Justifying the overdamped approximation for a bead on a rotating hoop

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We derived the equation of motion for $\phi(t)$ as

$$mr\frac{\mathrm{d}^2\phi}{\mathrm{d}t^2} + b\frac{\mathrm{d}\phi}{\mathrm{d}t} + mg\sin(\phi)\left(1 - \frac{r\omega^2}{g}\cos(\phi)\right) = 0. \tag{1}$$

To understand the relative sizes of inertia compared to the drag, gravitational and centrifugal forces, we introduce a dimensionless timescale $\tau = t/T$, where the typical time T is to be chosen so that

$$\frac{\mathrm{d}\phi}{\mathrm{d}\tau} \quad \text{and} \quad \frac{\mathrm{d}^2\phi}{\mathrm{d}\tau^2}$$

are both of size $O(1)$. By the chain rule, we have

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = \frac{1}{T}\frac{\mathrm{d}\phi}{\mathrm{d}\tau} \quad \text{and} \quad \frac{\mathrm{d}^2\phi}{\mathrm{d}t^2} = \frac{1}{T^2}\frac{\mathrm{d}^2\phi}{\mathrm{d}\tau^2}.$$

Substituting into (1) and re-arranging, we obtain

$$\frac{mr}{bT}\frac{\mathrm{d}^2\phi}{\mathrm{d}\tau^2} + \frac{\mathrm{d}\phi}{\mathrm{d}\tau} + \frac{mgT}{b}\sin(\phi)\big(1 - \gamma\cos(\phi)\big) = 0, \tag{2}$$

where $\gamma = r\omega^2/g$. For drag to balance the gravitational and centrifugal forces, we choose T so that $mgT/b = 1$, implying that $T = b/(mg)$. We then define the dimensionless parameter ϵ as the coefficient of the inertia time, namely

$$\epsilon = \frac{mr}{bT} = \frac{m^2}{b^2}gr.$$

Equation (2) is thus recast as

$$\epsilon\frac{\mathrm{d}^2\phi}{\mathrm{d}\tau^2} + \frac{\mathrm{d}\phi}{\mathrm{d}\tau} + \sin(\phi)\big(1 - \gamma\cos(\phi)\big) = 0. \tag{3}$$

If the dynamics are in a regime where $\epsilon \ll 1$ and $\frac{\mathrm{d}^2\phi}{\mathrm{d}\tau^2} = O(1)$ then we may neglect inertia effects, yielding the overdamped approximation

$$\frac{\mathrm{d}\phi}{\mathrm{d}\tau} = \sin(\phi)\big(\gamma\cos(\phi) - 1\big). \tag{4}$$

Paradoxically, this overdamped equation is only first-order in time, where as we started with a second-order equation. How do we resolve the disparity of the number of initial conditions?

The answer comes from the fact that $\frac{\mathrm{d}^2\phi}{\mathrm{d}\tau^2}$ may in fact be large over short timescales, and so our approximation for neglecting inertia is not justified (even if $\epsilon \ll 1$). To see this effect, we introduce a new timescale $\sigma = O(1)$ so that $\tau = \epsilon\sigma$. The chain rule supplies that

$$\frac{\mathrm{d}\phi}{\mathrm{d}\tau} = \frac{1}{\epsilon}\frac{\mathrm{d}\phi}{\mathrm{d}\sigma} \quad \text{and} \quad \frac{\mathrm{d}^2\phi}{\mathrm{d}\tau^2} = \frac{1}{\epsilon^2}\frac{\mathrm{d}^2\phi}{\mathrm{d}\sigma^2}.$$

Substituting into (3) yields

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}\sigma^2} + \frac{\mathrm{d}\phi}{\mathrm{d}\sigma} + \epsilon\sin(\phi)\big(1 - \gamma\cos(\phi)\big) = 0. \tag{5}$$

Over this short timescale, the gravitational and centrifugal effects are negligible when $\epsilon \ll 1$, and inertia is balanced by drag. Hence, the approximate second-order equation of motion for $\sigma = O(1)$ is

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}\sigma^2} + \frac{\mathrm{d}\phi}{\mathrm{d}\sigma} = 0.$$

Note that this equation is second-order in time! Solving this equation supplies that the velocity decays over an $O(1)$ timescale in σ, or, equivalently, an $O(\epsilon)$ timescale in τ. So any initial velocity supplied to the system is rapidly damped. After this initial transient, the inertia terms become negligible and the overdamped approximation is valid.

Conclusion: The overdamped limit neglects the short timescale effects immediately following initialisation in which the velocity rapidly changes, but the approximation is valid over long timescales.

This type of problem is known as a *singular perturbation problem* and arises naturally in many physical systems, such as viscous boundary layers in fluid mechanical systems.