Bifurcations for a Driven-Damped pendulum

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Abstract

These notes are a supplement for the stuff in §8.6 of Strogatz's book.

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1 Equations and basic analysis

The a-dimensional equation for a damped pendulum with an applied toque can be written in the form

$$\ddot{\phi} + \alpha\,\dot{\phi} + \sin\phi = I, \tag{1.1}$$

where $\phi = \phi(t)$ is the pendulum angle (measured from the bottom position, increasing in the same direction as the applied torque), $\boldsymbol{\alpha > 0}$ is the dissipation coefficient, and $\boldsymbol{I > 0}$ is the applied torque. Alternatively, we can write the equation as a system

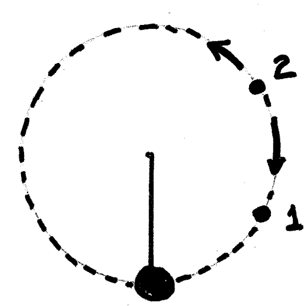
$$\dot{\phi} = y \quad \text{and} \quad \dot{y} = I - \sin\phi - \alpha\,y. \tag{1.2}$$

Note: because ϕ is an angle, the appropriate phase space for this system is the cylinder. However, it is often convenient to use the plane, remembering that ϕ and $\phi + 2\,\pi$ should be identified as the same angle.

Remark 1.1 (Intuition). *In terms of $z = \alpha\,y$ and $\tilde{t} = t/\alpha$ the system has the form*

$$\dot{\phi} = z \quad \textit{and} \quad \dot{z} = \alpha^2\,(I - \sin\phi - z). \tag{1.3}$$

Thus, when $\alpha \gg 1$, the dynamics becomes one dimensional $\qquad \dot{\phi} = I - \sin\phi. \qquad (1.4)$
As seen earlier, this 1-D system exhibits a saddle-node bifurcation where the periodic solution (ϕ goes around the circle with some period) is destroyed at $I = 1$.



For the full system, if α is large and $I < 1$, it should be "physically" obvious that: if the unstable equilibrium at $\phi = \phi_2$ (see §1.1 and the picture on the right) is very slightly perturbed, the pendulum will move to the equilibrium at $\phi = \phi_1$, either by swinging over the top, or going down. An it will do so in a monotone way, without any overshoot at $\phi = \phi_1$. The reason is that the net force is nonzero everywhere but at the equilibrium points, and the dissipation prevents the pendulum from accelerating.
As I grows the pattern persists and at $I = 1$, when ϕ_1 and ϕ_2 merge, the orbits described above yield an homoclinic connection. Beyond that the torque exceeds gravity everywhere, and the pendulum will start spinning. But there should be a critical solution where dissipation and the energy input from the torque exactly balance [yielding a periodic behavior]. Away from this, either dissipation lowers the mean energy, or the torque increases it. Hence this periodic solution should be attracting. ♣

We will show that this intuition is correct, and that *this system exhibits an infinite period bifurcation of a limit cycle,* at $I = 1$ and α large enough (with a globally attracting limit cycle for $I > 1$). In fact *the system also exhibits an homoclinic bifurcation of a limit cycle* when α is not large enough to sustain an infinite period bifurcation (at a fixed α below this critical value, this bifurcation happens at some $I < 1$).

1.1 Analysis of the critical points

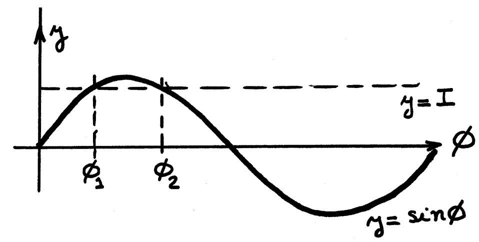
The conditions for a critical point are $0 = y = I - \sin\phi$. Hence:
There are no critical points for $I > 1$, there are two critical points for $I < 1$,
and a saddle-node bifurcation of critical points occurs at $I = 1$. (1.5)

Note that $I = 1$ *is the critical value at which the applied torque equals the maximum torque that gravity can provide* (at $\phi = \pi/2$).

Assume now that $I < 1$. Then there are two critical points:

1. $0 < \phi_1 < \frac{1}{2}\pi$, with $\cos\phi_1 = +\sqrt{1 - I^2} > 0$.

2. $\frac{1}{2}\pi < \phi_2 < \pi$, with $\cos\phi_2 = -\sqrt{1 - I^2} < 0$.



Both have $y = 0$, of course.

The matrix from the linearization near the critical points is
$$A = \begin{pmatrix} 0 & 1 \\ -\cos\phi_j & -\alpha \end{pmatrix}. \tag{1.6}$$

The eigenvalue equation is $0 = \det(\lambda - A) = \lambda^2 + \alpha\,\lambda + \cos\phi_j$,

with eigenvalues
$$\lambda = -\frac{1}{2}\,\alpha \pm \sqrt{\frac{1}{4}\,\alpha^2 - \cos\phi_j}. \tag{1.7}$$

Thus

3. ϕ_1 is a **stable spiral** for $\alpha^2 < 4\,\sqrt{1 - I^2} = 4\,\cos\phi_j$. (1.8)

4. ϕ_1 is a **stable node** for $\alpha^2 > 4\,\sqrt{1 - I^2} = 4\,\cos\phi_j$. (1.9)
This is *always the case* for $0 < 1 - I \ll 1$.

5. ϕ_2 is **always a saddle.** (1.10)

It is easy to check that, in all cases, the eigenvector is
$$\vec{v} = (1,\,\lambda)^T. \tag{1.11}$$

This leads to the phase portraits in figure 1.1 (the node requires $I^2 > 1 - \frac{1}{16}\alpha^4$) — note that the saddle is always to the right of the node ($\phi_2 > \phi_1$). For both the saddle and the node, as $I \uparrow 1$, the eigenvalue with the smallest size approaches zero (the attracting direction for the node, and the unstable direction for the saddle, both become parallel to the ϕ axis), while the other eigenvalue approaches $-\alpha$.

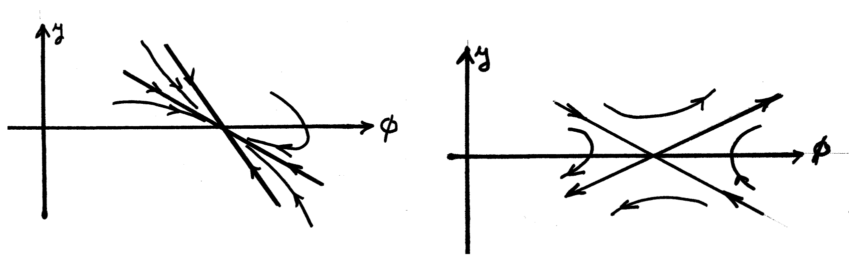


Figure 1.1: Phase portraits near the critical points ($I < 1$). Left picture: saddle. Right picture: node (assume $I^2 > 1 - \frac{1}{16}\alpha^4$).

1.2 Pseudo Energy

The system in (1.2) has the "pseudo energy" [1]

$$E = \frac{1}{2}\,y^2 + (1 - \cos\phi) - I\,(\phi - \pi) = E(y,\,\phi), \quad \text{which satisfies} \quad \boldsymbol{\dot{E} = -\alpha\, y^2 \leq 0}. \tag{1.12}$$

Figure 1.2 illustrates the level curves for E. Three cases occur: $\boldsymbol{(I < 1)}$ Then E has closed level curves near the stable critical point. Any orbit that starts in this area approaches the stable critical point as $t \to \infty$ — see (1.16). $\boldsymbol{(I = 1)}$ The closed level set curves disappear as the two critical points coalesce into a "strange" singularity — see §2. $\boldsymbol{(I > 1)}$ There are no closed level set curves, nor any singularities.

[1] We call E a "pseudo energy" because it is not defined in the proper phase space for the system: E is **not periodic** in ϕ.

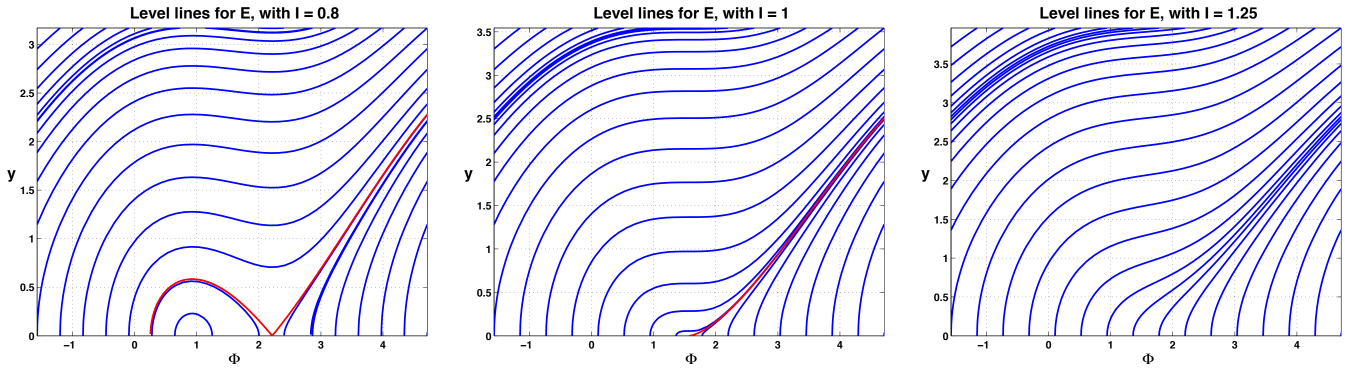


Figure 1.2: Level curves for the pseudo-energy E, defined in equation (1.12). From left to right, the three cases: $I < 1$ [$I = 0.8$], $I = 1$, and $I > 1$ [$I = 1.25$]. The curves in red are the special ones that go through the saddle for $I < 1$, or the degenerate critical point for $I = 1$. Finally, note: (i) The curves are invariant under $y \to -y$, hence only $y \geq 0$ is shown. (ii) $E(y, \phi + 2\pi) = E(y, \phi) - 2\pi I$, hence only $-\pi/2 \leq \phi \leq 3\pi/2$ is shown. (iii) For E large and ϕ bounded, the level curves satisfy $y \sim \sqrt{2E} + \frac{\cos(\phi)-1}{\sqrt{2E}} + \frac{I(\phi-\pi)}{\sqrt{2E}} + \ldots$ Hence only $y \leq 2\sqrt{I\pi}$ is shown (why this value: for any curve, if $y = 0$ at some ϕ_0, then $y = \pm 2\sqrt{I\pi}$ at $\phi_0 + 2\pi$).

Remark 1.2 (The limit $\alpha = 0$). *Then $\dot{E} = 0$, so it would seem that the system is conservative. However, this is not strictly correct: E is multiple valued in phase space, and a single valued function is required for a conservative system. Yet it is still true that the level curves of E are the orbits for the system.*
For solutions where the variation in ϕ is less than 2π, we can pick a single value for E, and the resulting "restricted" system is conservative. This applies for $I < 1$, in the region near the stable critical point where the level curves of E are closed — and only there. The solutions there are periodic, and are called "librations". Within this region the term $I(\pi - \phi)$ in E acts as a potential energy. However, "globally" $I(\pi - \phi)$ is not a potential energy, because knowing $\phi \bmod 2\pi$ is not enough to determine it, the history of the trajectory is needed to determine how many times the pendulum has "gone around".
Physics: the system is not isolated. The torque inputs, or subtracts, mechanical energy[2] as long as there is motion. For the librations the net input is zero, but for the other orbits E_{mech} varies from ∞ to some minimum (when $y = 0$), and then back up to ∞ as t varies in the range $-\infty \to \infty$. ♣

1.2.1 Properties of the pseudo energy E

p1. $E(y, \phi + 2\pi) = E(y, \phi) - 2\pi I.$ (1.13)
Proof. This is obvious.

p2. *E is strictly monotone decreasing along any orbit that is not a critical point.* (1.14)
Proof. $\dot{E} < 0$ unless $y = 0$. But, for an orbit $\neq$ critical point, $y = 0$ requires $I - \sin\phi = \dot{y} \neq 0$.
Hence, when $y = 0$: $\dot{E} = 0$, $\ddot{E} = -2\alpha y \dot{y} = 0$, and $\dddot{E} = -2\alpha \dot{y}^2 - 2\alpha y \ddot{y} = -2\alpha \dot{y}^2 < 0$.

Now write $$E = \frac{1}{2} y^2 + V(\phi), \text{ where } V(\phi) = 1 - \cos\phi - I(\phi - \pi). \quad (1.15)$$
Then the local minimums for V yield local minimums for E, while the local maximums yield saddles. Since $\boldsymbol{V' = \sin\phi - I}$ and $\boldsymbol{V'' = \cos\phi}$, a local minimum corresponds to $\sin\phi = I$ and $\cos\phi > 0$, while a local maximum is given by $\sin\phi = I$ and $\cos\phi < 0$. It follows that:

[2] That is: $E_{\text{mech}} = \frac{1}{2} y^2 + 1 - \cos\phi$.

p3. For $I < 1$, E has a local minimum at $(y = 0,\ \phi = \phi_1)$, and a saddle at $(y = 0,\ \phi = \phi_2)$. (1.16)
Then, because of (1.14), any orbit that starts close enough[3] to the stable critical point, approaches it as $t \to \infty$ (but see remark **1.3**).
What happens with the other orbits? Later, see §1.3, we will show that two cases are possible: (i) For α large enough, all the orbits (except for the stable manifolds for the saddles) approach the stable critical point as $t \to \infty$. (ii) For α small enough, a stable "limit cycle" exists as well.

p4. For $I = 1$ E has a "strange" extrema at $(y = 0,\ \phi = \pi/2)$ — see §2. (1.17)

p5. For $I > 1$ E has no extrema, and $\dot{E} < 0$ along all solutions. (1.18)
Later, see (1.24–1.26), we will show that in this case all the solutions approach a limit cycle as $t \to \infty$. As $t \to -\infty$ either: (i) $y \to \infty$ and $\phi \to -\infty$; or (ii) $y \to -\infty$ and $\phi \to \infty$.

Note that the "limit cycles" mentioned in **p3** and **p5** are not librations (i.e.: ϕ oscillates periodically), but periodic solutions that "wrap around" phase space (ϕ increases by 2π in each period). In fact librations are not possible for $\alpha > 0$ — see (1.19).

Remark 1.3 **(E decreasing is not enough to conclude that it has a limit as $t \to \infty$, even for $I < 1$).** *That is, because E is decreasing along orbits, we cannot conclude that the orbits approach a critical point as $t \to \infty$. The reason is that E is not bounded from below, so it can decrease forever, without reaching any limit. For example, consider a curve $y = \tilde{y}(\phi) > 0$, where y stays bounded and "does not oscillate too much" (i.e.: $d\tilde{y}/d\phi$ is never too large). From figure 1.2 it should be clear that E decreases as ϕ grows along such a curve. This is, in fact, what many of the solutions to the system do for t large enough (if α is not too large). And what all the solutions do when $I > 1$.* ♣

1.2.2 Consequences of the existence of the pseudo energy E

c1. Only non-trivial[4] periodic orbits that satisfy
$$\dot{\phi} = y > 0, \quad \phi(t+T) = \phi(t) + 2\pi, \quad \text{and} \quad y(t+T) = y(t), \tag{1.19}$$
where $T > 0$ is the period, are
possible (these orbits are periodic in the cylinder, not in the phase plane). *Librations cannot occur.*

Proof. (pr1) A periodic orbit that satisfies $\phi(t+T) = \phi(t)$ and $y(t+T) = y(t)$ violates (1.14). (pr2) A periodic orbit such that $\phi(t+T) = \phi(t) - 2\pi$ and $y(t+T) = y(t)$ also violates (1.14), because of (1.13). (pr3) It follows that a periodic orbit must satisfy $\phi(t+T) = \phi(t) + 2\pi$ and $y(t+T) = y(t)$. Hence, at least somewhere it must be $\dot{\phi} = y > 0$, say $\dot{\phi}(t_0) > 0$. Follow the orbit and suppose that at some $t_0 < t_1 < t_0 + T$, $\dot{\phi}(t_1) = 0$. Then, beyond t_1 the orbit would cross[5] to the region $\dot{\phi} = y < 0$, at which point ϕ would start to decrease. Clearly, it must cross back to $y > 0$ at some $t_2 > t_1$ to be able to satisfy $\phi(t_0 + T) = \phi(t_0) + 2\pi$. However, this creates a "trapping" situation that makes it impossible for the orbit to satisfy the periodicity criteria — see figure 1.3. We conclude that it must be $\dot{\phi} > 0$ everywhere.

c2. A corollary of (1.19) is that: *any periodic orbit can be written in the form $y = \tilde{y}(\phi)$, where $\tilde{y}$ is a positive 2π-periodic function.* (1.20)

The same type of arguments leading to (1.19–1.20) also yield:

[3] Specifically: within the region where the level curves for E are closed — see the left panel in figure 1.2.
[4] Critical points are trivially periodic, and we exclude them here.
[5] It has to cross. If not the point at t_1 is a critical point, and the orbit is a critical point, not a nontrivial periodic orbit.

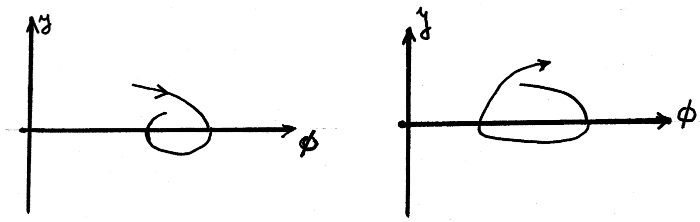


Figure 1.3: Proof of (1.19); the two scenarios that can occur once a "supposedly periodic" orbit crosses from $y > 0$ to $y < 0$. In both cases the orbit ends up trapping itself into a situation that prevents periodicity. Note that the "intermediate" scenario where the orbit closes on itself is not allowed — see (pr1).

c3. *Any cycle graph can be written in the form $y = \tilde{y}(\phi)$, where $\tilde{y}$ is a non-negative 2π-periodic function ($\tilde{y} > 0$ except at the critical point embedded in the cycle graph).* (1.21)
Note: a cycle graph is a sequence of orbits (all in the same sense) connecting critical points and making a closed curve. Here the only possibility is an homoclinic connection (for the saddle when $I < 1$, and for the degenerate critical point when $I = 1$).

c4. Uniqueness. *There can be at most one periodic orbit or a cycle graph, but not both.* (1.22)

This is a consequence of the fact that
$$\int_0^{2\pi} \tilde{y}\, d\phi = \frac{2\,\pi\,I}{\alpha}, \tag{1.23}$$
for any such orbit (or cycle graph), where $\tilde{y}$ is as in (1.20) or (1.21). If two existed, it would be $\tilde{y}_1 < \tilde{y}_2$ (they cannot cross), which negates (1.23).

Proof. Consider first a periodic orbit, and use (1.2) to write $\int_0^{2\pi} \dot{y}\, d\phi = 2\,\pi\,I - \alpha \int_0^{2\pi} y\, d\phi$. On the other hand, we can also write $\int_0^{2\pi} \dot{y}\, d\phi = \int_0^T \dot{y}\, \dot{\phi}\, dt = \int_0^T \dot{y}\, y\, dt = 0$, where T is the period. Then (1.23) follows.
The proof for the cycle graph is the same, except that $\int_0^T$ gets replaced by $\int_{-\infty}^{\infty}$ along the homoclinic connection for the critical point (where y vanishes).

1.3 Existence of periodic orbits, and bifurcations

Here we analyze what happens with the driven-damped pendulum as I varies, and characterize the bifurcations involving limit cycles that occur.

Let us **consider first the case $I > 1$** — see figure 1.4.

e1. *If $I > 1$ a periodic orbit, as in (1.19), exists.* (1.24)

Proof. Poincaré-Bendixson on the cylinder provides a short proof: For $y > (I+1)/\alpha$, $\dot{y} < 0$, and for $y < (I-1)/\alpha$, $\dot{y} > 0$. Now take constants $y_1 < (I-1)/\alpha$ and $y_2 > (I+1)/\alpha$. Then the orbits in the range $y_1 \le y \le y_2$ are trapped, and there are no critical points in this region. It follows that, as $t \to \infty$, the orbits must approach a periodic orbit (which therefore must exist).

Strogatz's book (§8.5, "Existence of a closed orbit") presents a different proof, based on a Poincaré map.

e2. *The periodic orbit in (1.24) is unique, and there are no cycle graphs. In fact, it is a globally attracting limit cycle.* (1.25)

Proof. The first statement follows from (1.22). Then, once we know it is unique (hence isolated, i.e.: a limit cycle) Poincaré-Bendixson tells us that every orbit in the region $y_1 \le y \le y_2$ approaches it as $t \to \infty$ (y_1 and y_2 as defined in the proof of (1.24)). But we can take y_1 arbitrarily large and negative, and y_2 arbitrarily large and positive. Hence all orbits approach the limit cycle as $t \to \infty$.

e3. *Let $I > 1$ and $t \to -\infty$. Then the orbits above (larger y) the limit cycle satisfy $y \to \infty$ and $\phi \to -\infty$. The orbits below satisfy $y \to -\infty$ and $\phi \to +\infty$.* (1.26)

Proof. Consider an orbit above the limit cycle. Then, as $t \to -\infty$ it either (i) must escape to infinity; or (ii) approach the limit cycle [the other options in Poincaré-Bendixson do not apply]. But it cannot be (ii), because this would imply a self-crossing of the orbit:

Think of $\Delta y = y_o - y_p > 0$, where $y_o(t)$ is y for the orbit and $y_p(t)$ is y for the limit cycle. If Δy vanishes as both $t \to \pm\infty$, there is no way to wrap the orbit on the cylinder without a self-crossing.

Hence it is (i), and because the orbit is above the limit cycle, this means $y \to \infty$, thus $\phi \to -\infty$ as well.

The proof for the orbits below the limit cycle is entirely similar.

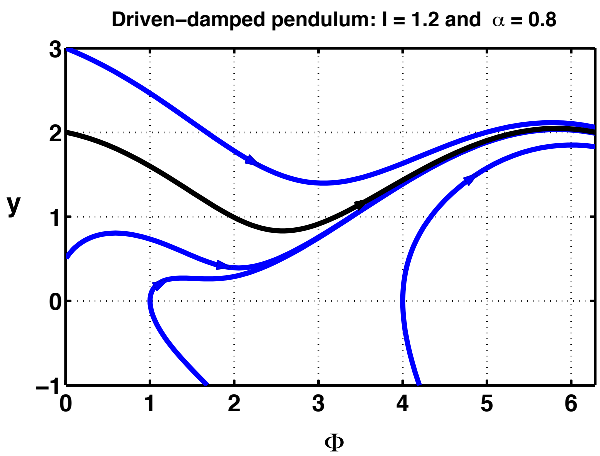


Figure 1.4: Phase plane portrait for the driven-damped pendulum with super-critical torque: $I > 1$. The portrait is dominated by a (stable and unique) globally attracting limit cycle, shown in black here.

Next let us **assume that $I = 1$** — see figure 1.5.
Then there is a degenerate critical point at $\phi = \frac{1}{2}\pi$ and $y = 0$. *Consider the unstable manifold leaving this critical point to the right* (saddle-like side, see §2.1). This orbit leaves the critical point with zero velocity (at $t = -\infty$) and moves towards $y > 0$ and ϕ increasing (see figure 2.8). **(#1)** Because the applied torque exceeds the force by gravity everywhere but at the critical point, this orbit cannot "stop" anywhere but at the critical point [after having ϕ grow by 2π] or beyond the critical point [overshoot] — see note to #1 below. **(#2)** If α is too small, then the overshoot scenario occurs, and the orbit arrives at $\phi = 5\pi/2$ with a positive speed $y > 0$. **(#3)** If we increase α from zero, it should be clear that the overshoot will get smaller and smaller, till a critical value α_{c1} is reached at which the overshoot vanishes and the orbit returns to the critical point along the stable manifold for the saddle-like side (specifically (a) in §2.1). *At this point a cycle graph is born.* **(#4)** Beyond α_{c1}, the dissipation is more than enough to prevent an overshoot, but the orbit must still reach $\phi = 5\pi/2$ (see (#1)). The only difference with (#3) is that it now returns to the critical point along the attracting stable manifold from the node-like side (specifically (c) in §2.1). *The cycle graph persists.*

Note to #1. The argument in #1 is the "physical" argument. The math. argument is: For t large and negative, the orbit is below the nullcline $y = (1 - \sin\phi)/\alpha$, and above $y = 0$. Hence both y and ϕ grow. This only stops once the orbit reaches the nullcline $y = (1 - \sin\phi)/\alpha$, at which point y starts to decrease. But the orbit cannot cross the nullcline $y = (1 - \sin\phi)/\alpha$ till $\phi \geq 5\pi/2$, so ϕ continues to increase.

Note to #2. The net torque on the pendulum is positive throughout. If the damping is too small, it will accelerate. Mathematically: for small α the situation should resemble the one for $\alpha = 0$, where the orbits are given by $E =$ constant. In particular, the orbit of interest here is the red curve in figure 1.2, middle panel, which clearly overshoots.

From the above we conclude

e4. For $\alpha > \alpha_{c1}$, at $I = 1$, an *infinite period bifurcation of the limit cycle occurs.* (1.27)

Note that, from (1.22), when the cycle graph appears, it must replace the limit cycle that exists for $I > 1$. Hence the infinite period bifurcation.

e5. *For $\alpha < \alpha_{c1}$, the limit cycle persists for $I = 1$.* (1.28)

Proof. For $I = 1$: use the overshooting orbit in **# 3** as the lower limit of a trapping region, to replace $y = y_1$ in the proof of item **e1**, and show that the limit cycle exists.

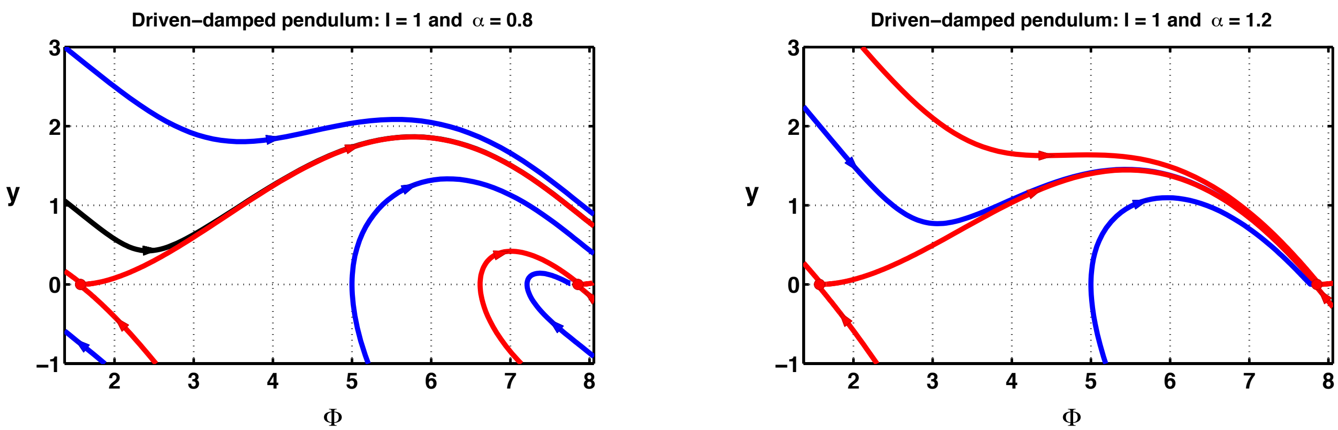


Figure 1.5: Phase plane portrait for the driven-damped pendulum with critical torque: $I = 1$. Red: critical point's stable and unstable manifolds. Black: limit cycle (if any). Blue: typical orbits. **Left panel: sub-critical damping, $\alpha < \alpha_{c1}$.** The unstable manifold from the critical point overshoots the critical point after one turn ($\Delta\phi = 2\pi$). A unique stable limit cycle exists. As $t \to \infty$ the solutions approach either the limit cycle or the critical point. However, the critical point is unstable. Thus, *in the presence of noise,* all the solutions eventually end up tracking the limit cycle (which, effectively, is then a global attractor). **Right panel: super-critical damping, $\alpha > \alpha_{c1}$.** The unstable manifold from the critical point reaches the critical point after one turn ($\Delta\phi = 2\pi$), and a cycle-graph results via an homoclinic connection. The limit cycle is replaced by the cycle-graph as an infinite period bifurcation occurs. As $t \to \infty$ the solutions approach the critical point. However, the critical point is unstable. Thus, *in the presence of noise:* (i) All the solutions eventually end up tracking the cycle-graph (which, effectively, is then a global attractor). (ii) The time period over which ϕ goes up by 2π is both long and random (since it depends on the noise kicking the solution from the stable side of the critical point, to the unstable side — see figure 2.8).

Now **consider what happens for $\alpha < \alpha_{c1}$, as I crosses below $I = 1$** — see figure 1.6.
Then the degenerate critical point splits into a sink and a saddle, and we can make the same arguments above in **#1-4** for the unstable manifold leaving the saddle to the right.[6] For $\alpha < \alpha_{c1}$ and I only slightly below 1, this orbit will overshoot when it goes around the cylinder (ϕ increases by 2π), and (just as in the proof of (1.28)) we can use it to show that the limit cycle still exists. However, as I decreases from $I = 1$,

[6]This orbit "merges" into the one considered in **#1-4** as $I \uparrow 1$.

the balance between applied torque and dissipation shifts in favor of the dissipation (so that the critical value of α at which the overshoot is lost gets smaller and smaller). Thus we conclude:

e6. *For $I < 1$ there exists $\alpha = \alpha_c(I)$, increasing from 0 to α_{c1} as I grows from 0 to 1, such that the limit cycle persists for $\alpha < \alpha_c$. There is no limit cycle (nor cycle graph) for $\alpha > \alpha_c$. At $\alpha = \alpha_c$ the limit cycle is destroyed through an homoclinic bifurcation.* (1.29)

The last statement follows because at $\alpha = \alpha_c$ the unstable manifold for the saddle becomes an homoclinic connection, and the limit cycle is replaced by a cycle graph. For α beyond α_c the dissipation is too large, and the unstable manifold from the saddle "cannot make it" around the cylinder; there is no cycle graph; nor any limit cycle (*physically:* the applied torque is too small to be able to make the pendulum go around continuously, forever). Math.: **to prove that there are no limit cycles (nor cycle graphs) for $\alpha > \alpha_c$,** consider the stable manifold

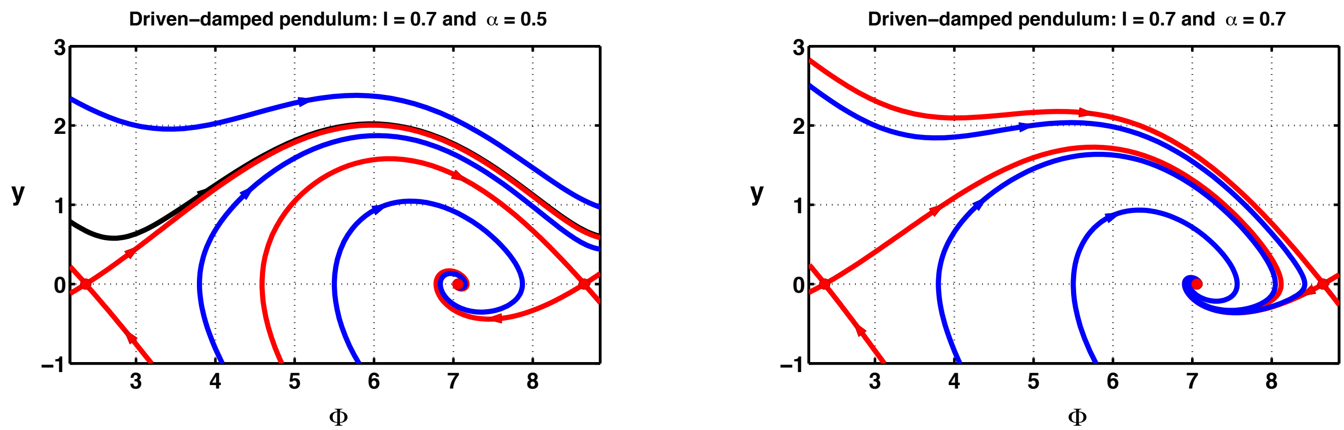


Figure 1.6: Phase plane portrait for the driven-damped pendulum with sub-critical torque: $I < 1$. Red: saddle's stable and unstable manifolds. Black: limit cycle (if any). Blue: typical orbits. **Left: sub-critical damping, $\alpha < \alpha_c$** (the right unstable manifold from the saddle overshoots the saddle after one turn: $\Delta\phi = 2\pi$). A unique stable limit cycle exists. As $t \to \infty$ the solutions approach either the limit cycle or the other critical point (a stable spiral or node). **Right: super-critical damping, $\alpha > \alpha_c$** (the right-unstable manifold from the saddle is unable to perform one turn and reach the saddle again). Instead, it turns around and starts approaching the other critical point (a stable spiral or node). Except for the saddle's stable manifolds, all the solutions approach the stable critical point as $t \to \infty$. **Not shown: Critical damping, $\alpha = \alpha_c$.** The right unstable manifold from the saddle reaches again the saddle after one turn. The limit cycle is replaced by a cycle graph (saddle homoclinic connection) and an homoclinic bifurcation of a the limit cycle occurs.

(on $y > 0$) for the saddle at $\phi = 2\pi + \phi_2$, $y = 0$. As we follow this curve backwards it arrives at $\phi = \phi_2$ (say, at $t = t_*$) with some value $y = y_* > 0$ (see the right panel in figure 1.6). Let $y = \overline{y}(\phi)$ be the curve, and repeat for it the calculation leading (1.23). The only difference is that in this case: $\int_{\phi_2}^{\phi_2+2\pi} \dot{y}\, d\phi = \int_{t_*}^{\infty} \dot{y}\, y\, dt = -\frac{1}{2}\, y_*^2$. Thus

$$\int_{\phi_2}^{\phi_2+2\pi} \overline{y}\, d\phi = \frac{1}{\alpha}\left(2\pi I + \frac{1}{2}\, y_*^2\right). \quad [A]$$

Any limit cycle (or cycle graph) would have to be above $\overline{y}$ (since it must satisfy $y \geq 0$, see (1.19–1.20)), and would not be able to satisfy (1.23) — because of [A].

The Poincaré-Bendixson theorem now yields (because the solutions cannot have $y \to \pm\infty$ as $t \to \infty$)

e7. For *$I < 1$ and $\alpha < \alpha_c$ the system is bi-stable:* (1.30)

As $t \to \infty$ the solutions are either attracted to the limit cycle, or the stable critical point (spiral or node). The only exceptions are the stable manifolds for the saddle.

e8. For $I < 1$ *and* $\alpha > \alpha_c$ *the stable critical point is a global attractor:* (1.31)
As $t \to \infty$ the solutions are attracted to the stable critical point (spiral or node). The only exceptions are the stable manifolds for the saddle.

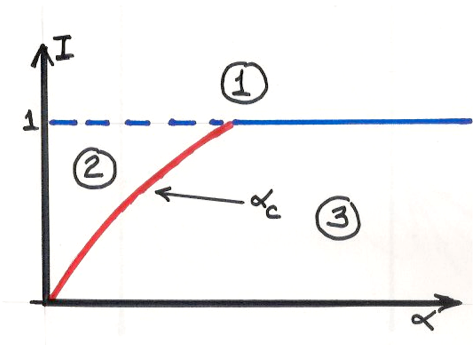


Figure 1.7: **Stability:** [Region (1), $\boldsymbol{I > 1}$] Stable, globally attracting, limit cycle. [Region (2), $\alpha < \alpha_c$ and $I < 1$] Bi-stable; solutions are attracted to either a stable limit cycle or a critical point (node or spiral). [Region (3), $\alpha > \alpha_c$ and $I < 1$] Stable, globally attracting, critical point (node or spiral). **Bifurcations:** (a) [$I = 1$ and $\alpha > \alpha_c$] Infinite period bifurcation of a limit cycle. (b) [$I = 1$ and $\alpha < \alpha_c$] Saddle-node bifurcation of a critical point. (a) [$I < 1$ and $\alpha = \alpha_c$] Homoclinic bifurcation of a limit cycle.

1.3.1 Hysteresis

Let $\alpha < \alpha_{c1} = \alpha(1)$ and vary I up an down (slowly). If we start with $I > 1$, the system settles into a limit cycle. As I decreases, the system stays in the limit cycle till $\alpha = \alpha_c(I)$ is reached, at which point the system settles to equilibrium at the stable critical point. On the other hand, if we start with I small, and increase it, the system stays at equilibrium till $I = 1$, and only then does it switch to a limit cycle.
Note that there is no hysteresis when $\alpha > \alpha_{c1}$.

2 The critical point at the critical torque I = 1

Here we study the (linearly degenerate) critical point for the driven-damped pendulum at critical torque (i.e.: the applied torque equals the maximum torque provided by gravity). Thus we look at the critical point at $\phi = \pi/2$ for $\ddot{\phi} + \alpha\,\dot{\phi} + \sin\phi = I$, with $I = 1$ ($\alpha > 0$ is the damping coefficient and $I > 0$ is the applied torque). At $I = 1$ a saddle-node bifurcation of critical points occurs, with no critical points for $I > 1$.

The critical point we are considering is degenerate, hence (to analyze it) we need to consider the leading order nonlinear effects. The equation for the the damped-driven pendulum at critical torque is

$$\ddot{\phi} + \alpha\,\dot{\phi} + \sin\phi = 1, \tag{2.1}$$

where $\phi = \phi(t)$ is the pendulum angle, $\alpha > 0$ is the damping coefficient and the critical torque is $I = 1$. We now write $\phi = \frac{1}{2}\pi + \varphi$, where φ is small, and expand. This yields

$$\ddot{\varphi} + \alpha\,\dot{\varphi} - \frac{1}{2}\,\varphi^2 = O(\varphi^4). \tag{2.2}$$

Keeping only the leading order contribution from the potential, and writing this as a system leads to:

$$\dot{\varphi} = y \quad \text{and} \quad \dot{y} = \frac{1}{2}\varphi^2 - \alpha\,y. \tag{2.3}$$

Next we do a **scale change:** $\varphi = \alpha^2 u$, $y = \alpha^3 v$, and $t = \tilde{t}/\alpha$. This yields (drop the tilde in the time derivatives)

$$\dot{u} = v \quad \text{and} \quad \dot{v} = \frac{1}{2}u^2 - v, \tag{2.4}$$

which shows that **the nature of the critical point does not depend on the amount of dissipation,** which only plays a scaling role. The **orbits** for this last system can also be characterized by the scalar ode

$$\frac{dv}{du} = \frac{1}{2\,v}\,u^2 - 1. \tag{2.5}$$

Figure 2.8 provides a numerically computed phase portrait for the system in (2.4). We justify and explain what is behind this picture in §2.1.

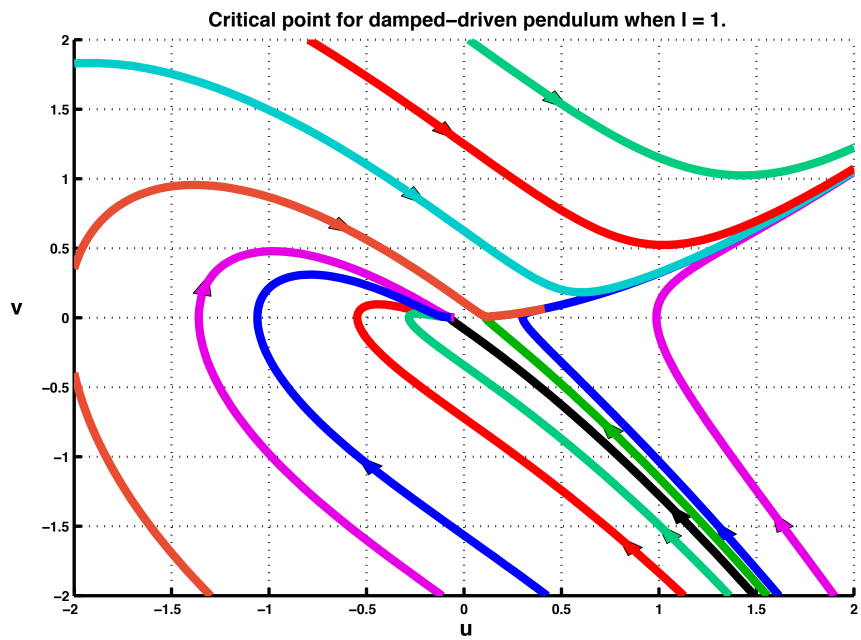


Figure 2.8: Phase plane portrait for equation (2.4). On the right side the portrait resembles that of a saddle, with an unstable manifold and two "bounding" stable manifolds. To the left of the stable manifolds, the portrait resembles that of a stable node.

2.1 Invariant manifolds and attracting curves

The nullclines, $v = 0$ and $v = \frac{1}{2}\,u^2$, for equation (2.4) play a fundamental role in the phase portrait, and can be clearly seen in figure 2.8. We also note that there are four special orbits:

(a) Two orbits/stable manifolds, which approach the critical point along the diagonal $v \sim -u$.

(b) An orbit/unstable manifold leaving the critical point (to the right) slightly below the nullcline $v = \frac{1}{2}\,u^2$.

(c) An orbit/stable manifold approaching the critical point (from the left) slightly above the nullcline $v = \frac{1}{2}\,u^2$. All orbits to the left of the special orbits in (a) end up coalescing with this orbit.

Note that *the critical point, roughly, resembles a saddle-point to the right of the special orbits in (a), and a node to the left of these special orbits.*

Node-like behavior: The solutions to the left of the stable manifolds in (a) first go up and to the left [region where $\dot{v} > 0$ and $\dot{u} < 0$], till they reach the nullcline $v = 0$, where they turn and start moving right (and still up). Then they reach the nullcline $v = \frac{1}{2}\,v^2$, where they turn again to move down. There they end up trapped from below by the nullcline and asymptotically approach it as $t \to \infty$ — we explain why this happens below in §2.1.1.

Saddle-like behavior: The solutions to the right of the stable manifolds in (a) either:

(d) Go down and to the right, till they cross the nullcline $v = \frac{1}{2}\,v^2$, where they turn and start going up. There they end up trapped from below by the nullcline, and asymptotically approach (from above) the orbit in (b) as $t \to \infty$.

(e) Go up and to the left, till they cross the nullcline $v = 0$, where they turn right, and also asymptotically approach (from below) the orbit in (b) as $t \to \infty$.

2.1.1 Scaling to detect the attracting curve near the critical point

In (2.4) scale the variables as follows $u = \epsilon\,\tilde{u}$ and $v = \epsilon^2\,\tilde{v}$, where $0 < \epsilon \ll 1$. Then the system becomes:

$$\dot{\tilde{u}} = \epsilon\,\tilde{v} \quad \text{and} \quad \dot{\tilde{v}} = \frac{1}{2}\tilde{u}^2 - \tilde{v}. \tag{2.6}$$

This system demonstrates what happens when the solutions get near the nullcline $v = \frac{1}{2}\,u^2$, with u small. We see that then u changes very slowly, while v is driven towards the nullcline. This is the behavior observed in figure 2.8 to the left of the stable manifolds in (a). It is also the behavior to the right, except that here we cannot consider the limit $t \to \infty$, because then u becomes large and the approximation fails.[7]

2.1.2 Expansions for the invariant manifolds

To find (approximate) expressions for the invariant manifolds, we turn to (2.4) and seek solutions where we expand v in powers of u — i.e.: we seek solutions that are smooth curves through the critical point. Using the method of dominant balances, we see that two expansions are possible:

At leading order $\frac{dv}{du} \sim -1$. This leads to
$$v = -u - \tfrac{1}{4}\,u^2 + O(u^3). \tag{2.7}$$
This corresponds to the stable manifolds in (a).

At leading order $\frac{1}{2}\,\frac{u^2}{v} \sim 1$. This leads to
$$v = \tfrac{1}{2}\,u^2 - \tfrac{1}{2}\,u^3 + O(u^4). \tag{2.8}$$
This corresponds to the manifolds in (b) and (c).

At leading order $\frac{dv}{du} \sim \frac{1}{2}\,\frac{u^2}{v}$. This balance leads to $v = O(u^{1.5})$, which is **not consistent** with $\frac{1}{2}\,\frac{u^2}{|v|} \gg 1$ (needed for the balance).

The End.

[7] In fact, (2.4) is valid only as long as u stays small.