Lectures 9–11: Phase Planes

- We now study 2D nonlinear systems!

$$\boxed{\begin{cases}\dot{x}_1 = f_1(x_1,x_2)\\ \dot{x}_2 = f_2(x_1,x_2)\end{cases}} \longrightarrow \boxed{\dot{\underline{x}} = \underline{f}(\underline{x})}\ , \text{ where } \underline{x} = \begin{pmatrix}x_1\\x_2\end{pmatrix},\ \underline{f} = \begin{pmatrix}f_1\\f_2\end{pmatrix}.$$

↑ First-order, so we have an initial condition for $\underline{x}(0)$ → each initial condition determines a trajectory.

→ Aim: Determine the qualitative behavior of the system!

- fixed points $\underline{x}_*$: where $\underline{f}(\underline{x}_*) = \underline{0}$ (so $\dot{\underline{x}} = \underline{0}$)
- closed orbits corresponding to periodic solutions $\underline{x}(t+T) = \underline{x}(t)$ $\forall t$.
- Behavior near to the fixed points and their stability

Note on numerical computations of $\dot{\underline{x}} = \underline{f}(\underline{x})$

- the methods for scalar systems all extend to vector systems, e.g. fourth-order Runge–Kutta with timestep $h>0$

$$\underline{x}_{n+1} = \underline{x}_n + \tfrac{1}{6}\left[\underline{k}_1 + 2\underline{k}_2 + 2\underline{k}_3 + \underline{k}_4\right]$$

where
$$\begin{cases}\underline{k}_1 = h\underline{f}(\underline{x}_n)\\ \underline{k}_2 = h\underline{f}(\underline{x}_n + \tfrac{1}{2}\underline{k}_1)\\ \underline{k}_3 = h\underline{f}(\underline{x}_n + \tfrac{1}{2}\underline{k}_2)\\ \underline{k}_4 = h\underline{f}(\underline{x}_n + \underline{k}_3)\end{cases}$$

Example. $\begin{cases}\dot{x} = x + e^{-y}\\ \dot{y} = -y\end{cases}$ → Fixed points (x_*, y_*) satisfy $\begin{cases}x_* + e^{-y_*} = 0\\ -y_* = 0\end{cases}$

$$\Rightarrow \underline{(x_*, y_*) = (-1, 0)}.$$

For an initial condition $y(0) = y_0$, we have $y(t) = y_0 e^{-t}$ ∴ $y \to 0$ as $t \to \infty$

$\Rightarrow e^{-y} \to 1$ as $t \to \infty$

So $\underbrace{\dot{x} \simeq x + 1}$ as $t \to \infty$

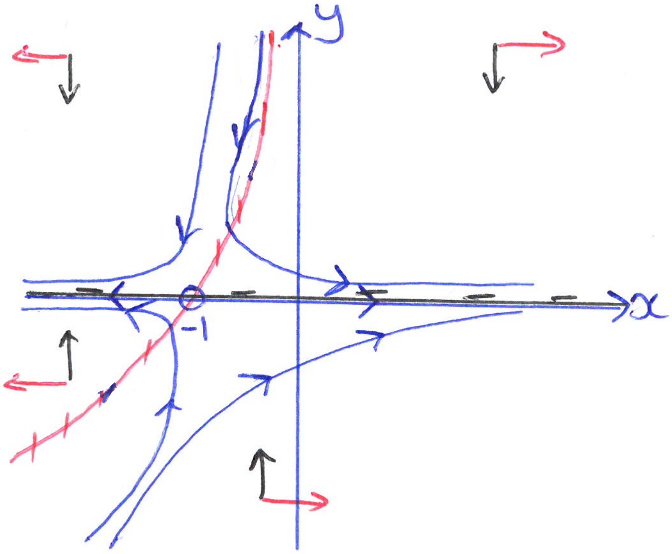
suggests that $x(t)$ blows-up as $t \to \infty$.

To sketch the phase portrait, we first draw the nullclines (curves along which $\dot{x}=0$ or $\dot{y}=0$)

$$\begin{cases}\dot{x}=x+e^{-y}\\ \dot{y}=-y\end{cases}$$

— Nullcline $y=-\log(-x)$

— Nullcline $y=0$



⊛ A nonlinear version of a saddle point ⊛

⊛ Nullclines are not trajectories, in general ⊛

Existence & Uniqueness.

Does $\underline{\dot{x}}=\underline{f}(\underline{x})$ even have solutions?

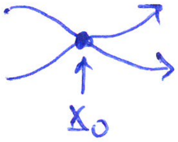
[For n dimensions]

Consider the Initial Value Problem $\dot{\underline{x}}=\underline{f}(\underline{x})$, $\underline{x}(0)=\underline{x}_0$. Suppose that $\underline{f}$ is continuous and all its partial derivatives $\frac{\partial f}{\partial x_j}$ $\forall j=1,..,n$ are continuous in some open connected set $D\in\mathbb{R}^n$. Then for $\underline{x}_0\in D$, the IVP has solution $\underline{x}(t)$ on some time interval $t\in(-\tau,\tau)$ about $t=0$, and the solution is unique.

So basically, we need sufficiently smooth vector fields!

Corollary: Two trajectories never intersect!

Suppose they do intersect



Then we have two different trajectories propagating from the same initial condition ↯

So if we have a trajectory that lies along a closed curve C then any trajectory that is within C remains within C forever - We study closed orbits later on!

Fixed Points and Linearization.

We now extend the method of linearising about a fixed point in order to determine the asymptotic linear stability.

Consider $\begin{cases} \dot{x} = f(x,y) \\ \dot{y} = g(x,y) \end{cases}$ with a fixed point (x_*, y_*) s.t. $\left. \begin{array}{r} f(x_*,y_*)=0 \\ \underline{\text{and}}\ g(x_*,y_*)=0 \end{array} \right\}$

Let $x = x_* + u$, $y = y_* + v$, where u and v are small perturbations, namely $0<|u|,|v|<<1$.

$$\text{So } \frac{d}{dt}(x_* + u) = f(x_* + u, y_* + v)$$

Taylor Expansion

$$\Rightarrow \dot{u} = \cancel{f(x_*,y_*)} + u\frac{\partial f}{\partial x}(x_*,y_*) + v\frac{\partial f}{\partial y}(x_*,y_*) + \underline{O(u^2, v^2, uv)}$$

0 as fixed point

$$\text{Similarly} \Rightarrow \dot{v} = \cancel{g(x_*,y_*)} + u\frac{\partial g}{\partial x}(x_*,y_*) + v\frac{\partial g}{\partial y}(x_*,y_*) + \underline{O(u^2, v^2, uv)}$$

↑ quadratic terms are extremely small

Define the Jacobian matrix $J(x,y)$ so that

$$J = \begin{pmatrix} \partial f/\partial x & \partial f/\partial y \\ \partial g/\partial x & \partial g/\partial y \end{pmatrix}$$

$$\text{Then } \underline{\underline{\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = J(x_*, y_*) \begin{pmatrix} u \\ v \end{pmatrix}}}. \left(+ \text{ quadratic terms}\right)$$

Neglecting quadratic terms gives

$$\underline{\underline{\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = A \begin{pmatrix} u \\ v \end{pmatrix}}} \quad \text{where } A = J(x_*, y_*)$$

↑ The stability of the fixed point is characterised by the trace and determinant of A.

⊛ Beware of neglecting quadratic terms for borderline cases: centers, stars, non-isolated fixed points, degenerate nodes ⊛

Example.

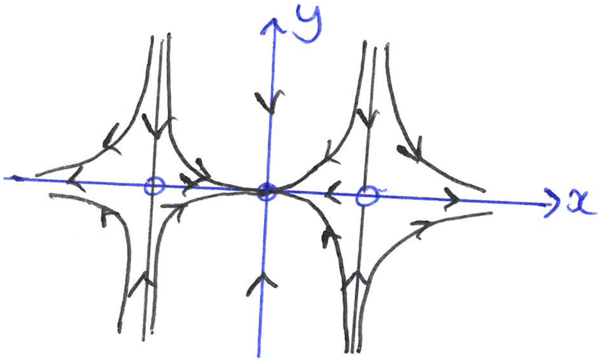
$$\left.\begin{cases} \dot{x} = -x(1-x)(1+x) = -x+x^3 \\ \dot{y} = -2y. \end{cases}\right\} \textcircled{\dagger}$$

Fixed points: $(0,0)$, $(1,0)$, $(-1,0)$.

$$J(x,y) = \begin{pmatrix} 3x^2-1 & 0 \\ 0 & -2 \end{pmatrix} \leftarrow \text{Jacobian}$$

- At $(0,0)$, $A = J(0,0) = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} \Rightarrow$ stable node,
- At $(\pm1,0)$, $A = J(\pm1,0) = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \Rightarrow$ saddle.

} not borderline cases, so linear stability is robust!



<u>Note</u>: ∘ system symmetric about $x=0$ since $x \mapsto -x$ in $\textcircled{\dagger}$ gives same system.

∘ Similarly, system is invariant under $y \mapsto -y$ ∴ symmetry about $y=0$

Example [effect of nonlinear terms for centers]

$$\textcircled{\dagger} \begin{cases} \dot{x} = -y + ax(x^2+y^2) \\ \dot{y} = x + ay(x^2+y^2) \end{cases}$$

$a \in \mathbb{R}$ is a parameter.

- Note, $(0,0)$ is a fixed point, with <u>linearized system</u> $\dot{x} = -y$, $\dot{y} = x$,
  i.e. $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \leftarrow \operatorname{Tr}(A) = 0$, $\det(A) = 1 > 0 \Rightarrow$ center.

∘ To study the dynamics of the nonlinear system, we perform a change of variables

$r = r(t)$
$\theta = \theta(t)$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \Rightarrow x^2+y^2 = r^2 \xRightarrow{\frac{d}{dt}} \underline{x\dot{x} + y\dot{y} = r\dot{r}}$$

Using $\textcircled{\dagger}$, we have

$$r\dot{r} = x(\cancel{-y} + ax\overbrace{(x^2+y^2)}^{=r^2}) + y(\cancel{x} + ay\overbrace{(x^2+y^2)}^{=r^2})$$
$$= ar^2(x^2+y^2) = ar^4$$

$$\therefore \underline{\dot{r} = ar^3}$$

Also, $\tan\theta = y/x$

$$\frac{d}{dt} \Rightarrow \dot\theta(1+\underbrace{\tan^2\theta}_{=(y/x)^2}) = \frac{\dot y}{x} - \frac{y\dot x}{x^2} = \frac{1}{x^2}(x\dot y - \dot x y)$$

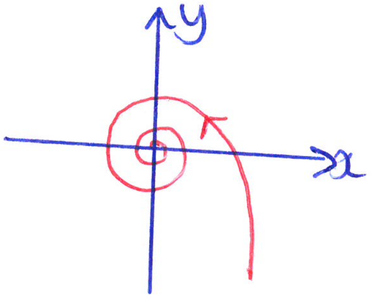
$$\Rightarrow \dot\theta(x^2+y^2) = x\dot y - \dot x y \Rightarrow \underline{\underline{\dot\theta = \frac{1}{r^2}(x\dot y - \dot x y)}}$$

Using (†), we have $\dot\theta = \frac{1}{r^2}\left[x\{x + a\cancel{y(x^2+y^2)}\} - y\{-y + a\cancel{x(x^2+y^2)}\}\right]$

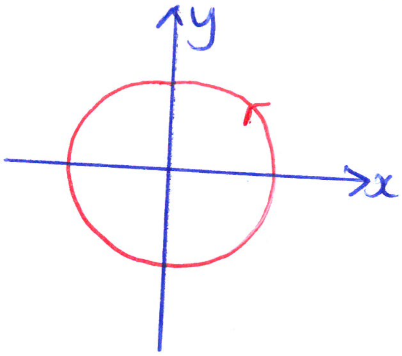
$$\Rightarrow \underline{\underline{\dot\theta = 1}}.$$

So $\begin{cases} \dot r = ar^3 \\ \dot\theta = 1 \end{cases}$ ⇒ All trajectories rotate about origin with angular velocity $\dot\theta = 1$.

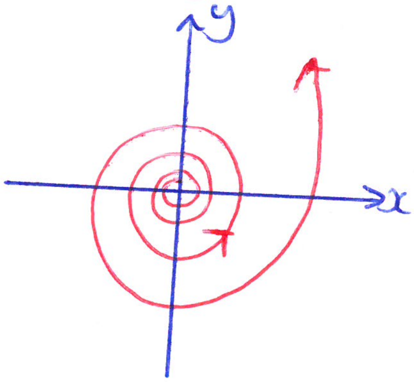
- $a<0$ → $\dot r$ is decreasing so trajectories spiral inwards
- $a>0$ → r is increasing so trajectories spiral outwards.



$a<0$



$a=0$



$a>0$

⊛ centers are very delicate: any mismatch after one cycle leads to a spiral! ⊛

When linear stability is robust and trajectories near to fixed points are accurate!

- Repellors, Attractors, saddles

Marginal cases (when Linear stability gives spurious trajectories):

- centers and non-isolated fixed points.

⊛ See Strogatz p156 for more mathematical details).

Example: Rabbits vs. Sheep. $0 \le x(t)$ = rabbit population

$0 \le y(t)$ = sheep population

[assume a continuum model]

$$\begin{cases} \dot{x} = \overbrace{3x - x^2}^{\text{logistic growth}} - 2xy \\ \dot{y} = 2y - xy - y^2 \end{cases}$$

conflict!

logistic growth.

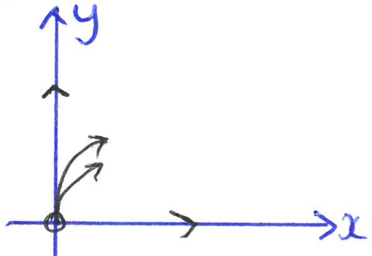
- Sheep inhibit rabbits by nudging them aside
- rabbits eat the grass that the sheep want.
(rabbits suffer more)

Fixed points: $(0,0), (0,2), (3,0), (1,1)$

Jacobian $J(x,y) = \begin{pmatrix} 3-2x-2y & -2x \\ -y & 2-x-2y \end{pmatrix}.$

- **Fixed point $(0,0)$**: $A = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \Rightarrow$ eigenvalues 2, 3 ⤳ unstable node

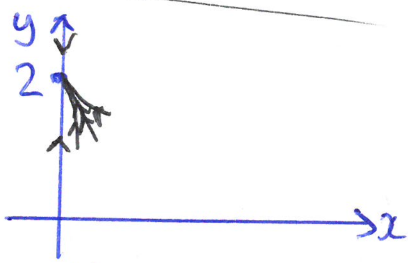
  - Trajectories leave tangential to eigenvector corresponding to slowest eigenvalue, i.e. $\lambda = 2$, so an eigenvector is $\underline{v} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.



- **Fixed point $(0,2)$**: $A = \begin{pmatrix} -1 & 0 \\ -2 & -2 \end{pmatrix} \Rightarrow$ eigenvalues $-1, -2$

  $\therefore$ stable node, Trajectories approach by slowest eigenvalue $(\lambda = -1)$ $\therefore \underline{v} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

  $y = -2x$

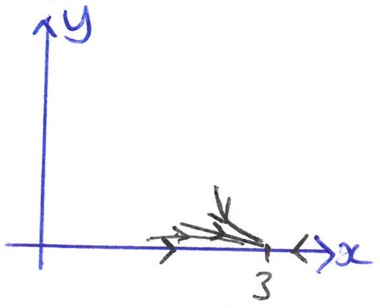


- **Fixed point $(3,0)$**: $A = \begin{pmatrix} -3 & -6 \\ 0 & -1 \end{pmatrix}$ ⤳ eigenvalues; $-1, -3$

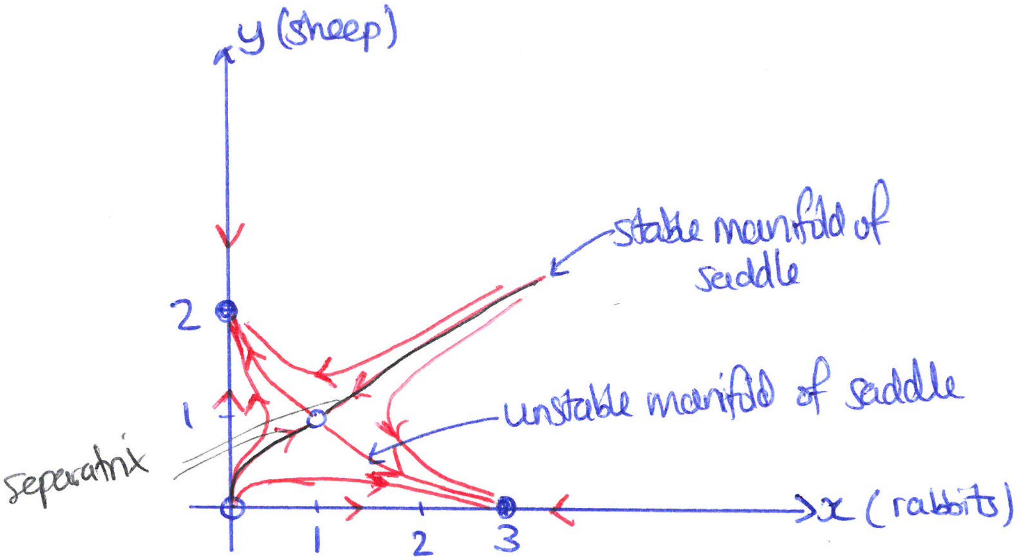
  $\therefore$ stable node. slowest eigenvalue

  $\underline{v} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$ is an eigenvector

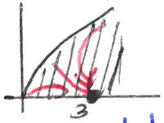
  $y = -\frac{1}{3}x$

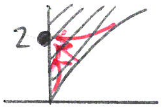


- Fixed point (1,1): $A = \begin{pmatrix} -1 & -2 \\ -1 & -1 \end{pmatrix} \rightsquigarrow \lambda = -1 \pm \sqrt{2}$ ∴ saddle



– separatrix along the two parts of the stable manifold forms part of the basin of attraction for each of the stable nodes





↳ if the initial rabbit population is sufficiently larger compared to the initial sheep population, then rabbits will eventually dominate and sheep will die out [and vice-versa].

⊛ principle of competitive exclusion ⊛

Conservative Systems.

Newton's Second Law $\rightsquigarrow m\ddot{x} = F(x)$ [Nonlinear force]

↑ independent of $\dot{x}$ and t. [no damping or inhomogeneity]

Find a (non-unique) potential V(x) s.t. $F(x) = -V'(x)$

$$\Rightarrow m\ddot{x}(t) + V'(x(t)) = 0$$

Multiply both sides by $\dot{x}$ and note that ① $\dot{x}\ddot{x} = \frac{1}{2}\frac{d}{dt}(\dot{x}^2)$

② $\frac{d}{dt}V(x(t)) = \dot{x}\frac{dV}{dx}(x(t))$ [by chain rule]

$$\Rightarrow \frac{d}{dt}\left[\frac{1}{2}m\dot{x}^2 + V(x(t))\right] = 0$$

$$\therefore E = \underbrace{\frac{1}{2}m\dot{x}^2}_{\text{kinetic energy}} + \underbrace{V(x(t))}_{\text{potential energy}}$$ is conserved [value of E is determined by the initial conditions]

⊛ This is an example of a conservative system as a system is conserved ⊛

In general, consider $\dot{\underset{\sim}{x}} = \underset{\sim}{f}(\underset{\sim}{x})$

↳ a conserved quantity $E(\underset{\sim}{x})$ [E is scalar] is a real-valued continuous function that is constant along trajectories, i.e. $\frac{dE}{dt} = 0$

⊛ We preclude the trivial cases $E = \text{constant}$ on any open set of $\mathbb{R}^2$ ⊛

Property: A conservative system cannot have any attracting fixed points

Proof: Assume that there is a fixed point x_*, where x_* is attracting.

$\Rightarrow$ all points in the basin of attraction also have energy $E(x_*)$

$\therefore$ E is constant on an open set $\#$

So we have a contradiction $\square$

→ So we cannot have attracting fixed points, but we can have saddles and centers, etc.

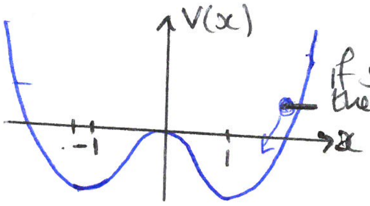
Example.

Consider $\ddot{x} + V'(x(t)) = 0$, where $V(x) = -\frac{1}{2}x^2 + \frac{1}{4}x^4$

$$\Rightarrow \ddot{x} = x - x^3 \quad \Rightarrow \begin{cases} \dot{x} = y \\ \dot{y} = x - x^3 = x(1-x)(1+x) \end{cases}$$

Fixed points $(0,0), (\pm 1, 0)$.

Jacobian $J(x,y) = \begin{pmatrix} 0 & 1 \\ 1-3x^2 & 0 \end{pmatrix}$



if start at rest then V is the energy level.

At (0,0): $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\therefore T = 0, D = -1 < 0$ ⤳ saddle point

At (±1, 0): $A = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix}$ $\therefore T = 0, D = 2$ $\Rightarrow$ centers [marginal case -warning!]

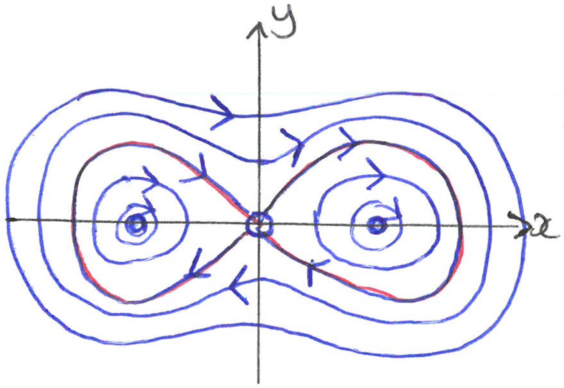
Are these centers a spurious result of linearization?]

↳ Not in this case! ☺

Recall, $E = \frac{1}{2}\dot{x}^2 - \frac{1}{2}x^2 + \frac{1}{4}x^4$ is a conserved quantity

- so we have trajectories that are closed curves, which are contours of constant energy!

Phase portrait.



Note: Each unstable manifold of the saddle becomes a stable manifold!

Not a periodic orbit as the trajectory takes forever to reach the fixed point

So we have two trajectories that both start $(t \to -\infty)$ and finish $(t \to +\infty)$ at the same point!

"homoclinic orbit"

Fact [Robustness of nonlinear centers]: If an isolated fixed point x_* is a minimum (or maximum) of a conserved quantity $E(\underline{x})$, then all trajectories sufficiently close to x_* are closed

Cautionary example (importance of the fixed point being isolated)

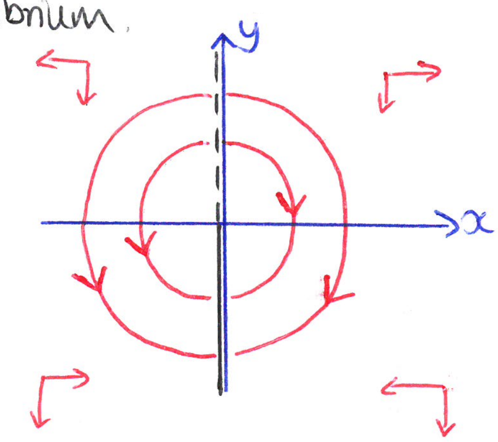
Consider $\begin{cases} \dot{x} = xy \\ \dot{y} = -x^2 \end{cases}$

Note: $E = x^2 + y^2$ is conserved since

$$\frac{dE}{dt} = 2x\dot{x} + 2y\dot{y} = 2x^2y - 2x^2y = 0$$

Also, we have a line of points $(x,y) = (0,\alpha)$ $\forall \alpha \in \mathbb{R}$ along which the system is in equilibrium.

Note: $(x,y) = (0,0)$ is a minimum of E are is a fixed point. (but non-isolated).



[Trajectories follow circular arcs, but are not closed]

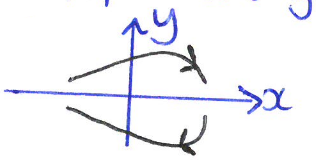
Reversible systems. "time-reversal symmetry"

Consider $m\ddot{x} = F(x)$ $\rightsquigarrow$ $\begin{cases}\dot{x} = y\\ \dot{y} = \frac{1}{m}F(x)\end{cases}$

if we map $t \mapsto -t$ (use chain rule) and $y \mapsto -y$ then both equations remain the same.

So if $(x(t), y(t))$ is a solution then so is $(x(-t), -y(-t))$

↳ each trajectory has a twin! Flip about $y=0$ and change direction of arrows.



More generally: $\begin{cases}\dot{x} = f(x,y)\\ \dot{y} = g(x,y)\end{cases}$ is reversible if the system is invariant under the mapping $t \mapsto -t, y \mapsto -y$,

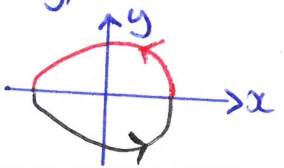
$\begin{cases}\text{i.e. } f \text{ is odd in } y \quad f(x,-y) = -f(x,y)\\ \text{and } g \text{ is even in } y\end{cases}$

← $g(x,-y) = g(x,y)$

Like conservative systems, centers are robust in reversible systems.

- Suppose $\underline{x}_* = \underline{0}$ is a linear center of a reversible system. Then close to the origin, all trajectories are closed curves.

↳ Consider a trajectory that starts on positive x-axis sufficiently near the origin. By local swirling of the vector field (from the center), we expect the trajectory to reach the negative x-axis at some later time. By reversibility, the twin trajectory yields a closed orbit.

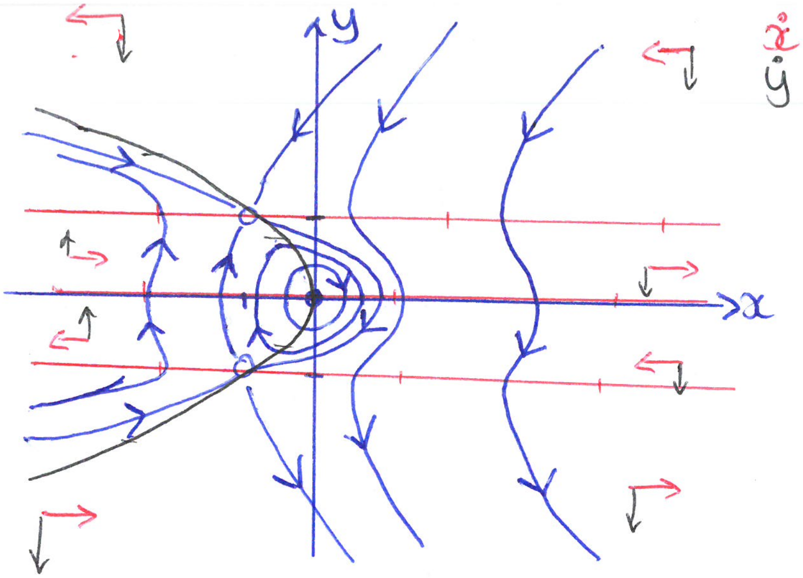


Example $\begin{cases}\dot{x} = y - y^3 = y(1-y)(1+y)\\ \dot{y} = -x - y^2\end{cases}$ reversible! ∴ Fixed points $(0,0)$, $(-1,\pm 1)$

Note: Jacobian $J(x,y) = \begin{pmatrix} 0 & 1-3y^2 \\ -1 & -2y \end{pmatrix}$ ∴ at $(0,0)$, $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ ∴ $\mathrm{Tr}(A) = 0$, $\det(A) = 1$ ⇒ center at $(0,0)$

⊛ As the system is reversible, the origin is a nonlinear center! ⊛ ←

- At $(-1,\pm 1)$, $A = \begin{pmatrix} 0 & -2 \\ -1 & \mp 2 \end{pmatrix}$ ∴ $\det(A) = -2 < 0$ ⇒ saddle points at $(-1,\pm 1)$



$\left.\begin{array}{l}\dot{x} = 0 \\ \dot{y} = 0\end{array}\right\}$ nullclines.

Note the symmetry!

Note: the saddle points are joined by a pair of trajectories
↳ "heteroclinic trajectories" or "saddle conditions"
[reminiscent of heteroclinic orbits for conservative systems].

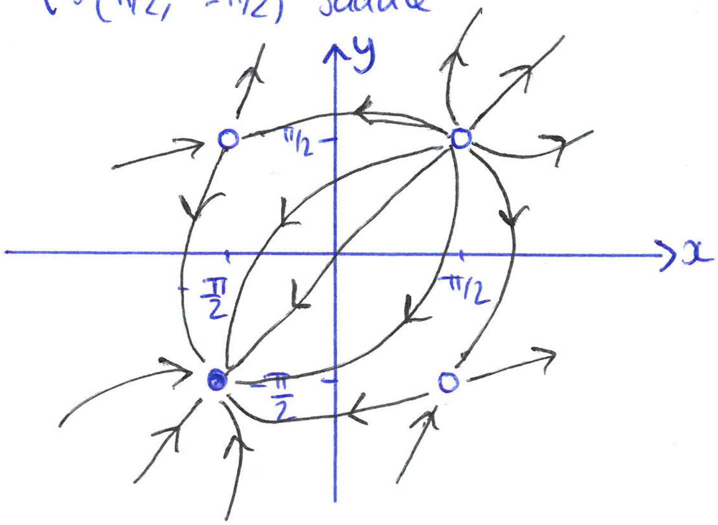
Example. $\begin{cases} \dot{x} = -2\cos x - \cos y \\ \dot{y} = -2\cos y - \cos x \end{cases}$ → reversible system. [and 2π-periodic in x and y]

Fixed points satisfy $\cos x_* = \cos y_* = 0$

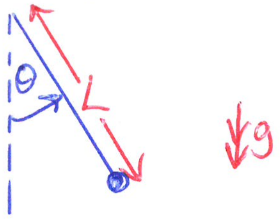
∴ fixed points $\left(\frac{\pi}{2}, \pm\frac{\pi}{2}\right)$, $\left(-\frac{\pi}{2}, \pm\pi/2\right)$

Jacobian $J(x,y) = \begin{pmatrix} 2\sin x & \sin y \\ \sin x & 2\sin y \end{pmatrix}$

We find that
- $(\pi/2, \pi/2)$ unstable node
- $(-\pi/2, -\pi/2)$ stable node → existence of an attractor means that the system is not conservative.
- $(-\pi/2, +\pi/2)$ saddle
- $(\pi/2, -\pi/2)$ saddle



Nonlinear pendulum.



$\theta(t)$ evolves according to

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0$$

Let $\tau = \sqrt{\frac{g}{L}}\, t$ be a dimensionless timescale $\Rightarrow$ $\dfrac{d^2\theta}{d\tau^2} + \sin\theta = 0$

So $\begin{cases} \dfrac{d\theta}{d\tau} = \omega \quad \leftarrow \text{dimensionless angular velocity.} \\ \dfrac{d\omega}{d\tau} = -\sin\theta \end{cases}$

Fixed points (θ,ω) are $(0,0)$ and $(\pi,0)$ [and 2π rotations in θ]

Jacobian $J(\theta,\omega) = \begin{pmatrix} 0 & 1 \\ -\cos\theta & 0 \end{pmatrix}$

So at $(0,0)$; $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \Rightarrow \mathrm{Tr}(A) = 0,\ \det(A) = 1$

∴ linear center at origin.

↳ Is the origin also a nonlinear center?

Yes $\Big\{$ ① We have a time-reversible system

② We have a conservative system due to the potential $V(\theta) = -\cos\theta$

so that $\dfrac{d^2\theta}{d\tau^2} = -V'(\theta)$ ∴ $\boxed{E = \frac{1}{2}\dot{\theta}^2 - \cos\theta \text{ is a conserved quantity.}}$

Furthermore $E(\theta,\omega) = \frac{1}{2}\omega^2 - \cos\theta$ has a minimum at $(\theta,\omega) = (0,0)$.

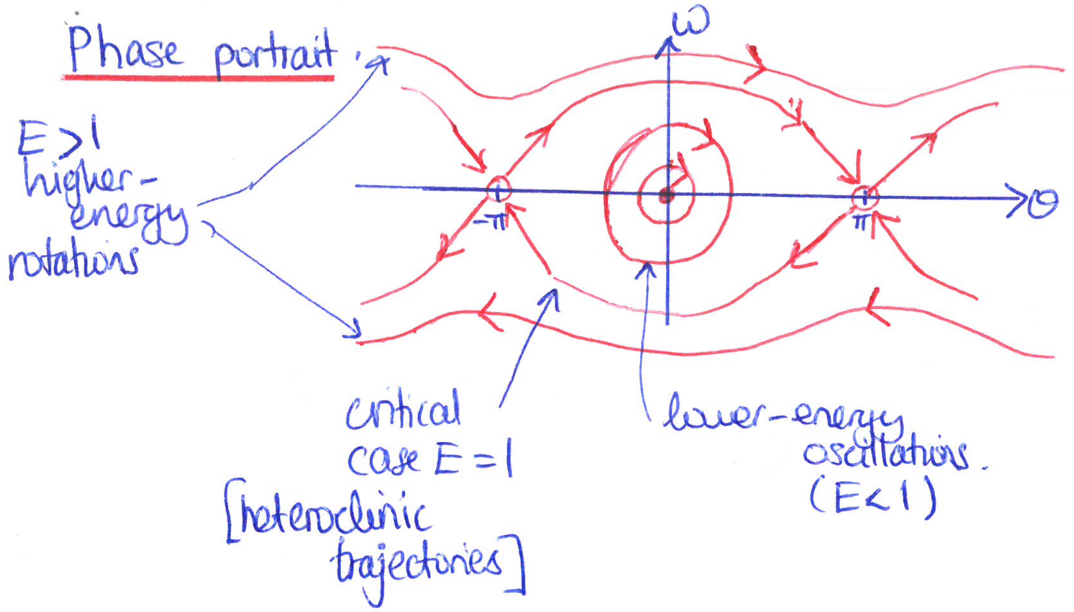
↳ Furthermore, for $|\theta| \ll 1$ (and $|\omega| \ll 1$), we have $E(\theta,\omega) \approx \frac{1}{2}(\theta^2 + \omega^2) - 1 + O(\theta^4)$

∴ nearly circular orbits close to the origin.

At $(\pi,0)$; $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\det A < 0 \Rightarrow$ Saddle!

- Eigenvalues are $\lambda_1 = -1$, $\lambda_2 = +1$ with eigenvectors $\underline{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $\underline{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Phase portrait



(2π-periodic in θ)

Note the symmetry!

$E = \frac{1}{2}\omega^2 - \cos\theta$ constant along trajectories

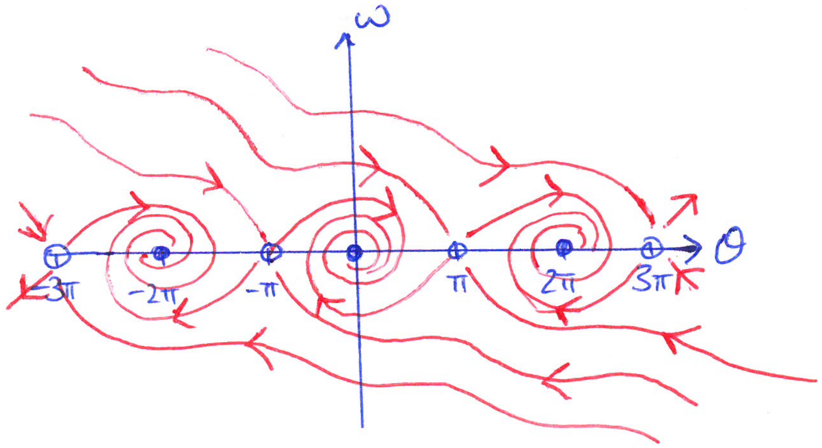
- $E=-1$: lowest energy state at $(\theta,\omega)=(0,0)$ [pendulum hanging downwards]
- $E=1$: heteroclinic trajectories.

The effect of damping.

$\ddot\theta + b\dot\theta + \sin\theta = 0$, $b>0$ is the damping parameter.

The system continually loses energy, i.e. $E=\frac{1}{2}\dot\theta^2 - \cos\theta$ decreases over time [along trajectories]

- $\frac{dE}{dt} = \frac{d}{dt}\left(\frac{1}{2}\dot\theta^2 - \cos\theta\right) = \dot\theta(\ddot\theta + \sin\theta) = -b\dot\theta^2 < 0$
- Linear stability analysis shows that for $b>0$, $(\theta_*,\omega_*)=(0,0)$ becomes a stable spiral, whilst $(\theta_*,\omega_*)=(0,\pi)$ remains a saddle



If $E(0)<1$, the oscillations are damped

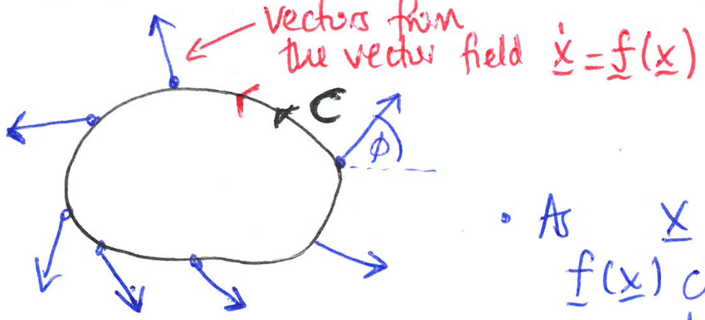
If $E(0)>1$, the pendulum whirls around, until $E(\tau)=1$, at which point no more rotations occur and there are damped oscillations about the origin.

Index Theory.

- Linear stability analysis only gives us information locally to a fixed point.
- Index theory gives us global information.
  - does a closed trajectory encircle a fixedpoint?
    - ↳ If yes, what kind of fixed points?
  - What types of trajectory can coalesc at bifurcations?
  - can rule out the existence of closed orbits.

The index of a closed curve.

∘ I_c is An integer that describes the net winding of a vector field on a simple closed curve C.



- not necessarily a trajectory.
- C does not intersect itself

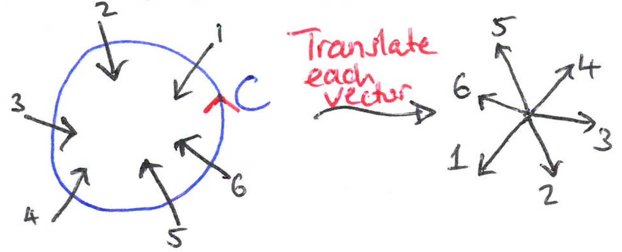
- As $\underline{x}$ moves counterclockwise around C, $\underline{f}(\underline{x})$ changes continuously, so ϕ varies continuously also.
- After ones loop, $\underline{x}$ returns to its original direction but ϕ has changed by an integer multiple of 2π

Index of C

$$\boxed{\text{Define } \quad I_c = \frac{1}{2\pi}[\phi]_c} \leftarrow \frac{\text{change in } \phi \text{ over one loop}}{2\pi}$$

↳ I_c is the net number of counterclockwise revolutions as $\underline{x}$ moves once counterclockwise around C.

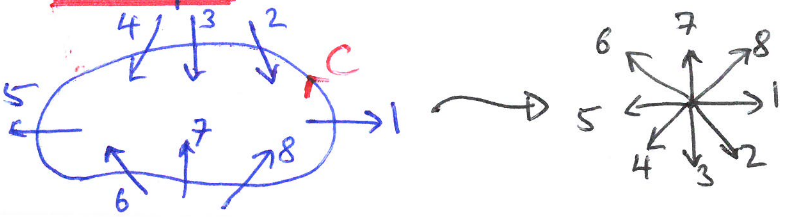
Example/method



I_c = net number of counterclockwise revolutions made by the numbered vectors

So $I_c = +1$

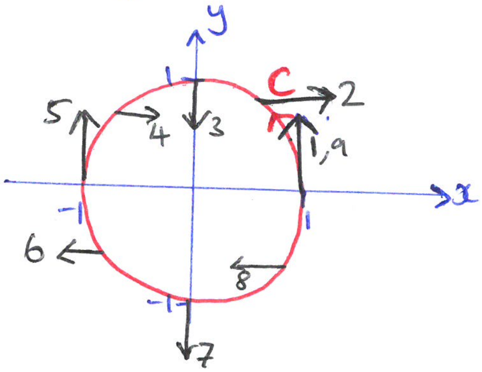
Example.

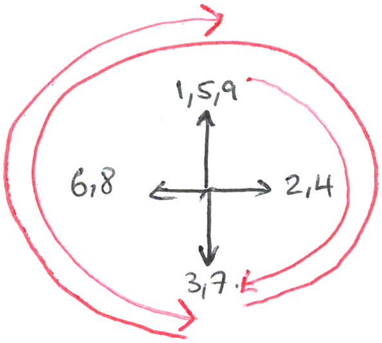


→ 1 clockwise revolution

$\therefore I_c = -1$.

Example. $\begin{cases}\dot{x} = x^2 y \\ \dot{y} = x^2 - y^2\end{cases}$, C is the unit circle $x^2+y^2=1$.





No net revolution
$\therefore I_c = 0$.

Properties.

① If C can be continuously deformed to C' without passing through a fixed point then $I_C = I_{C'}$.

- Proof: Deforming C continuously means that ϕ varies continuously, and hence I_c varies continuously. But I_c only takes integer values, and can only change by jumping. Therefore I_c is constant □.

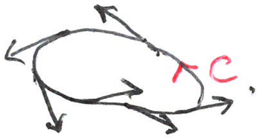
② If C doesn't include any fixed points then $I_C = 0$

- Proof: we can continuously deform C to a tiny circle without changing I_c. But ϕ is ≈ constant on this circle by continuity/smoothness of $\underline{f}(\underline{x}) \Rightarrow [\phi]_c = 0 \Rightarrow I_c = 0$ □

③ If we map $t \mapsto -t$ then the arrows change direction but the index is unchanged.

- Proof: $\phi \mapsto \phi + \pi$ $\therefore [\phi]_c$ remains constant.

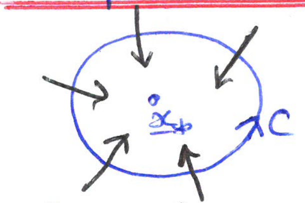
④ Suppose the closed curve C is actually a trajectory of the system, i.e. C is a closed orbit. Then $I_c = +1$



The vector field is everywhere tangent to C
$\therefore$ As $\underline{x}$ winds around C once, the tangent vector also rotates once in the same sense.

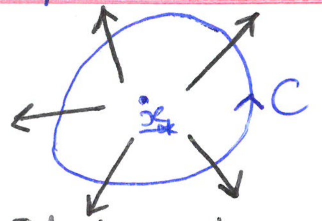
Index of a point: Suppose x_* is an isolated fixed point.
The index I of x_* is defined as I_C, where C is any closed curve that encloses x_* and no other fixed points.
By Property (1), I_C is independent of C and is only a property of x_*.

Example: Stable node



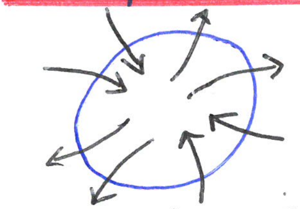
(from before)
$\Rightarrow I = +1$

Example: Unstable node



Set $t \mapsto -t$ to get stable nod
$\Rightarrow I = +1$

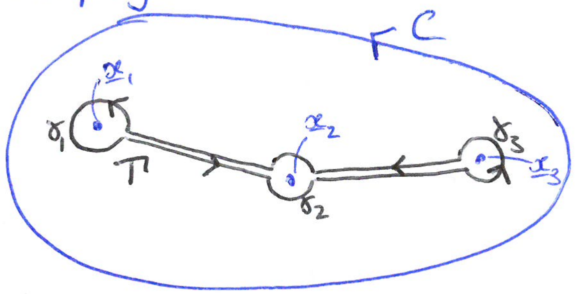
Example: saddle.



Similar to previous examples
$\Rightarrow I = -1$.

Theorem: If C is a closed curve that surrounds n isolated fixed points $x_1, \dots, x_n$ then $I_C = I_1 + \dots + I_n$, where I_j is the index of x_j for $j = 1, \dots, n$.

Idea of proof: Deform contour C to a new countour Γ: $\Rightarrow \boxed{I_C = I_\Gamma}$



- γ_j is a small circle about x_j with order I_j.
- Each γ_j is connected by a two-way bridge.
  ↳ contributions to I_Γ cancel out as bridges become narrower.
  $\Rightarrow$ only need contribution from circles.

$$\therefore I_C = I_\Gamma = \sum_{k=1}^{N} I_k.$$

Corollary. A closed orbit must enclose fixed points whose indices sum to $+1$.

Proof: Let C denote the closed orbit. By property 4; $I_C = +1$
By Thm, $\sum_{k=1}^{n} I_k = I_C = +1$ □.

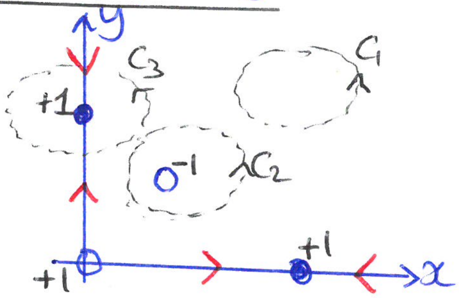
↳ So there is always a fixed point inside a closed orbit.
- If there is only one fixed point inside a closed orbit, the fixed point cannot be a saddle.

Rabbits vs. sheep example: Impossibility of closed orbits.

$$\begin{cases}\dot{x} = x(3-x-2y)\\ \dot{y} = y(2-x-y)\end{cases} \qquad x,y \geq 0.$$

Recall the fixed points:
$$\begin{cases}(0,0): \text{ unstable node} & (\text{index} = +1)\\ (3,0) \text{ and } (0,2): \text{ stable nodes} & (\text{index} = +1)\\ (1,1): \text{ saddle point} & (\text{index} = -1)\end{cases}$$

Candidate closed orbits



C_1: Impossible as no fixedpoints enclosed

C_2: violates condition that interior indices sum to 1

C_3: Orbit crosses the y-axis, which has straight-line trajectories. But trajectories cannot cross!

— other candidate closed orbits are rejected by similar arguments.