Hopf bifurcations using two timing and complex notation

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Abstract

In these notes we illustrate how the use of complex notation can dramatically simplify calculations for (at least some) problems. In particular, we show a Hopf bifurcation calculation. You should compare these notes with the section *Hopf bifurcation for second order scalar equations* in the *Hopf bifurcation* notes.

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1 Problem formulation

Consider a system in the plane dependent on a parameter r

$$\frac{d\vec{u}}{dt} = \vec{f}(\vec{u},\, r), \quad \vec{f} \text{ is smooth.} \tag{1.1}$$

Assume now that (1.1) *has an isolated critical point that, at some value $r = r_c$, changes stability: from a stable to an unstable spiral* (or the reverse). Without loss of generality, we will assume that the critical point is the origin, and that $r_c = 0$. Then, for $\vec{u}$ and r small we can write [1]

$$\frac{d\vec{u}}{dt} = A\,\vec{u} + \vec{I}_2(\vec{u}) + \vec{I}_3(\vec{u}) + r\,B\,\vec{u} + O(\epsilon^4,\, r^2\,\epsilon), \tag{1.2}$$

where $\epsilon = \|\vec{u}\|$, A and B are 2×2 matrices, $\vec{I}_2$ involves only quadratic terms in $\vec{u}$, and $\vec{I}_3$ involves only cubic terms in $\vec{u}$. Furthermore, because the origin is a center for $r = 0$, we know that A has a (complex) eigenvector $\vec{v}$, with eigenvalue $i\,\mu$ (where $\mu > 0$). That is:

$$A\,\vec{v} = i\,\mu\,\vec{v} \quad \Longleftrightarrow \quad A\,\vec{v}_1 = \mu\,\vec{v}_2 \text{ and } A\,\vec{v}_2 = -\mu\,\vec{v}_1, \tag{1.3}$$

where $\vec{v} = \vec{v}_1 - i\,\vec{v}_2$ ($\vec{v}_j$ real) and we can write any vector as a linear combination of the $\vec{v}_j$. In particular

$$\vec{u} = x\,\vec{v}_1 + y\,\vec{v}_2, \quad \text{so that} \quad A\,\vec{u} = \mu\,(-y\,\vec{v}_1 + x\,\vec{v}_2). \tag{1.4}$$

Note then that, in terms of the complex number $z = x + i\,y$, the action by A is equivalent to multiplication by $i\,\mu$. Hence we can write (1.2) in the equivalent *complex form*

[1] Because $\vec{u} = 0$ is assumed to be a critical point for all r small, there are no $O(r^n)$ terms.

$$\dot{z} = i\,\mu\,z + r\,(a_1\,z + a_2\,\overline{z}) + (b_1\,z^2 + b_2\,z\,\overline{z} + b_3\,\overline{z}^2) + (c_1\,z^3 + c_2\,z^2\,\overline{z} + c_3\,z\,\overline{z}^2 + c_4\,\overline{z}^3) + O(\epsilon^4,\, r^2\,\epsilon), \quad (1.5)$$

where (i) $\overline{z}$ denotes the complex conjugate and (ii) the a_j, b_j, and c_j are complex constants (for a generic system, they are unrestricted).

2 Two times expansion – no quadratic terms

Consider the situation where the quadratic terms in (1.5) vanish.[2] Then we assume $\boldsymbol{r = \nu\,\epsilon^2}$, $\nu = \pm 1$, so that the linear perturbation term and the cubic nonlinearity have the same size, and propose a two-times expansion of the form

$$z = \epsilon\,z_1(t,\,\tau) + \epsilon^3\,z_3(t,\,\tau) + \ldots \quad (2.1)$$

where $\boldsymbol{\tau = \epsilon^2\,t}$ and the dependence on t is periodic — it should be easy to see that no $O(\epsilon^2)$ terms are needed.[3] Then $\boldsymbol{z_0 = \mathcal{A}(\tau)\,e^{i\,\mu\,t}}$ and the $O(\epsilon^3)$ yield

$$\dot{z}_3 - i\,\mu\,z_3 = -\dot{\mathcal{A}}\,e^{i\,\mu\,t} + \nu\,a_1\,\mathcal{A}\,e^{i\,\mu\,t} + c_2\,|\mathcal{A}|^2\,\mathcal{A}\,e^{i\,\mu\,t} + \text{NRT}, \quad (2.2)$$

where the Non Resonant Terms (NRT) have t-dependences proportional to $e^{-i\,\mu\,t}$, $e^{-i\,3\,\mu\,t}$, and $e^{i\,3\,\mu\,t}$. Suppressing resonant terms then yields

$$\frac{d\mathcal{A}}{d\tau} = \nu\,a_1\,\mathcal{A} + c_2\,|\mathcal{A}|^2\mathcal{A}. \quad (2.3)$$

Write now $\mathcal{A} = \rho\,e^{i\,\phi}$, with ρ and ϕ real. Then (2.3) becomes

$$\frac{d\rho}{d\tau} = \left(\nu\,\text{Re}(a_1) + \text{Re}(c_2)\,\rho^2\right)\,\rho \quad \text{and} \quad \frac{d\phi}{d\tau} = \nu\,\text{Im}(a_1) + \text{Im}(c_2)\,\rho^2. \quad (2.4)$$

Note now that $\mathbf{Re}(\boldsymbol{a_1}) \neq \mathbf{0}$, because of the assumption that the origin switches stability and the expansion in (1.5) — which yields linearized equations $\dot{z} = i\,\mu\,z + r\,(a_1\,z + a_2\,\overline{z}) + O(r^2\,\epsilon)$. Thus, **the condition for a Hopf bifurcation is $\mathbf{Re}(\boldsymbol{c_2}) \neq \mathbf{0}$.** Then

1. If $\text{Re}(a_1)/\text{Re}(c_2) = \kappa^2 > 0$, a limit cycle with radius $\rho = \kappa$ arises for $\nu = -1$. The bifurcation is supercritical (soft) if $\text{Re}(a_1) < 0$, and subcritical (hard) if $\text{Re}(a_1) > 0$.

2. If $\text{Re}(a_1)/\text{Re}(c_2) = -\kappa^2 < 0$, a limit cycle with radius $\rho = \kappa$ arises for $\nu = 1$. The bifurcation is supercritical (soft) if $\text{Re}(a_1) > 0$, and subcritical (hard) if $\text{Re}(a_1) < 0$.

3. In either case, the second equation in (2.4) indicates that the angular frequency for the limit cycle, up to the order considered, is $\omega = \mu + r\,\text{Im}(a_1) + \epsilon^2\,\text{Im}(c_2)\,\kappa^2$.

Recall that for a supercritical bifurcation the limit cycle that arises is stable, while it is unstable for a subcritical bifurcation.

The End.

[2] We leave it as an exercise to consider the general case.

[3] They are needed when the quadratic terms in (1.5) do not vanish.