On the existence of limit cycles

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Gradient systems

Suppose there exists a continuously differentiable, single-valued, scalar potential function $V(\boldsymbol{x})$ so that $\dot{\boldsymbol{x}} = -\nabla V(\boldsymbol{x})$. Then no closed orbits exist.

Lyapunov functions

Consider $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$ with a fixed point $\boldsymbol{x}_*$. Suppose that a Lyapunov function $V(\boldsymbol{x})$ exists, where $V(\boldsymbol{x})$ is continuously differentiable and real, and satisfies the properties:

1. $V(\boldsymbol{x}) > 0$ for all $\boldsymbol{x} \neq \boldsymbol{x}_*$ and $V(\boldsymbol{x}_*) = 0$ [i.e. V is positive definite];
2. $\frac{\mathrm{d}}{\mathrm{d}t} V(\boldsymbol{x}(t)) < 0$ for all $\boldsymbol{x} \neq \boldsymbol{x}_*$ [i.e. all trajectories flow 'down hill'].

Then no closed orbits exist. In fact, $\boldsymbol{x}_*$ is globally attracting!

Dulac's Criterion

Let $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$ be a continuously differentiable vector field on a simply-connected subset R of the plane. If there exists a continuously differentiable, real-valued function $g(\boldsymbol{x})$ such that

$$\nabla \cdot (g(\boldsymbol{x})\dot{\boldsymbol{x}}) \quad \text{has one sign throughout } R$$

then there are no closed orbits lying entirely in R.

Poincaré-Bendixson Theorem

Suppose that:

1. R is a closed, bounded subset of the plane;
2. $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$ is a continuously differentiable vector field on an open set contained in R;
3. R does not contain any fixed points;
4. There is a trajectory C that is confined in R for all $t > 0$.

Then either C itself is a closed orbit or C approaches a closed orbit as $t \to \infty$. In either case, R contains a closed orbit!

Liénard's Theorem

Consider the equation $\ddot{x} + f(x)\dot{x} + g(x) = 0$. Suppose that f and g satisfy the following conditions:

1. $f(x)$ and $g(x)$ are continuously differentiable for all x;
2. $g(-x) = -g(x)$ for all x [i.e. $g(x)$ is an odd function];
3. $g(x) > 0$ for all $x > 0$;
4. $f(-x) = f(x)$ for all x [i.e. $f(x)$ is an even function];
5. The odd function
$$F(x) = \int_0^x f(u)\,\mathrm{d}u$$
has exactly one positive zero at $x = a$, is negative for $0 < x < a$, is positive and nondecreasing for $x > a$, and $F(x) \to \infty$ as $x \to \infty$.

Then the system has a unique, stable limit cycle surrounding the origin in the phase plane.