Example 1: The weakly-damped oscillator

Consider the problem

$$\ddot{x} + 2\epsilon\dot{x} + x = 0, \tag{1}$$

with initial conditions $x(0) = 0$ and $\dot{x}(0) = 1$. We note that there are two timescales: the oscillatory $O(1)$ timescale and the slow $O(1/\epsilon)$ damping timescale. We define

$$\tau = t, \qquad T = \epsilon t,$$

and express the solution $x(t, \epsilon)$ as

$$x(t, \epsilon) = x_0(\tau, T) + \epsilon x_1(\tau, T) + O(\epsilon^2) \quad \forall \epsilon > 0. \tag{2}$$

Note that τ is the fast timescale and t is the slow timescale. By the chain rule, derivatives are transformed as

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial \tau} + \epsilon \frac{\partial}{\partial T}, \qquad \frac{\mathrm{d}^2}{\mathrm{d}t^2} = \frac{\partial^2}{\partial \tau^2} + 2\epsilon \frac{\partial^2}{\partial \tau \partial T} + \epsilon^2 \frac{\partial^2}{\partial T^2}. \tag{3}$$

Substituting the expansions (2)–(3) into (1) gives

$$\Big[\partial_{\tau\tau} x_0 + x_0\Big] + \epsilon\Big[\partial_{\tau\tau} x_1 + x_1 + 2\partial_{\tau T} x_0 + 2\partial_\tau x_0\Big] = O(\epsilon^2), \quad \forall \epsilon > 0 \tag{4}$$

and the initial conditions give (at $\tau = T = 0$)

$$\begin{aligned} 0 &= x_0 + \epsilon x_1 + O(\epsilon^2), \quad \forall \epsilon > 0, \\ 1 &= \partial_\tau x_0 + \epsilon[\partial_T x_0 + \partial_\tau x_1] + O(\epsilon^2), \quad \forall \epsilon > 0. \end{aligned}$$

It may appear that we have made the problem more complicated by introducing this additional timescale, but we have included the flexibility required to avoid the issues created by secular terms in the regular perturbation theory.

By grouping together powers of ϵ in (4), to leading order (i.e. $O(1)$ terms), we have

$$\partial_{\tau\tau} x_0 + x_0 = 0, \quad x_0(0,0) = 0, \quad \partial_\tau x_0(0,0) = 1.$$

We may solve this partial differential equation to obtain

$$x_0(\tau, T) = A(T)\sin\tau + B(T)\cos\tau, \tag{5}$$

where $A(T)$ and $B(T)$ are functions to be determined. The initial conditions require that $A(0) = 1$ and $B(0) = 0$.

Moving onto the $O(\epsilon)$ terms, we obtain

$$\partial_{\tau\tau} x_1 + x_1 = -2\big(\partial_{\tau T} x_0 + \partial_\tau x_0\big), \quad x_1(0,0) = 0, \quad \partial_\tau x_1(0,0) = -\partial_T x_0(0,0).$$

Using the previously found solution $x_0(\tau, T)$ given in equation (5), the partial differential equation becomes

$$\partial_{\tau\tau} x_1 + x_1 = -2\big(A'(T) + A(T)\big)\cos\tau + 2\big(B'(T) + B(T)\big)\sin\tau.$$

We note that this equation has secular terms, which would give undesirable solutions like $\tau \sin\tau$, etc. But here lies the beauty of the method of multiple scales: we are free to choose $A(T)$ and $B(T)$, so we define $A(T)$ and $B(T)$ so that the coefficients of the secular terms are zero. Specifically, we have

$$A'(T) + A(T) = 0, \qquad B'(T) + B(T) = 0,$$

where we recall that $A(0) = 1$ and $B(0) = 0$. We then solve for A and B to obtain $A(T) = e^{-T}$ and $B = 0$ for all T. Having used the secular terms at $O(\epsilon)$, we can now go back to (5) and write down the $O(1)$ solution:

$$x_0(\tau, T) = e^{-T} \sin \tau.$$

Hence, we have $x(t, \epsilon) = e^{-\epsilon t} \sin t + O(\epsilon)$, which yields the correct behavior for all time (unlike the regular perturbation method, which was only valid for $\epsilon t \ll 1$). We note that to improve the accuracy of the approximation, we could solve for $x_1(\tau, T)$ by finding the secular terms that would appear in the inhomogeneity of the partial differential equation for x_2.

Example 2: The van der Pol oscillator

We now consider the limit cycle of the van der Pol oscillator $\ddot{x} + \epsilon(x^2 - 1)\dot{x} + x = 0$ with $\epsilon \ll 1$. As we are considering the long-time behavior, we don't need to worry about initial conditions, but the method still works when we include initial values (see the handwritten notes). Using the expansions (2)–(3) and grouping together powers of ϵ, we obtain

$$\begin{aligned} O(1) &: \partial_{\tau\tau} x_0 + x_0 = 0, \\ O(\epsilon) &: \partial_{\tau\tau} x_1 + x_1 = -2\partial_{\tau T} x_0 - (x_0^2 - 1)\partial_\tau x_0. \end{aligned}$$

It's convenient here to write the solution $x_0(\tau, T)$ in terms of the complex amplitude $A(T) \in \mathbb{C}$, namely

$$x_0(\tau, T) = A(T)e^{i\tau} + \bar{A}(T)e^{-i\tau},$$

where the complex conjugate $\bar{A}$ ensures that x_0 is real.

At $O(\epsilon)$, we obtain

$$\begin{aligned} \partial_{\tau\tau} x_1 + x_1 &= -2i\big(A' e^{i\tau} - \bar{A}' e^{-i\tau}\big) - i\big(A^2 e^{2i\tau} + 2|A|^2 + \bar{A}^2 e^{-2i\tau} - 1\big)\big(Ae^{i\tau} - \bar{A}e^{-i\tau}\big) \\ &= \Big\{ ie^{i\tau}\big[-2A' - A(|A|^2 - 1)\big] + \text{c.c.} \Big\} + \text{non-secular terms}, \end{aligned}$$

where c.c. denotes the complex conjugate of the preceding term. To remove secular terms, we thus require $A(T)$ to satisfy

$$2A' = A - A|A|^2. \tag{6}$$

This equation is known as the *Stuart-Landau equation*. To make sense of the complex amplitude, we express A in complex polar form, namely $A(T) = r(T)e^{i\phi(T)}$ for real ϕ and $r > 0$. By substituting into (6) and considering real and imaginary parts, we obtain the system of differential equations

$$\begin{aligned} 2r' &= r - r^3, \\ \phi' &= 0. \end{aligned}$$

A stable fixed point of the system is $r = 1$ and $\phi = \phi_0$ (a constant). We then have

$$x_0(\tau, T) = \big[e^{i(\tau + \phi_0)} + e^{-i(\tau+\phi_0)}\big] = 2\cos(\tau + \phi_0).$$

As $x(t, \epsilon) = 2\cos(t + \phi_0) + O(\epsilon)$, the limit cycle has approximate period 2π and amplitude 2.