

# 18.385 MIT

## Hopf Bifurcations.

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September 25, 1999

### Abstract

In two dimensions a Hopf bifurcation occurs as a Spiral Point switches from stable to unstable (or vice versa) and a periodic solution appears. There are, however, more details to the story than this: **The fact that a critical point switches from stable to unstable spiral (or vice versa) alone does not guarantee that a periodic solution will arise,**<sup>1</sup> though one almost always does. Here we will explore these questions in some detail, using the method of multiple scales to find precise conditions for a limit cycle to occur and to calculate its size. We will use a second order scalar equation to illustrate the situation, but the results and methods are quite general and easy to generalize to any number of dimensions and general dynamical systems.

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<sup>1</sup>Extra conditions have to be satisfied. For example, in the damped pendulum equation:  $\ddot{x} + \mu\dot{x} + \sin x = 0$ , there are **no** periodic solutions for  $\mu \neq 0$  !

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# 1 Hopf bifurcation for second order scalar equations.

## 1.1 Reduction of general phase plane case to second order scalar.

We will consider here equations of the form

$$\ddot{x} + h(\dot{x}, x, \mu) = 0, \quad (1.1)$$

where  $h$  is a smooth and  $\mu$  is a parameter.

**Note 1** *There is not much loss of generality in studying an equation like (1.1), as opposed to a phase plane general system. For let:*

$$\dot{x} = f(x, y, \mu) \quad \text{and} \quad \dot{y} = g(x, y, \mu). \quad (1.2)$$

Then we have

$$\ddot{x} = f_x \dot{x} + f_y \dot{y} = f_x f + f_y g = F(x, y, \mu). \quad (1.3)$$

Now, from  $\dot{x} = f(x, y, \mu)$  we can, at least in principle,<sup>2</sup> write

$$y = G(\dot{x}, x, \mu). \quad (1.4)$$

Substituting then (1.4) into (1.3) we get an equation of the form (1.1).<sup>3</sup>

## 1.2 Equilibrium solution and linearization.

Consider now an **equilibrium solution**<sup>4</sup> for (1.1), that is:

$$x = X(\mu) \quad \text{such that} \quad h(0, X, \mu) = 0, \quad (1.5)$$

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<sup>2</sup>We can do this in a neighborhood of any point  $(x_*, y_*)$  (say, a critical point) such that  $f_y(x_*, y_*, \mu) \neq 0$ , as follows from the Implicit Function theorem. If  $f_y = 0$ , but  $g_x \neq 0$ , then the same ideas yield an equation of the form  $\ddot{y} + \tilde{h}(\dot{y}, y, \mu) = 0$  for some  $\tilde{h}$ . The approach will fail only if both  $f_y = g_x = 0$ . But, for a critical point this last situation implies that the eigenvalues are  $f_x$  and  $g_y$ , that is: **both real**! Since we are interested in studying the behavior of phase plane systems near a **non-degenerate** critical point switching from stable to unstable spiral behavior, this **cannot happen**.

<sup>3</sup>Vice versa, if we have an equation of the form (1.1), then defining  $y$  by  $y = G(\dot{x}, x, \mu)$ , for any  $G$  such that the equation can be solved to yield  $\dot{x} = f(x, y, \mu)$  (for example:  $G = \dot{x}$ ), then  $\dot{y} = G_{\dot{x}} \ddot{x} + G_x \dot{x} = g(x, y)$  upon replacing  $\dot{x} = f$  and  $\ddot{x} = -h$ .

<sup>4</sup>i.e.: a critical point.

so that  $x \equiv X$  is a solution for any fixed  $\mu$ . There is **no loss of generality** in assuming

$$X(\mu) \equiv 0 \quad \text{for all values of } \mu, \quad (1.6)$$

since we can always change variables as follows:  $x_{\text{old}} = X(\mu) + x_{\text{new}}$ .

The linearized equation near the equilibrium solution  $x \equiv 0$  (that is, the equation for  $x$  infinitesimal) is now:

$$\ddot{x} - 2\alpha\dot{x} + \beta x = 0, \quad (1.7)$$

where  $\alpha = \alpha(\mu) = -\frac{1}{2}h_x(0, 0, \mu)$  and  $\beta = \beta(\mu) = h_x(0, 0, \mu)$ .

The critical point is a **spiral point** if  $\beta > \alpha^2$ . The eigenvalues and linearized solution are then

$$\lambda = \alpha \pm i\tilde{\omega} \quad (1.8)$$

(where  $\tilde{\omega} = \sqrt{\beta - \alpha^2}$ ) and

$$x = ae^{\alpha t} \cos(\tilde{\omega}(t - t_0)), \quad (1.9)$$

where  $a$  and  $t_0$  are constants.

### 1.3 Assumptions on the linear eigenvalues needed for a Hopf bifurcation.

**Assume now:** At  $\mu = 0$  the **critical point changes from a stable to an unstable spiral point** (if the change occurs for some other  $\mu = \mu_c$ , one can always redefine  $\mu_{\text{old}} = \mu_c + \mu_{\text{new}}$ ).

Thus

$$\alpha < 0 \text{ for } \mu < 0 \text{ and } \alpha > 0 \text{ for } \mu > 0, \text{ with } \beta > 0 \text{ for } \mu \text{ small.}$$

In fact, **assume:**

- I.  $h$  is smooth.
- II.  $\alpha(0) = 0$ ,  $\beta(0) > 0$  and  $\frac{d}{d\mu}\alpha(0) > 0$ .<sup>5</sup>

We point out that, in addition, there are some restrictions on the behavior of the nonlinear terms near the critical point that are needed for a Hopf bifurcation to occur. See equation (1.22).

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<sup>5</sup>This last is known as the Transversality condition. It guarantees that the eigenvalues cross the imaginary axis as  $\mu$  varies.

## 1.4 Weakly Nonlinear things and expansion of the equation near equilibrium.

Our objective is to study what happens **near** the critical point, for  $\mu$  small. Since for  $\mu = 0$  the critical point is a **linear center**, the *nonlinear terms will be important in this study*. Since we will be considering the **region near** the critical point, the **nonlinearity will be weak**. Thus **we will use the methods** introduced in the *Weakly Nonlinear Things* notes.

For  $x, \dot{x}$ , and  $\mu$  small we can expand  $h$  in (1.1). This yields

$$\begin{aligned} \ddot{x} + \omega_0^2 x &+ \left\{ \frac{1}{2} A \dot{x}^2 + B \dot{x} x + \frac{1}{2} C x^2 \right\} + \\ &+ \frac{1}{6} \left\{ D \dot{x}^3 + 3E \dot{x}^2 x + 3F \dot{x} x^2 + G x^3 \right\} \\ &- 2p^2 \dot{x} \mu + \Omega x \mu + O(\epsilon^4, \epsilon^2 \mu, \epsilon \mu^2) = 0, \end{aligned} \quad (1.11)$$

where we have used that  $h(0, 0, \mu) \equiv 0$  and  $\alpha(0) = 0$ . In this equation we have:

- A.  $\omega_0^2 = \frac{\partial}{\partial x} h(0, 0, 0) = \beta(0) > 0$ , with  $\omega_0 > 0$ ,
- B.  $A = \frac{\partial^2}{\partial \dot{x}^2} h(0, 0, 0)$ ,  $B = \frac{\partial^2}{\partial \dot{x} \partial x} h(0, 0, 0)$ ,  $\dots$ ,
- C.  $p^2 = -\frac{1}{2} \frac{\partial^2}{\partial \dot{x} \partial \mu} h(0, 0, 0) = \frac{d}{d\mu} \alpha(0) > 0$ , with  $p > 0$ ,
- D.  $\Omega = \frac{\partial^2}{\partial x \partial \mu} h(0, 0, 0) = \frac{d}{d\mu} \beta(0)$ ,
- E.  $\epsilon$  is a measure of the size of  $(x, \dot{x})$ . Further: **both  $\epsilon$  and  $\mu$  are small**.

## 1.5 Explanation of the idea behind the calculation.

We now want to study the solutions of (1.11). The idea is, again: for  $\epsilon$  and  $\mu$  small the solutions are going to be dominated by the center in the linearized equation  $\ddot{x} + \omega_0^2 x = 0$ , with a *slow drift* in the amplitude and small changes to the period<sup>6</sup> caused by the higher order terms. Thus we will use an approximation for the solution like the ones in section 2.1 of the *Weakly Nonlinear Things* notes.

<sup>6</sup>We will not model these period changes here. See section 2.3 of the *Weakly Nonlinear Things* notes for how to do so.

## 1.6 Calculation of the limit cycle size.

An important point to be answered is: What is epsilon? (1.12)

This is a parameter that does not appear in (1.1) or, equivalently, (1.11). In fact, the only parameter in the equation is  $\mu$  (assumed small as we are close to the bifurcation point  $\mu = 0$ ).

Thus:

$\epsilon$  must be related to  $\mu$ .

 (1.13)

In fact,  $\epsilon$  **will be a measure of the size of the limit cycle**, which is a **property of the equation** (and thus a function of  $\mu$  and not arbitrary all).

However: We do not know  $\epsilon$  a priori! How do we go about determining it?

**The idea is:** If we choose  $\epsilon$  “too small” in our scaling of  $(x, \dot{x})$ , then we will be looking “too close” to the critical point and thus will find only spiral-like behavior, with no limit cycle at all. Thus, we **must choose  $\epsilon$  just large enough** so that the terms involving  $\mu$  in (1.11) (**specifically  $2p^2\mu\dot{x}$ , which is the leading order term in producing the stable/unstable spiral behavior**) are “balanced” by the nonlinearity in such a fashion that a limit cycle is allowed. In the context of Two–Timing this means we **want  $\mu$  to “kick in”** the damping/amplification term  $2p^2\mu\dot{x}$  at **“just the right level”** in the sequence of solvability conditions the method produces. Thus, going back to (1.11), we see that<sup>7</sup>

- The linear leading order terms  $\ddot{x} + \omega_0^2 x$  appear at  $O(\epsilon)$ .
- The first nonlinear terms (quadratic) appear at  $O(\epsilon^2)$ .

**However:** Quadratic terms produce **no resonances**, since  $\sin^2 \theta = \frac{1}{2}(1 - \cos 2\theta)$  and there are no sine or cosine terms. The same applies to  $\cos^2 \theta$  and to  $\sin \theta \cos \theta$ .

- Thus, the first resonances will occur when the cubic terms in  $x$  play a role  $\Rightarrow$  **we must have the balance**

$$O(x^3) = O(\mu\dot{x}), \quad (1.14)$$

$$\Rightarrow \mu = O(\epsilon^2).$$

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<sup>7</sup>This is a crucial argument that must be well understood. Else things look like a bunch of miracles!

## 1.7 The Two Timing expansion up to $O(\epsilon^3)$ .

We are now ready to start. The expansion to use in (1.11) is

$$x = \epsilon x_1(\tau, T) + \epsilon^2 x_2(\tau, T) + \epsilon^3 x_3(\tau, T) + \dots, \quad (1.15)$$

where  $0 < \epsilon \ll 1$ ,  $2\pi$ -periodicity in  $T$  is required,  $T = \omega_0 t$ ,  $\omega_0$  is as in (1.11)<sup>8</sup>,  $\tau$  is a *slow time* variable and  $\epsilon$  is related to  $\mu$  by  $\mu = \nu \epsilon^2$ , where  $\nu = \pm 1$  (which  $\nu$  we take depends on which “side” of  $\mu = 0$  we want to investigate).

What exactly is  $\tau$ ? Well, we need  $\tau$  to resolve resonances, which will not occur until the cubic terms kick in into the expansion  $\Rightarrow \tau = \epsilon^2 t$ . (This is exactly the same argument used to get (1.14)).

Then, with  $' = \frac{\partial}{\partial T}$ , (1.11) becomes:

$$\begin{aligned} \omega_0^2 x'' + \omega_0^2 x + \left\{ \frac{1}{2} A \omega_0^2 (x')^2 + B \omega_0 x x' + \frac{1}{2} C x^2 \right\} + \\ \frac{1}{6} \{ D \omega_0^3 (x')^3 + 3 E \omega_0^2 (x')^2 x + 3 F \omega_0 x' x^2 + G x^3 \} + \\ 2 \epsilon^2 \omega_0 x'_\tau - 2 \epsilon^2 \nu p^2 \omega_0 x' + \epsilon^2 \nu \Omega x + O(\epsilon^4) = 0. \end{aligned} \quad (1.16)$$

The rest is now a computational nightmare, but it is fairly straightforward. **Without getting into any of the messy algebra, this is what will happen:**

$$\begin{aligned} \text{At } O(\epsilon) \quad \omega_0^2 \{ x''_1 + x_1 \} = 0. \text{ Thus} \\ x_1 = a_1(\tau) e^{iT} + c.c. \end{aligned} \quad (1.17)$$

for some complex valued function  $a_1(\tau)$ . We use complex notation, as in the *Weakly Non-linear Things* notes.

$$\text{At } O(\epsilon^2) \quad \omega_0^2 \{ x''_2 + x_2 \} + \underbrace{\{ \text{quadratic terms in } x_1 \text{ and } x'_1 \}} = 0. \quad (1.18)$$

From the first bracket in (1.16), the quadratic terms here have the form:

$$C_1 a_1^2 e^{i2T} + C_2 |a_1|^2 + C_1^* (a_1^*)^2 e^{-2iT},$$

where  $C_1$  and  $C_2$  are **constants that can be computed in terms of  $\omega_0$ ,  $A$ ,  $B$  and  $C$** . Since the solution and equation are real valued,  $C_2$  is real. Here, as usual,  $*$  indicates the complex conjugate.

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<sup>8</sup>Same as the linear (at  $\mu = 0$ ) frequency. No attempt is made in this expansion to include higher order nonlinear corrections to the frequency.

No resonances occur and we have

$$x_2 = \left\{ \left( a_2(\tau)e^{iT} + \frac{1}{3}\omega_0^{-2}C_1a_1^2e^{i2T} \right) + c.c. \right\} - \omega_0^{-2}C_2 |a_1^2|. \quad (1.19)$$

$$\boxed{\text{At } O(\epsilon^3)} \quad \omega_0^2(x_3'' + x_3) + 2\omega_0x_{1\tau}' - 2\nu p^2\omega_0x_1' + \nu\Omega x_1 + \text{CNLT} = 0, \quad (1.20)$$

where **CNLT** stands for **Cubic Non Linear Terms**, involving products of the form  $x_2x_1$ ,  $x_2'x_1$ ,  $x_2x_1'$ ,  $(x_1')^3$ ,  $(x_1')^2x_1$ ,  $x_1'x_1^2$  and  $x_1^3$ . These will produce a term of the form

$\boxed{da_1^2a_1^*e^{iT} + c.c.}$  plus other terms whose  $T$  dependencies are: 1,  $e^{\pm 2iT}$  and  $e^{\pm 3iT}$ , none of which is resonant (forces a non periodic response in  $x_3$ ). Here

$$\boxed{d \text{ is a constant}} \text{ that can be computed in terms of } \omega_0, A, B, C, D, E, F \text{ and } G. \quad (1.21)$$

This is a big and messy calculation, but it involves only sweat. **In general**, of course,  $\text{Im}(d) \neq 0$ . The case  $\text{Im}(d) = 0$  is very particular, as it requires  $h$  in equation (1.1) to be just right, so that the particular combination of its derivatives at  $x = 0$ ,  $\dot{x} = 0$  and  $\mu = 0$  that yields  $\text{Im}(d)$  just happens to vanish. Thus

$$\boxed{\text{Assume a } \underline{\text{nondegenerate case}}: \text{Im}(d) \neq 0.} \quad (1.22)$$

For equation (1.20) to have solutions  $x_3$  periodic in  $T$ , the forcing terms proportional to  $e^{\pm iT}$  must vanish. This leads to the equation:

$$\boxed{2\omega_0i\frac{d}{d\tau}a_1 - 2\nu p^2\omega_0ia_1 + \nu\Omega a_1 + d|a_1^2|a_1 = 0.} \quad (1.23)$$

Then write

$$a_1 = \rho e^{i\theta}, \quad \text{with } \rho \text{ and } \theta \text{ real, } \rho > 0.$$

This yields

$$\boxed{\frac{d}{d\tau}\theta = \frac{1}{2}\nu\omega_0^{-1}\Omega + \frac{1}{2}\omega_0^{-1}\text{Re}(d)\rho^3} \quad (1.24)$$

and

$$\boxed{\frac{d}{d\tau}\rho = \nu p^2(1 - \nu q\rho^2)\rho,} \quad (1.25)$$

where  $\boxed{q = \frac{1}{2}\omega_0^{-1}p^{-2}\text{Im}(d).}$



Equation(1.24) provides a correction to the phase of  $x_1$ , since  $x_1 = 2\rho \cos(T + \theta)$ . The first term on the right of (1.24) corresponds to the changes in the linear part of the phase due to  $\mu \neq 0$ , away from the phase  $T = \omega_0 t$  at  $\mu = 0$ . The second term accounts for the nonlinear effects.

The second equation (1.25) above is more interesting. First of all, it reconfirms that for  $\mu < 0$  (that is,  $\nu = -1$ ) the critical point ( $\rho = 0$ ) is a stable spiral, and that for  $\mu > 0$  (that is,  $\nu = 1$ ) it is an unstable spiral. **Further**

If $\text{Im}(d) > 0$ .	Then a <b>stable limit cycle</b> exists for $\mu > 0$ (i.e. $\nu = 1$ ) with $\rho = \sqrt{2\omega_0 p^2 (\text{Im}(d))^{-1}}$ . <b>Supercritical (Soft) Hopf Bifurcation.</b>	}	(1.26)
If $\text{Im}(d) < 0$ .	Then an <b>unstable limit cycle</b> exists for $\mu < 0$ (i.e. $\nu = -1$ ) with $\rho = \sqrt{-2\omega_0 p^2 (\text{Im}(d))^{-1}}$ . <b>Subcritical (Hard) Hopf Bifurcation.</b>		

Notice that  $\rho$  here is equal to  $\frac{1}{2\epsilon}$  the radius of the limit cycle.

### 1.7.1 Remark on the situation at the critical bifurcation value.

Notice that, **for**  $\mu = 0$  (critical value of the bifurcation parameter)<sup>9</sup> we can do a two timing analysis as above to verify what the nonlinear terms do to the center.<sup>10</sup> The calculations are exactly as the ones leading to equations (1.23)–(1.25), except that  $\nu = 0$  and  $\epsilon$  is now a small parameter (unrelated to  $\mu$ , as  $\mu = 0$  now) simply measuring the strength of the nonlinearity near the critical point. Then we get for  $\rho = \frac{1}{2\epsilon}$  radius of orbit around the critical point

$$\frac{d}{d\tau}\rho = -\frac{1}{2}\omega_0^{-1}\text{Im}(d)\rho^3. \quad (1.27)$$

From this the behavior near the critical point follows.

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<sup>9</sup>**Then the critical point is a center in the linearized regime.**

<sup>10</sup>This is the way one would normally go about deciding if a linear center is actually a spiral point and what stability it has.

Clearly  $\left\{ \begin{array}{l} \bullet \operatorname{Im}(d) > 0 \iff \text{Soft bifurcation} \iff \text{Nonlinear terms stabilize.} \\ \quad \text{For } \mu = 0 \text{ critical point is a stable spiral.} \\ \bullet \operatorname{Im}(d) < 0 \iff \text{Hard bifurcation} \iff \text{Nonlinear terms de-stabilize.} \\ \quad \text{For } \mu = 0 \text{ critical point is an unstable stable spiral.} \end{array} \right.$

### 1.7.2 Remark on higher orders and two timing validity limits.

As pointed out in the *Weakly Nonlinear Things* notes, Two Timing is generally valid for some “limited” range in time, here probably  $|\tau| \ll \epsilon^{-1}$ . This is because we have no mechanism for incorporating the higher order corrections to the period the nonlinearity produces. If we are only interested in calculating the limit cycle in a Hopf bifurcation (not its stability characteristics), we can always do so using the Poincaré–Lindsteadt Method. In particular, then we can get the period to as high an order as wanted.

### 1.7.3 Remark on the problem when the nonlinearity is degenerate.

What about the degenerate case  $\operatorname{Im}(d) = 0$  ?

In this case there may be a limit cycle, or there may not be one. To decide the question one must look at the effects of nonlinearities higher than cubic (going beyond  $O(\epsilon^3)$  in the expansion) and see if they stabilize or destabilize. If a limit cycle exists, then its size will not be given by  $\sqrt{|\mu|}$ , but something else entirely different (given by the appropriate balance between nonlinearity and the linear damping/amplification produced by  $\alpha \neq 0$  when  $\mu \neq 0$  in equation (1.7)). The details of the calculation needed in a case like this can be quite hairy. One must use methods like the ones in Section 2.3 of the *Weakly Nonlinear Things* notes because: even though the nonlinearity may require a high order before it decides the issue of stability, modifications to the frequency of oscillation will occur at lower orders.<sup>11</sup> We will not get into this sort of stuff here.

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<sup>11</sup>Note that  $\operatorname{Re}(d) \neq 0$  in (1.24) produces such a change, even if  $\operatorname{Im}(d) = 0$  and there are no nonlinear effects in (1.25).