On the existence of limit cycles

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Gradient systems
Suppose there exists a continuously differentiable, single-valued, scalar potential function $V(x)$ so that $\dot{x} = -\nabla V(x)$. Then no closed orbits exist.

Lyapunov functions
Consider $\dot{x} = f(x)$ with a fixed point $x_*$. Suppose that a Lyapunov function $V(x)$ exists, where $V(x)$ is continuously differentiable and real, and satisfies the properties:

1. $V(x) > 0$ for all $x \neq x_*$ and $V(x_*) = 0$ [i.e. $V$ is positive definite];
2. $\frac{d}{dt} V(x(t)) < 0$ for all $x \neq x_*$ [i.e. all trajectories flow 'down hill'].

Then no closed orbits exist. In fact, $x_*$ is globally attracting!

Dulac’s Criterion
Let $\dot{x} = f(x)$ be a continuously differentiable vector field on a simply-connected subset $R$ of the plane. If there exists a continuously differentiable, real-valued function $g(x)$ such that

$$\nabla \cdot (g(x) \dot{x}) \quad \text{has one sign throughout } R$$

then there are no closed orbits lying entirely in $R$.

Poincaré-Bendixson Theorem
Suppose that:

1. $R$ is a closed, bounded subset of the plane;
2. $\dot{x} = f(x)$ is a continuously differentiable vector field on an open set contained in $R$;
3. $R$ does not contain any fixed points;
4. There is a trajectory $C$ that is confined in $R$ for all $t > 0$.

Then either $C$ itself is a closed orbit or $C$ approaches a closed orbit as $t \to \infty$. In either case, $R$ contains a closed orbit!
Liénard’s Theorem

Consider the equation $\ddot{x} + f(x)\dot{x} + g(x) = 0$. Suppose that $f$ and $g$ satisfy the following conditions:

1. $f(x)$ and $g(x)$ are continuously differentiable for all $x$;
2. $g(-x) = -g(x)$ for all $x$ [i.e. $g(x)$ is an odd function];
3. $g(x) > 0$ for all $x > 0$;
4. $f(-x) = f(x)$ for all $x$ [i.e. $f(x)$ is an even function];
5. The odd function

$$F(x) = \int_0^x f(u) \, du$$

has exactly one positive zero at $x = a$, is negative for $0 < x < a$, is positive and nondecreasing for $x > a$, and $F(x) \to \infty$ as $x \to \infty$.

Then the system has a unique, stable limit cycle surrounding the origin in the phase plane.