# Nonlinear dynamical systems

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Consider the following nonlinear coupled system, where $x_1(t)$ and $x_2(t)$ evolve according to

$$\dot{x}_1 = f_1(x_1, x_2), \qquad \dot{x}_2 = f_2(x_1, x_2),$$

for given functions $f_1$ and $f_2$, and also given initial conditions for $x_1$ and $x_2$. By defining the vector $\boldsymbol{x}(t) = (x_1(t), x_2(t))^T$ and the function $\boldsymbol{F} = (f_1, f_2)^T$, this system may be recast in the form $\dot{\boldsymbol{x}} = \boldsymbol{F}(\boldsymbol{x})$ with some initial condition $\boldsymbol{x}(0) = \boldsymbol{x}_0$. Note that this form is the nonlinear version of $\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}$, where $\boldsymbol{A}$ is a matrix.

## Existence and uniqueness theorem (for $n$ dimensions)

*Consider the initial value problem $\dot{\boldsymbol{x}} = \boldsymbol{F}(\boldsymbol{x})$, $\boldsymbol{x}(0) = \boldsymbol{x}_0$. Suppose that $\boldsymbol{F}$ is continuous and all its partial derivatives $\partial \boldsymbol{F}/\partial x_j$ (for $j = 1, \ldots, n$) are continuous in some open connected set $D \subseteq \mathbb{R}^n$. Then for $\boldsymbol{x}_0 \in D$, the initial value problem has solution $\boldsymbol{x}(t)$ on some time interval $t \in (-\tau, \tau)$ about $t = 0$, and the solution is unique.*

Two important corollaries:

1. Two trajectories cannot intersect for a two-dimensional system, as otherwise the two trajectories would propagate from the same initial condition.
2. If a trajectory forms a closed curve $C$ then any other trajectory that is within $C$ remains within $C$ forever.

## Numerical solution

The numerical methods for evolving scalar systems all extend to vector systems! For a time step $h > 0$, times $t_n = nh$ and corresponding numerical solution $\boldsymbol{x}_n \approx \boldsymbol{x}(t_n)$ (where $\boldsymbol{x}_0$ is the initial condition), the fourth-order Runge-Kutta method is

$$\boldsymbol{x}_{n+1} = \boldsymbol{x}_n + \frac{1}{6}\big(\boldsymbol{k}_1 + 2\boldsymbol{k}_2 + 2\boldsymbol{k}_3 + \boldsymbol{k}_4\big),$$

where between each time step, the vectors $\boldsymbol{k}_j$ are defined as

$$\begin{aligned}
\boldsymbol{k}_1 &= h\boldsymbol{F}\left(\boldsymbol{x}_n\right), \\
\boldsymbol{k}_2 &= h\boldsymbol{F}\left(\boldsymbol{x}_n + \boldsymbol{k}_1/2\right), \\
\boldsymbol{k}_3 &= h\boldsymbol{F}\left(\boldsymbol{x}_n + \boldsymbol{k}_2/2\right), \\
\boldsymbol{k}_4 &= h\boldsymbol{F}\left(\boldsymbol{x}_n + \boldsymbol{k}_3\right).
\end{aligned}$$

For Problem Set 4, you will need to write code for the fourth-order Runge-Kutta method in two-dimensions. The one-dimensional version of the code (in MATLAB) is uploaded on Stellar, which may be of use!