Answers to P-Set # 07, (18.353/12.006/2.050)j

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1 Bifurcations of a Critical Point for a 1-D map

This problem has been “moved” to the next problem set.

2 Liapunov exponents for 1-D maps #01

2.1 Statement: Liapunov exponents for 1-D maps #01

Compute the Liapunov exponent, and produce a figure analog to figure 10.5.2 in Strogatz’s book (see example 10.5.3 there) for the 1-D maps $x_{n+1} = f(x_n)$ below. In all the cases perform the following task: given the range of $r$ selected, justify the selected range for $x$.

Meaning of "justify". Show that the $x$-region is such that it may contain an attractor, because either: (a) It is trapping; an orbit starting there stays there; or (b) Orbits starting outside the region diverge to infinity, so that any attractor has to be inside.

1. The sine map $f(x) = r \sin(x)$, with $-5 \leq r \leq 5$ and $-5 \leq x \leq 5$.
2. The cubic map $f(x) = r x - x^3$, with $0 \leq r \leq 3$ and $-2 \leq x \leq 2$.
3. The exponential #1 map $f(x) = x e^{r(1-x)}$, with $1 \leq r \leq 4$ and $0 \leq x \leq 5$.

Process: Explained in example 10.5.3 of Strogatz book.

Note: if you use MatLab, “vectorize” the operation, so that you do all the $r$ simultaneously. Also, in MatLab the command “print -dps FigureName” will save the figure as a small png file (and it is more reliable than trying to use the GUI in the figure window).

2.2 Answer: Liapunov exponents for 1-D maps #01

The answer to the three parts follows below.

Remark. The Liapunov exponent is negative within the “periodic windows” (where the attractor is a periodic orbit), and goes to $-\infty$ at the super-stable periodic orbit within each window — though lack of numerical resolution cuts these downward “spikes” to a finite value. In the chaotic regions the Liapunov exponent is positive. However the chaotic regions contain a fractal structure of periodic windows (which are very hard to resolve beyond the larger ones). This feature is evident in the plots of the Liapunov exponents below.

1. The Liapunov exponent calculation for the sine map $f(x) = r \sin(x)$, with $-5 \leq r \leq 5$ and $-5 \leq x \leq 5$, is shown on the left panel in figure 2.1. The right panel shows a detail near the onset of chaos for $r > 0$.

Justification of the $x$-range: since $|f(x)| \leq r$, the attractor must be contained within $|x| \leq r$.

2. The Liapunov exponent calculation for the cubic map $r x - x^3$, with $0 \leq r \leq 3$ and $-2 \leq x \leq 2$, is shown on the left panel in figure 2.2. The right panel shows a detail near the onset of chaos.

Justification of the $x$-range: Let $x^2 = r + 1 + \delta$ ($\delta > 0$), then $|f(x)| = |r - x^2| x = (1 + \delta) |x|$. This shows that: when $|x_0| > \sqrt{r + 1}$, $|x_n| \rightarrow \infty$. Hence the attractor must be contained within $|x| \leq \sqrt{r + 1}$.

3. The Liapunov exponent calculation for the exponential #1 map $f(x) = x e^{r(1-x)}$, with $1 \leq r \leq 4$ and $0 \leq x \leq 5$, is shown on the left panel in figure 2.3. The right panel shows a detail near the onset of chaos.

Justification of the $x$-range: for $x \geq 0$, $0 \leq f(x) \leq f_m = \frac{1}{r} e^{r-1}$ (with $f_m$ achieved at $x = 1/r$). In turn, the maximum of $f_m$ over the given range of $r$ is achieved at $r = 4$, where $f_m = f_M$. Thus the attractor

\[ f'(x) = (1 - r x) e^{r(1-x)}, \quad f''(x) = -r (2 - r x) e^{r(1-x)}. \]

1 Note that $f' = (1 - r x) e^{r(1-x)}$, and $f'' = -r (2 - r x) e^{r(1-x)}$. 


Figure 2.1: **Left panel:** sine map’s Liapunov exponent, $L_i(r)$, $-5 \leq r \leq 5$. **Right panel:** detail near the onset of chaos, $2.7 < r < 2.8$. Sensitive dependence on initial conditions, and chaos, occur for $L_i > 0$ (dashed red line).

Figure 2.2: **Left panel:** cubic map’s Liapunov exponent, $L_i(r)$, $0 \leq r \leq 3$. **Right panel:** detail near the onset of chaos, $2.25 < r < 2.35$. Sensitive dependence on initial conditions, and chaos, occur for $L_i > 0$ (dashed red line).

must be contained within $0 \leq x \leq f_M$. Further point: because $f$ is concave for $x < 0$ ($f'' < 0$), with $f'(0) > 1$, a cobweb shows that any orbit with $x_0 < 0$ diverges to $-\infty$.

3 Orbit diagrams for 1-D maps #01

3.1 Statement: Orbit diagrams for 1-D maps #01

Compute the orbit diagrams\(^2\) for the 1-D maps $x_{n+1} = f(x_n)$ below. Furthermore, in all the cases perform the following task: given the range of $r$ selected, justify the selected range for $x$.

\(^2\)An orbit diagram is what figure 10.2.7 in Strogatz’s book shows.
Figure 2.3: **Left panel:** exponential #1 map’s Liapunov exponent, $L_i(r)$, $1 \leq r \leq 4$. **Right panel:** detail near the onset of chaos, $2.65 < r < 2.72$. Sensitive dependence on initial conditions, and chaos, occur for $L_i > 0$ (dashed red line).

**Meaning of “justify”.** Show that the $x$-region is such that it may contain an attractor, because either: (a) It is trapping; an orbit starting there stays there; or (b) Orbits starting outside the region diverge to infinity, so that any attractor has to be inside.

1. The **sine map** $f(x) = r \sin(x)$, with $-5 \leq r \leq 5$ and $-5 \leq x \leq 5$.
2. The **cubic map** $f(x) = rx - x^3$, with $0 \leq r \leq 3$ and $-2 \leq x \leq 2$.
3. The **exponential #1 map** $f(x) = xe^{r(1-x)}$, with $1 \leq r \leq 4$ and $0 \leq x \leq 0.25e^3 = 5.0213\ldots$
4. The **tent map** $f(x) = r(1 - |2x - 1|)$, with $0 \leq r \leq 1$ and $0 \leq x \leq 1$. At first sight the tent map orbit diagram may look rather plain. However, **do a blow up of the region in the tent map orbit diagram given by**:
   - 4b. $0.5 \leq r \leq 0.6$ and $0.492 \leq x \leq 0.501$.
   - 4c. $0.5 \leq r \leq 0.525$ and $0.4999 \leq x \leq 0.5001$.

Note that doing these blow-ups will require to plot far more iterates than for the main plot, because you will not be plotting the whole attractor, but just a small region within it, which means most points will be outside the plotted region.

**Process:** Pick a grid in $r$; say $r_m = r_0 + m \Delta r$, for some small $\Delta r$, covering the desired range for $r$. Then, for every $r_m$, take a random starting point $x_0$, and calculate the resulting $x_n$ for a large number of iterates $1 \leq n \leq N$. Finally: throw away the first few hundred iterates (to eliminate transients as the attractor is approached) and plot the rest as points $(r_m, x_n)$ in a 2-D diagram.

Note: if you use MatLab, “vectorize” the operation, so that you do all the $r$ simultaneously (MatLab will be very slow if you use a “for loop” in both $n$ and $m$).

Also, in MatLab the command “print -dpng FigureName” will save the figure as a small png file (and it is more reliable than trying to use the GUI in the figure window).

### 3.2 Answer: Orbit diagrams for 1-D maps #01

Note: the answer below is much more detailed than what you were expected to turn in. I am using this as an opportunity to examine 1-D maps a little deeper.

1. The orbit diagram for the **sine map** $f(x) = r \sin(x)$, with $-5 \leq r \leq 5$ and $-5 \leq x \leq 5$, is shown on the left panel in figure 3.1. **Justification of the $x$-range:** since $|f(x)| \leq r$, the attractor must be contained within $|x| \leq r$. Note that the figure shows that the chaotic attractors explore the whole range $-r < x < r$. 

Figure 3.1: The left panel shows the orbit diagram for the sine map, where (for every value $-5 \leq r \leq 5$) $x_0$ is a random point in $-5 \leq x_0 \leq 5$. Here the dashed red line indicates the limits $x = \pm r$. For the right panel the initial data is restricted by $0 \leq x_0 \leq r$. Here the green lines correspond to $x = \pm \pi$.

Remark 1a: For $0 \leq r \leq \pi$, the interval $[0, r]$ is mapped onto itself by the sine map — and the same happens for $[-r, 0]$. Hence, for $0 \leq r \leq \pi$, the left and right sides of the orbit diagram correspond to separate, independent, attractors. Only for $r > \pi$ that the two sides “mix” — see the right panel in figure 3.1.

Remark 1b: The sine map is odd. Hence the orbit diagram is invariant under $x \mapsto -x$ (left panel in figure 3.1). However: why is the orbit diagram shown in the figure also invariant under $r \mapsto -r$? The reason is as follows: consider an orbit $\{x_n\}_{n=0}^{\infty}$ for the map, corresponding to some value $r = r_0$. Then $\{(−1)^n x_n\}_{n=0}^{\infty}$ is an orbit of the map corresponding to $r = -r_0$ (the proof is trivial). In addition, $\{-x_n\}_{n=0}^{\infty}$ is also an orbit corresponding to $r = r_0$, so that $\{−(−1)^n x_n\}_{n=0}^{\infty}$ is also an orbit for $r = -r_0$. It follows that the attractors for $r = r_0$ and $r = -r_0$ have the same points (even though the orbit’s even and odd terms map separately). ♦

2. The left panel in figure 3.2 shows the orbit diagram for $f(x) = r \cdot x - x^3$ (cubic map), with $0 \leq r \leq 3$ and $|x| \leq 2$. Justification of the x-range: Let $x^2 = r + 1 + \delta$ ($\delta > 0$), then $|f(x)| = |r - x^2| \cdot |x| = (1 + \delta)|x|$. This shows that: when $|x_0| > \sqrt{r + 1}$, $|x_n| \to \infty$. Hence the attractor must be contained within $|x| \leq \sqrt{r + 1}$.

Remark 2a: The figure shows that the attractor is contained within $|x| \leq x_\mu$, where $x_\mu = 2 \left(\frac{r}{3}\right)^{1.5}$ is the value of $f$ (local maximum) at $x = \sqrt{r/3}$.

Remark 2b: For $0 \leq x \leq \sqrt{r}$, $0 \leq f(x) \leq x_\mu$. Thus if $x_\mu \leq \sqrt{r}$ (i.e., $r \leq 1.5 \sqrt{3}$), the interval $0 \leq x \leq x_\mu$ is mapped onto itself by the cubic map — and the same happens for the interval $-x_\mu \leq x \leq 0$. Hence, for $0 \leq r \leq 1.5 \sqrt{3}$, the left and right sides of the orbit diagram correspond to separate, independent, attractors. It is only for $r > 1.5 \sqrt{3}$ that the two sides “mix” — as shown by the right panel in figure 3.2.

Remark 2c: The cubic map is odd. Hence the orbit diagram should be invariant under $x \mapsto -x$, which the left panel in figure 3.1 confirms. However, unlike what happens for the sine map (see remark 1b) there is no $r \mapsto -r$ symmetry: for $r < -1$, $|f(x)| = |r - x^2| \cdot |x| > |r| \cdot |x|$, so that all non-zero orbits diverge to infinity (while $x = 0$ is unstable). In fact, figure 3.3 shows that, for $-1 < r < 1$ the attractor is $x = 0$.

Remark 2d: The computations show that, beyond $r = 3$ there is no attractor — see figure 3.3. I do not have a proof for this. If you can figure out an argument for this, I would like to know it.

3. The orbit diagram for the exponential #1 map $f(x) = e^{r(1-x)}$, with $1 \leq r \leq 4$ and $0 \leq x \leq f_M$ (where $f_M = 0.25 e^3 = 5.0213 \ldots$), is shown on the left panel in figure 3.4.

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I do not have a proof of this. I could not think of an “elegant” way to do it, and did not want to get involved into a lot of messy algebra.
Figure 3.2: The **left panel** shows the orbit diagram for the cubic map, where (for every value $0 \leq r \leq 3$) $x_0$ is a random point in $-2 \leq x_0 \leq 2$. Here the dashed red line indicates the curves $x = \pm 2 (r/3)^{1/5}$. For the **right panel** the initial data is restricted by $0 \leq x_0 \leq \sqrt{r}$. Here the green line corresponds to $r = x^2$.

Figure 3.3: Cubic map.

Orbit diagram for $f(x) = rx - x^3$ (cubic map), for the range $-1 < r < 4$. Note that for $r > 3$ there is no attractor (orbits are either unstable or diverge to infinity), and the same happens for $r \leq -1$ (this is shown in the text). For $-1 < r < 1$ the attractor is just $x = 0$. At $r = 1$ a pitchfork bifurcation happens, followed by a period doubling cascade.

**Justification of the $x$-range**: for $x \geq 0$, $0 \leq f(x) \leq f_m = \frac{1}{r} e^{r-1}$ (with $f_m$ achieved at $x = 1/r$). In turn, the maximum of $f_m$ over the given range of $r$ is achieved at $r = 4$, where $f_m = f_M$. Thus the attractor must be contained within $0 \leq x \leq f_M$. Further point: because $f$ is concave for $x < 0$ ($f'' < 0$), with $f'(0) > 1$, a cobweb shows that any orbit with $x_0 < 0$ diverges to $-\infty$.

Note that figure 3.4 shows that the attractors are contained $\dagger$ within $f(f_m) \leq x \leq f_m$, and that the chaotic attractors explore the whole range. Note that $f(f_m) = \frac{1}{r} e^{2r-1-e^{r-1}}$ vanishes very fast as $r$ grows.

$\dagger$: The reason why the attractors must be within $x \leq f_m$ was explained earlier. The additional restriction $f(f_m) \leq x$ arises as follows: It is easy to see that any point $0 \leq x \leq 1$ satisfies $f(x) > x$ ($x = 0$ and $x = 1$ are the fixed points of the map) so the map pushes these points to the right — do the cobweb. The only way that these points can be in the attractor is if they are mapped back there from points beyond the maximum of $f$ at $x = 1/r$, where $f$ is decreasing. Clearly the

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If you can think of an elegant way to do so, I will appreciate it.

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4 Note that $f' = (1 - r \ x) e^{r (1-x)}$, and $f'' = -r (2 - r \ x) e^{r (1-x)}$. 

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If you can think of an elegant way to do so, I will appreciate it.
18.385 MIT, (Rosales) Orbit diagrams for 1-D maps #01.

Figure 3.4: Exponential #1 map.
Orbit diagram for \( f(x) = x e^{r(1-x)} \) (exponential #1 map). Here, for every value \( 1 \leq r \leq 4 \), \( x_0 \) is a random point in \( 0 \leq x_0 \leq f_m \). The dashed red curves are: \( x = f_m = \frac{1}{r} e^{r-1} \) on the right, and \( x = f(f_m) \) on the left.

The smallest point thus achievable is the image of the largest \( x \) possible, i.e.: \( x = f_m \), where \( f \) reaches its minimum value in the allowed range.

4. The orbit diagram for the tent map \( f(x) = r \left(1 - \left|2x - 1\right|\right) \), with \( 0 \leq r \leq 1 \) and \( 0 \leq x \leq 1 \), is shown on the left panel in figure 3.5. **Justification of the \( x \)-range:** follows because \( 0 \leq f(x) \leq r \) for \( 0 \leq x \leq 1 \).

Figure 3.5: Tent map.
Orbit diagram for \( f(x) = r \left(1 - \left|2x - 1\right|\right) \) (tent map). Here, for every value \( 0 \leq r \leq 1 \), \( x_0 \) is a random point in \( 0 \leq x_0 \leq 1 \). The dashed green curves are: \( x = 2r \left(1 - r\right) \) on the left (that is: \( x = f(r) \) for \( r > 0.5 \)), and \( x = r \) on the right. Note that for \( 0 \leq r < 0.5 \), \( x = 0 \) is a global attractor. It is only for \( r > 0.5 \), where a new (and unstable) fixed point at \( x = 2r/(1 + 2r) \) appears, that "the show" begins.

This diagram appears quite featureless, except for the football shaped gap for \( 0.5 \lesssim r \lesssim 0.7 \). However: see figure 3.6.

Notice that the attractor is contained within the lines \( x = f(r) \) and \( x = r \) (for \( r > 0.5 \)). The explanation for this is exactly analog to the reasons given for the bounds on the attractor for the exponential #1 map.

Note also that the orbit diagram shows nothing for \( r = 0.5 \). This is because in this case every point \( 0 \leq x \leq 0.5 \) is a fixed point. Hence if the initial random point satisfies \( x_0 \leq 0.5 \), then \( x_n = x_0 \) for all \( n \). Else \( x_n = x_1 < 0.5 \) for all \( n > 0 \). This means that, at most, there are two points plotted for \( r = 0.5 \), which (of course) do not show up in the png file. Even in the original (full data) figure they are very hard to spot.

The tent map orbit diagram in figure 3.5 seems to have no structure. However, a blow of the “tongue” to the left of the gap in the attractor (specifically: the region \( 0.5 \leq r \leq 0.6 \) and \( 0.492 \leq x \leq 0.501 \)) shows that this is not so — see the left panel in figure 3.6. A (distorted) whole copy of the diagram appears in the blow up. A further blow up of the right “tongue” of the picture on the left panel in figure 3.6 (specifically: the region \( 0.5 \leq r \leq 0.525 \) and \( 0.4999 \leq x \leq 0.5001 \)) shows again the same phenomena — see the right panel.
Figure 3.6: Details of the orbit diagram for the tent map \( f(x) = r(1-|2x-1|) \). The left panel shows the region \( 0.5 \leq r \leq 0.6 \) and \( 0.492 \leq x \leq 0.501 \), while the right panel shows \( 0.5 \leq r \leq 0.525 \) and \( 0.4999 \leq x \leq 0.5001 \). These plots indicate that whole (distorted) copies of the full diagram exist within the tongues to the left and right of the gap in figure 3.5.

in figure 3.6. This shows that this orbit diagram has fractal structure, just like the prior ones.

Finally, we “explain” how/why the gaps in figures 3.5–3.6 form. We do this by observing how the iterates of the tent map behave, where the iterates are defined by \( f_1(x) = f(x) \) and \( f_{n+1}(x) = f(f_n(x)) \) — thus an orbit is given by \( x_n = f_n(x_0) \).

Figure 3.7 shows plots of iterates of the tent map for values of \( r \) corresponding to the main gap shown in

Figure 3.7: Iterates of the tent map for values of \( r \) corresponding to the main gap shown in figure 3.5. Specifically: Left panel: \( f_{24} \) for \( r = 0.54 \). Middle panel: \( f_{20} \) for \( r = 0.60 \). Right panel: \( f_{16} \) for \( r = 0.68 \).

figure 3.5. Specifically \( r = 0.54, 0.60, 0.68 \) (corresponding to the lower end, middle, and upper end of the gap, respectively). The take-away message is that the iterates develop a vaguely square wave shape, with sharp transitions between one set of “lower” values, and another set of “upper” values, leaving a gap between these two sets. This means that the values between the sets are extremely unstable (very large derivative) and will not show up in the attractor. You can check that the gaps in figure 3.7 correspond to the gaps in figure 3.5.

Next, figure 3.8 shows the structure of one of the “oscillatory” regions in figure 3.7 (where details are beyond the resolution of the png file used). Notice how the same pattern seen in figure 3.7 repeats here, leading to the smaller gaps seen in figure 3.6.

**Remark 4a:** In figure 3.8 we have applied a shift to the vertical and horizontal scales. This is to avoid the need to have axis labels involve 4 decimals (which would make them unreadable). If you blow up the pdf image, you will see that they are now readable.
4 Problem 14.10.03 - Newton’s method in the complex plane

4.1 Statement for problem 14.10.03

Suppose that you want to solve an equation, \( g(x) = 0 \). Then you can use Newton’s method, which is as follows: Assume that you have a “reasonable” guess, \( x_0 \), for the value of a root. Then the sequence

\[
x_{n+1} = f(x_n), \quad n \geq 0,
\]

where \( f(x) = x - \frac{g(x)}{g'(x)} \), (4.1)

converges (very fast) to the root.

**Remark 4.1 (The idea).** Assume an approximate solution \( g(x_a) \approx 0 \). Write \( x_b = x_a + \delta x \) to improve it, where \( \delta x \) is small.

Then \( 0 = g(x_a + \delta x) \approx g(x_a) + g'(x_a) \delta x \Rightarrow \delta x \approx -\frac{g(x_a)}{g'(x_a)} \), and (4.1) follows.

Of course, if \( x_0 \) is not close to a root, the method may not converge. Even if it converges, it may converge to a root that is far away from \( x_0 \), not necessarily the closest root. In this problem we investigate the behavior of Newton’s method in the complex plane, for arbitrary starting points.

Consider iterations of the map in the complex plane generated by Newton’s method for the roots of \( z^3 - 1 = 0 \). That is

\[
z_{n+1} = f(z_n) = \left(\frac{2}{3} + \frac{1}{3 z_n^3}\right) z_n, \quad n \geq 0,
\]

where \( 0 < |z_0| < \infty \) is arbitrary. Note that

\[
\zeta_1 = 1, \quad \zeta_2 = e^{i2\pi/3} = \frac{1}{2}(-1 + i \sqrt{3}), \quad \text{and} \quad \zeta_3 = e^{i4\pi/3} = \frac{1}{2}(-1 - i \sqrt{3}),
\]

are the roots of \( z^3 = 1 \).

**Your tasks:** Write a computer program to calculate the orbits \( \{z_n\}_{n=0}^{\infty} \). Then, for every initial point \( z_0 \), draw a colored dot at the position of \( z_0 \), where the colors are picked as follows:

\[
z_n \rightarrow \zeta_1, \text{ cyan.} \quad z_n \rightarrow \zeta_2, \text{ magenta.} \quad z_n \rightarrow \zeta_3, \text{ yellow.} \quad \text{No convergence, black.} \quad (4.4)
\]

**What do you see?** Do blow ups of the limit regions between zones.

**Hint.** Deciding that the sequence converges is easy: once \( z_n \) gets “close enough” to one of the roots, then the very design of Newton’s method guarantees convergence. Thus, given a \( z_0 \), compute \( z_N \) for some large \( N \), and check if \( |z_N - \zeta_j| < \delta \) for one

\(^5\) Numerically this means: choose a sufficiently fine grid in a rectangle, and pick every point in the grid. For example, select the square \(-2 < x < 2 \) and \(-2 < y < 2 \), where \( z_0 = x + iy \).
of the roots and some “small” tolerance $\delta$ — which does not have to be very small, in fact $\delta = 0.25$ is good enough. You can get pretty good pictures with $N = 50$ iterations on a $150 \times 150$ grid. A larger $N$ is needed when refining near the boundary between zones.

Hint. If you use MatLab, do not plot “points”. Instead, plot “regions”, where the color of each pixel is decided by $z_0$ — use the command `image(x, y, C)` to plot. Why? Because using points leaves a lot of unpainted space in the figure, and gives much larger file sizes.

4.2 Answer for problem 14.10.03

Figures 4.1 and 4.2 show the results of our calculations. Note the fractal nature of the boundary between the basins of attraction for each root: as we zoom in, the object appears as a smaller (but distorted) copy of itself. Non-trivial

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**Figure 4.1:** (Problem 14.10.03). Convergence zones for the $z^3 = 1$ Newton’s map iterates. Color scheme in (4.4), with a $500 \times 500$ pixel grid. Left: $N = 100$ iterations, for $-2 < x, y < 2$. The crosses are the roots $\zeta_j$. Right: $N = 200$ iterations, for $0.2 < x < 0.6$ and $0.4 < y < 0.8$.

**Figure 4.2:** (Problem 14.10.03). See figure 4.1. Further blow ups with $N = 300$ iterations. Left: region $0.35 < x < 0.45$ and $0.4 < y < 0.5$. Right: region $0.39 < x < 0.42$ and $0.44 < y < 0.47$. 
self-similarity\textsuperscript{6} is the hallmark of a fractal. Sets like this (boundaries between convergence regions of complex analytic iterations) are called \textbf{Julia sets}. \par


The orbits within the Julia set are chaotic. These orbits are, generally, not periodic (but recurrent), and small differences in $z_n$ grow exponentially with $n$ (sensitive dependence on initial conditions). However, computing these orbits is extremely hard, as perturbations out of the Julia set make the resulting orbit convergent. \par

\textbf{THE END.}