Answers to P-Set # 05, (18.353/12.006/2.050)j  
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Contents

1 Problem 05.01.02 - Strogatz (Asymptotic behavior as \( t \to \infty \))  
  1.1 Problem 05.01.02 statement .................................................. 2  
  1.2 Problem 05.01.02 answer ...................................................... 2  

2 Problem 06.01.08 - Strogatz (Computer generated phase portrait)  
  2.1 Problem 06.01.08 statement .................................................. 2  
  2.2 Problem 06.01.08 answer ...................................................... 2  

3 Problem 21.10.15 - Dipole fixed point  
  3.1 Problem 21.10.15 statement .................................................. 3  
  3.2 Problem 21.10.15 answer ...................................................... 3  

4 Problem 06.01.11 - Strogatz (Computer generated phase portrait)  
  4.1 Problem 06.01.11 statement .................................................. 4  
  4.2 Problem 06.01.11 answer ...................................................... 4  

5 Problem 06.01.12 - Strogatz (Saddle connections)  
  5.1 Problem 06.01.12 statement .................................................. 5  
  5.2 Problem 06.01.12 answer ...................................................... 5  

6 Problem 07.02.06 - Strogatz (Find the potential for a gradient system)  
  6.1 Problem 07.02.06 statement .................................................. 7  
  6.2 Problem 07.02.06 answer ...................................................... 7  

7 Problem 16.10.06 Systems both potential and Hamiltonian  
  7.1 Problem 16.10.06 statement .................................................. 8  
  7.2 Problem 16.10.06 answer ...................................................... 8  

List of Figures

2.1 Problem 06.01.08. Phase plane portrait for the van der Pol oscillator ............... 2  
3.1 Problem 21.10.15. Phase plane portrait for the "Dipole fixed point" system .......... 3  
4.1 Problem 06.01.11. Phase plane portrait for the "Parrot" system ....................... 5  
5.1 Problem 06.01.12. Two connected saddles in the plane ........................... 6  
5.2 Problem 06.01.12. Two non-connected saddles in the plane ......................... 6  

Note: for problems from Strogatz’s book, if the version here differs from the one in the book do the version here.
1 Problem 05.01.02 - Strogatz (Asymptotic behavior as $t \to \infty$)

1.1 Statement for problem 05.01.02

Consider the system $\dot{x} = ax$, $\dot{y} = -y$, where $a < -1$. Show that all trajectories become parallel to the $y$-direction as $t \to \infty$, and parallel to the $x$-direction as $t \to -\infty$.

*Hint*: Examine the slope $\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}}$.

*Strictly speaking, not all trajectories satisfy these two statements. What are the exceptions?*

1.2 Answer for problem 05.01.02

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = -\frac{1}{a} \frac{y}{x} = -\frac{c_y}{a c_x} \exp\left(-(1 + a) t\right),$$

where we have used the fact that $x = c_x e^{at}$ and $y = c_y e^{-t}$ — for some constants $c_x$ and $c_y$. Hence, since $1 + a < 0$,

1. As $t \to \infty$, $\frac{dy}{dx} \to \pm \infty$, and the trajectory becomes parallel to the $y$-axis.

*Exception: $c_y \neq 0$ is needed for this to apply.*

2. As $t \to -\infty$, $\frac{dy}{dx} \to 0$, and the trajectory becomes parallel to the $x$-axis.

*Exception: $c_x \neq 0$ is needed for this to apply.*

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2 Problem 06.01.08 - Strogatz (Computer generated phase portrait)

2.1 Statement for problem 06.01.08

Plot a computer generated phase plane portrait for the van der Pol oscillator

$$\frac{dx}{dt} = y, \quad \text{and} \quad \frac{dy}{dt} = -x + y(1-x^2). \quad (2.1)$$

2.2 Answer for problem 06.01.08

See figure 2.1.

![van der Pol oscillator: $x_t = y$, $y_t = -x + (1-x^2)y$.](image)

*Figure 2.1: (Problem 06.01.08). Phase plane portrait for the van der Pol oscillator.*
3 Problem 21.10.15 - Dipole fixed point

3.1 Statement for problem 21.10.15

Plot a computer generated phase plane portrait for the “Dipole fixed point” system

\[
\frac{dx}{dt} = 2xy \quad \text{and} \quad \frac{dy}{dt} = y^2 - x^2. \tag{3.1}
\]

Extra Task #1. Find the critical points for this system, and linearize near them. What do the linearized equations tell you about the behavior near the critical points?

Extra Task #2. Use the generated phase plane portrait to compute the index of the critical points.

Extra Task #3. The generated phase plane portrait should suggest that the orbits for this system are circles.\(^1\) In fact any circle tangent to the y-axis at the origin would seem to be an orbit. Show that this is correct.

Hint for #3. Write a function \(E\) whose level curves are the circles tangent to the y-axis at the origin, and show that \(E\) is conserved. You will not be able to avoid the fact that \(E\) will have singularities somewhere on the y-axis\(^2\) (the best you can do is having \(E\) singular at the origin). This is related to the fact that the y-axis is the circle tangent to the y-axis at the origin, whose radius is infinity — while the origin itself corresponds to a zero radius.

3.2 Answer for problem 21.10.15

See figure 3.1 for a computed generated phase plane portrait. Note that, because (3.1) is invariant under \(x \mapsto ax, y \mapsto ay\), and \(t \mapsto t/a\) \((a > 0\) some constant), the phase plane portrait is invariant under stretching — hence once we know what happens in a neighborhood of the origin, we know what happens everywhere.

Extra Task #1. The system has only one critical point, the origin. The linearized equations there are \(\dot{x} = 0\), which tells us nothing about the behavior of the system near the critical point.

Extra Task #2. The index of the origin in figure 3.1 is 2. This follows because, as we go around the origin counterclockwise (say, starting on the positive real axis), in each quarter turn of the path, the flow vector rotates (also counterclockwise) by a half turn.

Extra Task #3. The circle of radius \(|r|\), tangent to the y-axis at the critical point, is given by \((x-r)^2 + y^2 = r^2\), where the sign of \(r\) determines which side of the y-axis the circle is. Solving for \(r\) yields

\[
r = \frac{x^2 + y^2}{2x}, \quad \text{or} \quad \frac{1}{r} = \frac{2x}{x^2 + y^2}. \tag{3.2}
\]

\(^1\) In MatLab, use “axis square” when plotting so there are no distortions.
\(^2\) Not an issue, since the y-axis can be analyzed separately with ease.
To show that these are conserved quantities, we use (3.1) to obtain: \( \frac{d}{dt}(x^2 + y^2) = 2y(x^2 + y^2) \). Hence
\[
\frac{\dot{x}}{x} = 2y = \frac{d(x^2 + y^2)}{x^2 + y^2}
\implies x \text{ and } x^2 + y^2 \text{ are proportional to each other along trajectories.} \tag{3.3}
\]
This proves that both \( E_1 = r \) and \( E_2 = 1/r \) are conserved quantities — note that \( E_1 \) is singular on the whole \( y \)-axis, while \( E_2 \) is singular at the critical point only. Hence the circles (with the origin missing) \( r = \text{constant}, -\infty < r \neq 0 < \infty \), are orbits — these circles are contained within either \( x > 0 \) or \( x < 0 \). The \( y \)-axis requires a separate argument.

Alternatively, we can look for solutions to the equation of the form
\[
x = r (1 + \cos \phi) \quad \text{and} \quad y = r \sin \phi, \tag{3.4}
\]
where \( r \) is a constant and \( \phi = \phi(t) \). It is easy to check that this reduces the system to the single equation
\[
\dot{\phi} = -2r (1 + \cos \phi), \tag{3.5}
\]
with a semi-stable critical point at \( \phi = \pm \pi \). It should be clear that (3.4 – 3.5) yield the phase portrait in figure 3.1, except for the \( y \)-axis, which requires a separate argument.

**Remark (on global attractors that are not Liapunov stable).** An inspection of figure 3.1 shows that there is exactly one orbit that never returns to the critical point (positive \( y \)-axis). Were it not for this orbit, this critical point would be an example of a global attractor that is not Liapunov stable. However, it is easy to construct an example by “projecting” the phase portrait in figure 3.1 onto a sphere, as follows:

1. Pick a point on the sphere and call it “the origin” (this will be the critical point).
2. Draw a straight line tangent to the sphere at the critical point, and select a direction along the line.
3. Let the orbits be the intersections of the sphere with any plane that contains the line selected in item 2.
4. Let the flow direction along any of the orbits in item 3 be the same as that selected for the line in item 2.

This second system is an example of a global attractor that is not Liapunov stable.

4 Problem 06.01.11 - Strogatz (Computer generated phase portrait)

4.1 Statement for problem 06.01.11

Plot a computer generated phase plane portrait for the “Parrot” system
\[
\frac{dx}{dt} = y + y^2 \quad \text{and} \quad \frac{dy}{dt} = -x + \frac{1}{5}y - xy + \frac{6}{5}y^2, \tag{4.1}
\]
in some “large” square, say: \(-6 \leq x, y \leq 6\).

**A.** Note how all the orbits eventually turn back towards the origin, to form the “parrot’s eye.”

**B.** Linearize the system near \((x, y) = (0, 0)\), and **show that it is an unstable spiral.**

How are **A** and **B** consistent? What is going on near the eye?

4.2 Answer for problem 06.01.11

The linearized system near the origin is given by \( \dot{x} = y \) and \( \dot{y} = -x + \frac{1}{5}y \). It has the eigenvalues \( \lambda = \frac{1+i\sqrt{29}}{10} \). Figure 4.1 shows the plots. A **stable limit cycle exists near the critical point;** it makes the “edge” of the eye.
Figure 4.1: (Problem 06.01.11). Phase plane portrait for the "Parrot" system in (4.1). The picture on the right shows a detail near the limit cycle (Parrot’s eye).

5 Problem 06.01.12 - Strogatz (Saddle connections)

5.1 Statement for problem 06.01.12

A certain system is known to have exactly two fixed points, both of which are saddles. Sketch phase portraits in which

a. There is a single trajectory that connects the saddles.

b. There is no trajectory that connects the saddles.

5.2 Answer for problem 06.01.12

We begin by noticing that

1. The only way that the two saddles can have a trajectory that connects them is if one of the saddle’s unstable manifolds is the stable manifold for the other saddle. Furthermore:

   1a. There cannot be any homoclinic connections. Why? Because this would require an extra fixed point inside the “loop” formed by the homoclinic connection (index theory).

   Thus if there is only one connection, the remaining stable/unstable manifolds must all be distinct (this means 7 separate curves: 6 semi-infinite ones starting/ending at one of the saddles, and one connecting them). Modulo deformations this yields the picture on the left panel of figure 5.1. Up to topology, this picture is the unique answer to part a. Furthermore:

   1b. No further connections can be added (i.e.: one is the maximum allowed). The reason is index theory. Two connections would create a “loop” (cycle graph), which would require an extra fixed point inside it.

2. On the other hand, if there is no connections between the saddles, all the 8 stable/unstable manifolds must be distinct curves. This is, in fact, enough to guarantee no connections. The picture on the left panel of figure 5.2 gives an example of this. However, the allowed topology in the answer to part b is not unique — can you think of other alternatives?
3. Finally, notice: from the above it should be obvious that: the key to answer this problem correctly is to sketch what the stable and unstable manifolds for the two saddles do. If your answer does not include the stable and unstable manifolds, then it is not answering the questions.

Instead of just sketching the phase portraits, below we construct systems with the desired characteristics. The easiest way to do this is to use conservative Hamiltonian systems:

\[
\frac{dx}{dt} = -\frac{\partial H}{\partial y}, \quad \text{and} \quad \frac{dy}{dt} = \frac{\partial H}{\partial x}.
\] (5.1)

Then we only need to specify the energy function \( H = H(x, y) \), with the orbits given by the level curves for the surface \( z = H(x, y) \).

When considering critical points for the system in (5.1), it is helpful to think of the surface \( z = H(x, y) \) as describing a mountain range. Then a saddle occurs at the low point of any ridge connecting two peaks, while the peaks themselves (or the very bottoms of valleys) correspond to centers. Thus, in order to get a specified phase plane portrait, the trick is to put peaks, valleys and ridges appropriately (note that centers can only be avoided by having the peaks and valley bottoms at “infinity”).

![Figure 5.1](Problem 06.01.12). Left: phase plane portrait for equation (5.1), with \( H = y(x^2 - 1) \). Stable and unstable manifolds for the saddles shown in thick solid (blue) lines. Typical orbits shown in thin solid (green) lines. Arrows indicate the flow direction. Right: the surface \( z = H(x, y) \).

![Figure 5.2](Problem 06.01.12). Left: phase plane portrait for equation (5.1), with \( H = y^3 - 3y(1 + x^2) \). Stable and unstable manifolds for the saddles shown in thick solid (blue) lines. Typical orbits shown in thin solid (green) lines. Arrows indicate the flow direction. Right: the surface \( z = H(x, y) \).

For part (a) of this problem, a simple way to achieve the desired result is to
A. Choose the curves that will be the stable and unstable manifolds for the saddles (i.e.: the level curves at the same level as the saddles, in the the surface \( z = H(x, y) \)).

B. Choose \( H \) so it switches sign across the curves selected in A, and grows in size away from them (this second condition is so \( H \) has no maximums or minimums — thus, no centers occur).

We implement this idea by selecting the saddles to be at \((x, y) = (\pm 1, 0)\), and the curves to be given by \( x \equiv \pm 1 \) and \( y \equiv 0 \). Then take \( H = y(x^2 - 1) \). Figure 5.1 shows the results of this choice.

Part (b) of this problem is a bit trickier. We need two saddles that occur at different levels in the mountain range, and nothing else. To get the first saddle, place a long ridge (from \( x = -\infty \) to \( x = \infty \)), with a low point at \( x = 0 \) (this will create a saddle at the position of the low point). Right next to this ridge, place a long valley (notice that there is a duality between valleys and ridges), with a high point at \( x = 0 \) (this will create a saddle at the position of the high point). If done carefully, the surface constructed in this fashion will have the right properties; since it will have a shape like the one shown on the right in figure 5.2. How do we get a formula that gives rise to such a surface? Well, notice that each of the \( x = \text{constant} \) cross-sections of the surface looks like a cubic, while each of the \( y = \text{constant} \) cross-sections looks like a parabola. Thus, take a cubic in \( y \), with coefficients that depend quadratically on \( x \). For example: \( H = y^3 - 3(1 + x^2)y \) does the trick. Figure 5.2 shows the results of this choice.

6 Problem 07.02.06 - Strogatz
(Find the potential for a gradient system)

6.1 Statement for problem 07.02.06

Given that a system is a gradient system, here is how to find its potential function \( V \).

\[ \dot{x} = f(x, y) \quad \text{and} \quad \dot{y} = g(x, y). \]

Then, \( \dot{x} = -\nabla V \) implies \( f(x, y) = -\frac{\partial V}{\partial x} \) and \( g(x, y) = -\frac{\partial V}{\partial y} \). These two equations may be ”partially integrated” to find \( V \). Use this procedure to find \( V \) for the following gradient systems.

\[ a) \dot{x} = y^2 + y \cos(x) \quad \text{and} \quad \dot{y} = 2xy + \sin(x). \]
\[ b) \dot{x} = 3x^2 - 1 - \exp(2y) \quad \text{and} \quad \dot{y} = -2x \exp(2y). \]

6.2 Answer for problem 07.02.06

\[ a) f = y^2 + y \cos(x) \quad \text{and} \quad g = 2xy + \sin(x). \]

Integrating \( -\frac{\partial V}{\partial x} = y^2 + y \cos(x) \), gives \( V(x, y) = -y^2x - y \sin(x) + Y(y) \). Taking the first derivative of this last equation with respect to \( y \), and setting the result equal to the expression for \( g(x, y) \) gives \( -\frac{\partial V}{\partial y} = 2yx + \sin(x) + \frac{dY}{dy} = 2xy + \sin(x). \) This is satisfied if \( Y \) is a constant. Thus we get the answer
\[ V(x, y) = -xy^2 - y \sin(x) + C, \quad \text{where} \quad C \text{ is an arbitrary constant}. \]

\[ b) f = 3x^2 - 1 - e^{2y} \quad \text{and} \quad g = -2xe^{2y}. \]

Following the same process as in part a, we first get (upon integrating the equation involving \( f \)) \( V(x, y) = -x^3 + x + xe^{2y} + Y(y). \) Substituting this into the equation involving \( g \) then gives \( -2xe^{2y} + \frac{dY}{dy} = -2xe^{2y}. \) Again, this is satisfied provided \( Y(y) \) is a constant. Thus we get the potential:
\[ V(x, y) = -x^3 + x + xe^{2y} + C, \quad \text{where} \quad C \text{ is an arbitrary constant}. \]
7 Problem 16.10.06 Systems both potential and Hamiltonian

7.1 Statement for problem 16.10.06
Consider the system \( \dot{x} = \cos x \cosh y = f \) and \( \dot{y} = \sin x \sinh y = g \).

a. Then \( f_y = g_x \), hence this is a gradient system.\(^3\) Find the potential \( V = V(x, y) \).

b. Show that the system is also Hamiltonian, and find the Hamiltonian \( H = H(x, y) \).

c. Show that the system has the complex form \( \dot{z} = \cos \bar{z} \), where \( z = x + iy \) and \( \bar{z} = x - iy \).

7.2 Answer for problem 16.10.06

a. We have \( f_y = \cos x \sinh y = g_x \). We can write \( f = -V_x \) and \( g = -V_x \), where \( V = -\sin x \cosh y \).

b. Since \( f_x = -\sin x \cosh y = -g_y \), the system is Hamiltonian,\(^4\) and we can write \( f = -H_y \) and \( g = H_x \), where \( H = -\cos x \sinh y \).

c. \( \dot{z} = \cos \bar{z} \) follows from the trigonometric identity \( \cos(x - iy) = \cos x \cosh y + i \sin x \sinh y \).

THE END.

\(^3\) Why? See exercise 7.2.5 (part a).

\(^4\) Why?