally\(^2\)) the following happens:

**a.** For $\frac{\partial f}{\partial x}(0, 0) = 1$, either:

   a1. **Transcritical bifurcation (no special symmetries assumed for $f$):** There exists another fixed point, $x_* = x_*(\mu) = O(\mu)$, such that: $x_* \neq 0$ is unstable for $\mu < 0$ and $x_* \neq 0$ is stable for $\mu > 0$. The two points “collide” at $\mu = 0$ and exchange stability.

   a2. **Subcritical or soft pitchfork bifurcation, assuming that $f$ is an odd function of $x$:** Two stable fixed points exist for $\mu > 0$, one on each side of $x = 0$, at a distance $O(\sqrt{\mu})$. All three points merge for $\mu = 0$.

   a3. **Subcritical or hard pitchfork bifurcation, assuming that $f$ is an odd function of $x$:** Two unstable fixed points exist for $\mu < 0$, one on each side of $x = 0$, at a distance $O(\sqrt{-\mu})$. All three points merge for $\mu = 0$.

**What does all this mean in the context of the Poincaré map for a limit cycle?**

**b.** For $\frac{\partial f}{\partial x}(0, 0) = -1$ (no special symmetries assumed for $f$), either:

   b1. **Supercritical or soft flip bifurcation:** For $\mu > 0$ two points $x_1(\mu) \approx -x_2(\mu) = O(\sqrt{\mu})$ exist, on each side of the fixed point $x = 0$, with $x_2 = f(x_1, \mu)$ and $x_1 = f(x_2, \mu)$. Thus $\{x_1, x_2\}$ is a period two orbit for the map (1.1). **Show that this orbit is stable.**

   b2. **Subcritical or hard flip bifurcation:** For $\mu < 0$ two points $x_1(\mu) \approx -x_2(\mu) = O(\sqrt{-\mu})$ exist, on each side of the fixed point $x = 0$, with $x_2 = f(x_1, \mu)$ and $x_1 = f(x_2, \mu)$. Thus $\{x_1, x_2\}$ is a period two orbit for the map (1.1). **Show that this orbit is unstable.**

In the context of the Poincaré map for a limit cycle, a flip bifurcation corresponds to a period doubling bifurcation of the limit cycle.

**Hint 1.1** If you expand $f$ in a Taylor expansion near $x = 0$ and $\mu = 0$, up to the leading order beyond the trivial first term ($f \sim \pm x$), and you make sure to keep all the relevant terms (and nothing else), and you make sure to identify certain terms that must vanish, the problem reduces to (mostly) trivial high-school algebra. In figuring out what to keep and what not to keep, note that you have two small quantities ($x$ and $\mu$), whose sizes are related. The process is very similar to the Hopf bifurcation expansion calculation (e.g. see course notes), but much simpler computationally.

\(^2\) There are special conditions under which all this fails. You must find them as part of your analysis. What are they?
For part (b) you will have to keep in the expansion not just the leading order terms beyond the trivial first term, but the next order as well. **Reason:** In this case, you will be looking for solutions to the equation \( f(f(x, \mu), \mu) = x \). But, when you calculate \( f(f(x, \mu), \mu) \), you will see that the second order terms cancel out — thus the need for an extra term in the expansion. This is the same phenomena that forces the Hopf bifurcation calculation to third order.

### IMPORTANT

To standardize the notation, use the following symbols in your answer:

\[
\begin{align*}
\nu &= \frac{\partial f}{\partial x}(0, 0) \quad \text{— note that } \nu = \pm 1, \\
a &= \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(0, 0), \\
b &= \frac{\partial^2 f}{\partial x \partial \mu}(0, 0), \\
c &= \frac{1}{6} \frac{\partial^3 f}{\partial x^3}(0, 0), \quad \text{and} \\
d &= \frac{1}{2} \frac{\partial^3 f}{\partial x^2 \partial \mu}(0, 0).
\end{align*}
\]

Then obtain leading order expressions for the fixed points and flip-bifurcation orbits in terms of these quantities.

### 1.2 Answer: Bifurcations of a Critical Point for a 1D map

We begin by writing the first few terms in the Taylor expansion for \( f \) near \( x = \mu = 0 \). We have

\[
f = \nu x + a x^2 + b \mu x + c x^3 + d \mu x^2 + e \mu^2 x + O(x^4, \mu x^3, \mu^2 x^2, \mu^3 x),
\]

where \( \nu, a, b, c, \) and \( d \), are as defined in the problem statement. Furthermore:

- **\( \nu = \pm 1 \).** This follows from (1.2) and (1.3).
- **\( b \neq 0 \),** as follows from (1.4).
- **The expansion has no terms involving powers of \( \mu \) alone.** This follows from \( f(0, \mu) \equiv 0 \).

#### a. Case \( \nu = 1 \). Then:

\[
f = x + a x^2 + b \mu x + c x^3 + d \mu x^2 + e \mu^2 x + O(x^4, \mu x^3, \mu^2 x^2, \mu^3 x),
\]

where \( b > 0 \), as follows from (1.2) and (1.3) — given that \( \frac{\partial f}{\partial x}(0, \mu) = 1 + b \mu + O(\mu^2) \).

We have the following sub-cases:

#### a1. **Generic sub-case:** \( a \neq 0 \). Then the equation \( x = f(x, \mu) \) has two solutions near \( x = \mu = 0 \), as follows from (1.6). These are

\[
x = 0 \quad \text{and} \quad x = x_*(\mu) = -\frac{b}{a} \mu + O(\mu^2).
\]

Furthermore, note that

\[
\frac{\partial f}{\partial x}(x_*, \mu) = 1 + 2 a x_* + b \mu + O(\mu^2) = 1 - b \mu + O(\mu^2).
\]

Thus \( x_* \) is an unstable fixed point for \( \mu < 0 \) and a stable one for \( \mu > 0 \). This sub-case corresponds to a TRANSCRITICAL bifurcation.

An expansion for \( x_* \) in powers of \( \mu \) (to any order) is easy to obtain by writing

\[
x_* = \alpha_1 \mu + \alpha_2 \mu^2 + \ldots,
\]

substituting this into \( f(x_*, \mu) = x_* \), using the expression in (1.6) for \( f \), and then equating equal powers of \( \mu \) to obtain the coefficients \( \alpha_n \).
a2. **Sub-case: $f$ is an odd function of $x$, and $c < 0$.** Then the expansion for $f$ in (1.6) reduces to

$$f = x + b\mu x + cx^3 + O(x^5, \mu x^3, \mu^2 x)$$

(1.7)

Clearly, for $0 < \mu \ll 1$, the equation $f(x, \mu) = x$ has three roots near $x = 0$. These are

$$x = 0 \quad \text{and} \quad x = x_1(\mu) = -x_2(\mu) = \sqrt[3]{\frac{b}{c} \mu + O(\mu^{3/2})}.$$  

Note that

$$\frac{\partial f}{\partial x}(x_n, \mu) = 1 + b\mu + 3cx_n^2 + O(\mu^2) = 1 - 2b\mu + O(\mu^2).$$

Thus both $x_1$ and $x_2$ are stable for $0 < \mu \ll 1$.

This **sub-case corresponds to a SUPERCritical (or soft) PITCHFORK bifurcation.**

a3. **Sub-case: $f$ is an odd function of $x$, and $c > 0$.** This sub-case is entirely analogous to the one in item a2, except that the two extra solutions occur for $\mu < 0$ and they are unstable — the algebra for showing this is an exact replica of the one in item a2.

This **sub-case corresponds to a SUBCRITICAL (or hard) PITCHFORK bifurcation.**

b. **Case $\nu = -1$.** Then:

$$f = -x + a x^2 + b\mu x + cx^3 + O(x^4, \mu x^2, \mu^2 x),$$

(1.8)

where $b < 0$, as follows from (1.2) and (1.3) — given that $\frac{\partial f}{\partial x}(0, \mu) = -1 + b\mu + O(\mu^2)$. It should then be clear that $f(x, \mu) = x$ has no roots other than $x = 0$ near $x = \mu = 0$, as follows from the implicit function theorem and the fact that $g = g(x, \mu) = f - x$ has a nonzero partial derivative with respect to $x$ at $x = \mu = 0$.

Thus consider the second iterate of $f$, namely $f^2 = f(f(x))$, and its expansion:

$$f^2 = x - 2b\mu x - 2(a^2 + c)x^3 + O(x^4, \mu x^2, \mu^2 x).$$

(1.9)

The leading order terms in this expansion have exactly the same form as the leading order terms in (1.7), except that the constants are different and ($f^2$ is not necessarily odd). The algebra that follows is thus very similar, except that fixed points of $f^2$ which are different from $x = 0$ correspond to period two orbits of $f$.

b1. **Sub-case: $a^2 + c > 0$.** Then $f^2$ has two nonzero fixed points for $0 < \mu \ll 1$, which correspond to a period two orbit. These are given by

$$x = x_1(\mu) = \sqrt[3]{\frac{b}{a^2 + c} \mu + O(\mu^{3/2})} \quad \text{and} \quad x = x_2(\mu) = -\sqrt[3]{\frac{b}{a^2 + c} \mu + O(\mu^{3/2})}.$$  

Note that, while $x_1 \approx -x_2$, this is (generally) only an approximate equality — with the two roots differing at higher orders. We also note that

$$\frac{\partial f^2}{\partial x}(x_n, \mu) = 1 - 2b\mu - 6(a^2 + c)x_n^2 + O(\mu^2) = 1 + 4b\mu + O(\mu^2).$$

Thus the orbit is stable for $\mu > 0$ (recall that $b < 0$).

This **sub-case corresponds to a SUPERCritical (or soft) FLIP bifurcation.**

b2. **Sub-case: $a^2 + c < 0$.** This case is entirely analogous to the one in item b1, except that the period two orbit occurs for $\mu < 0$ and it is unstable — the algebra for showing this is an exact replica of the one in item b1. This **sub-case corresponds to a SUBCRITICAL (or hard) FLIP bifurcation.**

b3. **Sub-case: $a^2 + c = 0$.** This is not generic. In the absence of any known symmetry of the equations guaranteeing this, it is not important (**structurally unstable situation**).
2 Coastline fractal

2.1 Statement: Coastline fractal

In this problem we construct a fractal that is a very idealized caricature of what a coastline looks like. The construction proceeds by iteration of a basic process, which we describe next.

We start with a simple curve, $\Gamma_0$, and apply to it a process, that yields a new curve $\Gamma_1$. This new curve is made up of several parts, each of which is a scaled down copy of $\Gamma_0$. The same process is then applied to each of these parts, yielding $\Gamma_2$. Iterating, we obtain a series of curves $\Gamma_n$, for $n = 0, 1, 2, 3 \ldots$. **The fractal is the limit of this process:**

\[ \Gamma = \lim_{n \to \infty} \Gamma_n \]  

provided that the limit exists.

For the *coastline fractal* we start by picking an angle $0 < \Theta < \pi$, and a length $R_0 > 0$. The first curve is:

\[ \Gamma_0 = \text{circular arc of radius } R_0, \text{ subtending an angle } \Theta. \]  

(2.1)

Next divide $\Gamma_0$ into three equal sub-arcs, each subtending an angle $\Theta/3$, and replace each of these pieces by a properly scaled version of $\Gamma_0$. **This yields $\Gamma_1$.** The process is then repeated on each of the three pieces making up $\Gamma_1$, so as to obtain $\Gamma_2$, and so on ad infinitum. The first two steps in this construction are illustrated in figure 2.1.

Now we show that the limit $\lim_{n \to \infty} \Gamma_n$ exists. Consider an arbitrary radial line within the circle sector associated with $\Gamma_0$, and the intersection of this line with $\Gamma_n$. It should be clear that this intersection is unique. Let $d_n$ be the distance of this intersection from the origin of the radial line. Then \{ $d_n$ \} is an increasing, bounded sequence — thus it has a limit. This limit defines a point along the radial line. The set of all these points is the fractal $\Gamma$. With a little more work one can show as well that $\Gamma$ is described via a continuous function $r = R(\theta)$.

Now do the following:

1. For each $n = 0, 1, 2, 3 \ldots$, calculate $\ell_n = \text{length of } \Gamma_n$. What do you conclude about the “length” of $\Gamma$?

2. Calculate the fractal dimension (self-similar or box) of $\Gamma$.

**Hint:** The first thing that you need to calculate is the “scaling” factor between $\Gamma_0$ and each of the three parts that make up $\Gamma_1$. With this scaling factor, which depends on $\Theta$ — $0 < S_c = S_c(\Theta) < 1$, everything else follows.

**Note:** Real coastlines are not this simple, of course. At the very least the number of parts into which each sector is divided should not be a constant (3 here), nor should the parts be equal in size, nor should they all subtend the same angle $\Theta$. Further: the sectors need not be exactly circular — though, this probably is not a terrible approximation.
2.2 Answer: Coastline fractal

Following the hint, we begin by calculating the scaling factor between the circular arc $\Gamma_0$, and the three circular arcs that make up $\Gamma_0$. In this process it is convenient to consider the chords associated with each of the circular arcs (subtending an angle $\Theta$). More precisely:

- Let $L_0$ be the length of the chord corresponding to the arc $\Gamma_0$; let $L_1$ be the (common) length of the chords corresponding to each of the three arcs making up $\Gamma_1$; and so on for $L_2$, $L_3$ . . .
- Let $R_0$ be the radius of the arc $\Gamma_0$; let $R_1$ be the (common) radius associated with each of the three arcs making up $\Gamma_1$; and so on for $R_2$, $R_3$ . . .

Consider the triangle made by the bisecting radius of a sector, one of the boundary radius, and one-half the corresponding cord. From it we get the relationship

$$L_n = 2R_n \sin \left( \frac{\Theta}{2} \right),$$  \hspace{1cm} (2.2)

which applies for all $n = 0, 1, 2 \ldots$ On the other hand, the cord whose length is $L_1$ spans an angle $\Theta/3$ in the circular arc corresponding to $\Gamma_0$. By an argument similar to the one leading to (2.2), this observation yields the relationship

$$L_1 = 2R_0 \sin \left( \frac{\Theta}{6} \right), \hspace{1cm} \text{easy to generalize to} \hspace{1cm} L_{n+1} = 2R_n \sin \left( \frac{\Theta}{6} \right),$$ \hspace{1cm} (2.3)

which applies for all $n = 0, 1, 2 \ldots$ Thus, from (2.2–2.3) it follows that

$$R_{n+1} = \left( \frac{\sin(\Theta/6)}{\sin(\Theta/2)} \right) R_n = S_c(\Theta) R_n,$$ \hspace{1cm} (2.4)

where $S_c = S_c(\Theta)$ (defined by the formula above) is the scaling factor between successive levels of the fractal $\Gamma$. Note that:

$$\frac{1}{3} < S_c(\Theta) < \frac{1}{2} \hspace{1cm} \text{for} \hspace{1cm} 0 < \Theta < \pi.$$ \hspace{1cm} (2.5)

The length $\ell_n$ of curve $\Gamma_n$ ................................................................. ♠

Clearly, $\ell_0 = R_0 \Theta$. Because $\Gamma_1$ is made by three equal arcs of radius $R_1$, $\ell_1 = 3S_c R_0 \Theta$ (upon using (2.4)). Because of the recursive construction of the fractal, it follows that

$$\ell_n = (3S_c)^n R_0 \Theta.$$ \hspace{1cm} (2.6)

From (2.5), $3S_c > 1$. Thus

$$\ell_n \to \infty \hspace{1cm} \text{as} \hspace{1cm} n \to \infty.$$  

Self-similar dimension of the fractal $\Gamma$ ................................................................. ♠

The fractal $\Gamma$ is made up by three equal copies of itself, scaled down by a factor $S_c$. The definition of self-similar fractal dimension $d$ is: $N = r^{-d}$, where $N$ is the number of copies, and $r$ is the scaling factor. Thus:

$$\dim(\Gamma) = -\frac{\ln 3}{\ln S_c}.$$ \hspace{1cm} (2.7)

Further, from (2.4–2.5)

$$1 < \dim(\Gamma) < \frac{\ln(3)}{\ln(2)}, \hspace{1cm} \text{and} \hspace{1cm} \dim(\Gamma) \text{ is monotone increasing with } \Theta.$$ \hspace{1cm} (2.8)

THE END.