Lectures 9-11: Phase Planes

- We now study 2D nonlinear systems!

\[
\begin{align*}
\dot{x}_1 &= f_1(x_1, x_2) \\
\dot{x}_2 &= f_2(x_1, x_2)
\end{align*}
\]

\[
\dot{x} = f(x), \quad \text{where } x = (x_1, x_2), \quad f = (f_1, f_2).
\]

\[\uparrow\text{First-order, so we have an initial condition for } x(0) \rightarrow \text{each initial condition determines a trajectory.}\]

→ Aim: Determine the qualitative behavior of the system!

- Fixed points \( x^*_+ \): where \( f(x^*_+) = 0 \) \( \Rightarrow \dot{x} = 0 \)
- Closed orbits corresponding to periodic solutions \( x(t+T) = x(t) \) \( \forall t \)
- Behavior near to the fixed points and their stability

Note on numerical computation of \( \dot{x} = f(x) \)

- The methods for scalar systems all extend to vector systems,
  e.g., fourth-order Runge-Kutta with timestep \( h > 0 \)

\( x_{n+1} = x_n + \frac{h}{6}[k_1 + 2k_2 + 2k_3 + k_4] \)

where

\[
\begin{align*}
k_1 &= h \cdot f(x_n) \\
k_2 &= h \cdot f(x_n + \frac{1}{2}k_1) \\
k_3 &= h \cdot f(x_n + \frac{1}{2}k_2) \\
k_4 &= h \cdot f(x_n + k_3)
\end{align*}
\]

Example

\[
\begin{align*}
\dot{x} &= x + e^{-y} \\
\dot{y} &= -y
\end{align*}
\]

→ Fixed points \((x^*_+, y^*_+)\) satisfy \( \begin{align*} x^*_+ + e^{-y^*_+} &= 0 \\
y^*_+ &= 0 \end{align*} \)

\( \Rightarrow (x^*_+, y^*_+) = (-1, 0) \).

For an initial condition \( y(0) = y_0 \), we have \( y(t) = y_0 e^{-t} \rightarrow 0 \) as \( t \rightarrow \infty \)

\( \Rightarrow e^{-y} \rightarrow 1 \) as \( t \rightarrow \infty \)

So, \( x(t) \approx x + 1 \) as \( t \rightarrow \infty \),

suggests that \( x(t) \) blows-up as \( t \rightarrow \infty \).
To sketch the phase portrait, we first draw the nullclines (curves along which $\dot{x} = 0$ or $\dot{y} = 0$)

\[
\begin{align*}
\dot{x} &= x + e^{-y} & \text{Nullcline } y = -\log(-x) \\
\dot{y} &= -y & \text{Nullcline } y = 0
\end{align*}
\]

- A nonlinear version of a saddle point
- Nullclines are not trajectories, in general

Existence & Uniqueness:
Does $\dot{x} = f(x)$ even have solutions?

[For n dimensions]
Consider the Initial Value Problem $\dot{x} = f(x)$, $x(0) = x_0$. Suppose that $f$ is continuous and all its partial derivatives $\frac{\partial f}{\partial x_j}$ for $j = 1, \ldots, n$ are continuous in some open connected set $D \subseteq \mathbb{R}^n$. Then for $x_0 \in D$, the IVP has solution $x(t)$ on some time interval $t \in (-\epsilon, \epsilon)$ about $t = 0$, and the solution is unique.

So basically, we need sufficiently smooth vector fields!

Corollary: Two trajectories never intersect!

Suppose they do intersect $x_0$. Then we have two different trajectories propagating from the same initial condition $x_0$.

So if we have a trajectory that lies along a closed curve $C$ then any trajectory that is within $C$ remains within $C$ forever.

- We study closed orbits later on!
Fixed Points and Linearization.

We now extend the method of linearizing about a fixed point in order to determine the asymptotic linear stability.

Consider \( \begin{cases} \dot{x} = f(x, y) \\ \dot{y} = g(x, y) \end{cases} \) with a fixed point \((x_*, y_*)\) s.t. \( f(x_*, y_*) = 0 \) and \( g(x_*, y_*) = 0 \)

Let \( x = x_* + u, \ y = y_* + v \), where \( u \) and \( v \) are small perturbations, namely \( O(|u|, |v|) \ll 1 \).

So \( \frac{d}{dt}(x_* + u) = f(x_* + u, y_* + v) \)

Taylor Expansion \( \Rightarrow \dot{u} = f(x_*, y_*) + u \frac{df}{dx}(x_*, y_*) + v \frac{df}{dy}(x_*, y_*) + O(u^2, v^2, uv) \)

\( \frac{d}{dt} \) \( \Rightarrow \dot{v} = g(x_*, y_*) + u \frac{dg}{dx}(x_*, y_*) + v \frac{dg}{dy}(x_*, y_*) + O(u^2, v^2, uv) \)

Define the Jacobian matrix \( J(x, y) \) so that

\[
J = \begin{pmatrix}
\frac{df}{dx} & \frac{df}{dy} \\
\frac{dg}{dx} & \frac{dg}{dy}
\end{pmatrix}
\]

Then \( \begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = J(x_*, y_*) \begin{pmatrix} u \\ v \end{pmatrix} \) \( + \) quadratic terms

Neglecting quadratic terms gives

\( \begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = A \begin{pmatrix} u \\ v \end{pmatrix} \) where \( A = J(x_*, y_*) \)

\( \uparrow \) The stability of the fixed point is characterized by the trace and determinant of \( A \).

\( \blacklozenge \) Beware of neglecting quadratic terms for borderline cases: centers, stars, non-saddle fixed points, degenerate nodes.
Example:
\[
\begin{cases}
\dot{x} = x(1-x)(1+x) = -x + x^3 \\
\dot{y} = -2y.
\end{cases}
\]
Fixed points: \((0,0), (1,0), (-1,0)\),
\[
J(x,y) = \begin{pmatrix} 3x^2 - 1 & 0 \\ 0 & -2 \end{pmatrix} \quad \text{\textless; Jacobian\textgreater;}
\]
- At \((0,0)\), \(A = J(0,0) = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}\) \(\Rightarrow\) stable node, \not\text{not borderline cases, so linear stability is robust!}
- At \((\pm 1,0)\), \(A = J(\pm 1,0) = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}\) \(\Rightarrow\) saddle,

\[\text{Note: system symmetric about } x=0 \text{ since } x \mapsto -x \text{ in } \oplus \text{ gives same system.}
\]
- Similarly, system is invariant under \(y \mapsto -y\). Symmetry about \(y=0\).

Example [effect of nonlinear terms for center]
\[
\oplus \begin{cases}
\dot{x} = -y + ax(x^2+y^2) \\
\dot{y} = x + ay(x^2+y^2)
\end{cases}
\]
\(a \in \mathbb{R}\) is a parameter.
- Note, \((0,0)\) is a fixed point, with linearized system \(\dot{x} = -y, \dot{y} = x\),
- i.e. \(\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}\) \(\Leftarrow\) \(\text{Tr}(A) = 0, \det(A) = 1 > 0 \Rightarrow\) center.

To better study the dynamics of the nonlinear system, we perform a change of variables
\[
\begin{aligned}
&\dot{r} = \dot{r}(t) \\
&\theta = \theta(t) \\
\end{aligned}
\]
\[
\begin{cases}
x = r \cos \theta \\
y = r \sin \theta
\end{cases} \Rightarrow x^2 + y^2 = r^2 \quad \Rightarrow \quad x\dot{x} + y\dot{y} = r \dot{r}
\]
Using \(\oplus\), we have
\[
\dot{r}^2 = x(-y + ax(x^2+y^2)) + y(x + ay(x^2+y^2)) = 2r^2 \quad \Rightarrow \quad \dot{r}^2 = ar^2
\]
\(a \in \mathbb{R}\).
Also, \( \tan \theta = \frac{y}{x} \)

\[
\frac{d}{dt} (1 + \tan^2 \theta) = \frac{y^2 - xy^2}{x^2} = \frac{1}{x^2} (xy - yx)
\]

\[
\Rightarrow \frac{d}{dt} (x^2 + y^2) = xy - xy \Rightarrow \dot{\theta} = \frac{1}{r^2} (xy - xy)
\]

Using \( \dot{\theta} \), we have

\[
\dot{\theta} = \frac{1}{r^2} \left[ x \{ x + ay(2y^2) \} - y \{ -y + ax(2y^2) \} \right]
\]

\[
\Rightarrow \dot{\theta} = 1.
\]

So \( \begin{cases} r = ar^3 \\ \dot{\theta} = 1 \end{cases} \) \Rightarrow All trajectories rotate about origin with angular velocity \( \dot{\theta} = 1 \).

\( a < 0 \rightarrow r \) is decreasing so trajectories spiral inwards.

\( a > 0 \rightarrow r \) is increasing so trajectories spiral outwards.

\( a < 0 \)

\begin{align*}
\text{centers are very delicate: any mismatch after one cycle leads to a spiral!}
\end{align*}

When linear stability is robust and trajectories near to fixed points are accurate:

- Repellors, Attractors, saddles

Marginal cases (when linear stability gives spurious trajectories):

- centers and non-isolated fixed points.

\( \oplus \) See Strogatz p.156 for more mathematical details.
Example: Rabbits vs. Sheep.

\[
\begin{align*}
\dot{x} &= 3x - x^2 - 2xy \\
y &= 2y - xy - y^2
\end{align*}
\]

- Sheep inhibit rabbits by nudging them aside.
- Rabbits eat the grass that the sheep went.

(Rabbits suffer more)

Fixed points: \((0,0), (0,2), (3,0), (1,1)\)

Jacobian \(J(x,y) = \begin{pmatrix} 3-2x-2y & -2x \\ -y & 2-x-2y \end{pmatrix} \)

- Fixed point \((0,0)\):
  \[A = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \Rightarrow \text{eigenvalues } 2,3 \Rightarrow \text{unstable node}\]

  Trajectories leave tangential to eigenvector corresponding to slowest eigenvalue, i.e., \(\lambda = 2\), so an eigenvector is \(\mathbf{v} = (1)\).

- Fixed point \((0,2)\):
  \[A = \begin{pmatrix} -1 & 0 \\ -2 & -2 \end{pmatrix} \Rightarrow \text{eigenvalues } -1,-2\]

  Stable node, trajectories approach by slowest eigenvalue \(\lambda = -1\) :: \(\mathbf{v} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \)

  \(y = -2x\)

- Fixed point \((3,0)\):
  \[A = \begin{pmatrix} -3 & -6 \\ 0 & -1 \end{pmatrix} \Rightarrow \text{eigenvalues } -1,-3\]

  Stable node, slowest

  \[\mathbf{v} = \begin{pmatrix} 3 \\ -1 \end{pmatrix} \] is an eigenvector

  \(y = -\frac{1}{3}x\)
**Fixed point (1, 1):**  
\[ A = \begin{pmatrix} -1 & -2 \\ -1 & -1 \end{pmatrix} \Rightarrow \lambda = -1 \pm \sqrt{2} \text{ saddle}\]

Separatrix along the two parts of the stable manifold forms part of the basin of attraction for each of the stable nodes.

\[ \Rightarrow \text{if the initial rabbit population is sufficiently larger compared to the initial sheep population, then rabbits will eventually dominate and sheep will die out [and vice-versa].} \]

- Principle of competitive exclusion

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**Conservative Systems.**

- Newton's Second Law \[ m \ddot{x} = F(x) \text{ [Nonlinear force]} \]
  - Independent of \( x \) and \( t \) \[ \text{[no damping or inhomogeneity]} \]

Find a (non-unique) potential \( V(x) \) such that \[ F(x) = -V'(x) \]

\[ \Rightarrow m \ddot{x} + V'(x(t)) = 0 \]

Multiply both sides by \( \dot{x} \) and note that

1. \[ \dot{x}^2 = \frac{1}{2} \frac{d}{dt} (x^2) \]
2. \[ \frac{d}{dt} V(x(t)) = \dot{x} \frac{dV}{dx} \text{ [by chain rule]} \]

\[ \Rightarrow \frac{d}{dt} \left[ \frac{1}{2} m \dot{x}^2 + V(x(t)) \right] = 0 \]

\[ \therefore E = \frac{1}{2} m \dot{x}^2 + V(x(t)) \text{ is conserved [value of } E \text{ is determined by the initial conditions]} \]

- This is an example of a conservative system as a system is conserved
In general, consider \( \dot{x} = f(x) \)

\[ \Rightarrow \text{a conserved quantity } E(x) \text{ [E is scalar]} \text{ is a real-valued continuous function that is constant along trajectories, i.e. } \frac{dE}{dx} = 0 \]

\[ \Rightarrow \text{We preclude the trivial cases } E = \text{constant on any open set of } \mathbb{R}^n \]

**Property:** A conservative system cannot have any attracting fixed points

**Proof:** Assume that there is a fixed point \( x_* \), where \( x_* \) is attracting.

\[ \Rightarrow \text{all points in the basin of attraction also have energy } E(x_*) \]

\[ \Rightarrow \text{E is constant on an open set } \]

\[ \Rightarrow \text{So we have a contradiction } \]

\[ \Rightarrow \text{So we cannot have attracting fixed points, but we can have saddles and centers, etc.} \]

**Example:**

Consider \( \ddot{x} + V'(x(t)) = 0 \), where \( V(x) = -\frac{1}{2}x^2 + \frac{1}{4}x^4 \)

\[ \Rightarrow \ddot{x} = x - x^3 \Rightarrow \begin{cases} \dot{x} = y \\ \dot{y} = x - x^3 = x(1-x)(1+x) \end{cases} \]

**Fixed points** \((0,0), (\pm 1, 0)\).

**Jacobian** \( J(x, y) = \begin{pmatrix} 0 & 1 \\ 1 - 3x^2 & 0 \end{pmatrix} \)

**At \((0,0)\):** \( A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) \( \Rightarrow T = 0, D = -1 < 0 \Rightarrow \text{saddle point} \)

**At \((\pm 1, 0)\):** \( A = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix} \) \( \Rightarrow T = 0, D = 2 \Rightarrow \text{center} \) (marginal case)

Are these centers a spurious result of linearization?

\[ \Rightarrow \text{Not in this case! } \]

Recall, \( E = \frac{1}{2}x^2 - \frac{1}{2}x^2 + \frac{1}{4}x^4 \) is a conserved quantity

\[ \Rightarrow \text{so we have trajectories that are closed curves, which are contours of constant energy!} \]
Phase portrait:

Not a periodic orbit as the trajectory takes forever to reach the fixed point.

So we have two trajectories that both start \((t \to -\infty)\) and finish \((t \to +\infty)\) at the same point! "Homoclinic orbit"

Fact (Robustness of nonlinear center): If an isolated fixed point \(x^*\) is a minimum (or maximum) of a conserved quantity \(E(x)\), then all trajectories sufficiently close to \(x^*\) are closed.

Cautionary example (Importance of the fixed point being isolated):

Consider \(\begin{cases} x &= xy \\ y &= -x^2 \end{cases}\) Note: \(E = x^2 + y^2\) is conserved since \(\frac{dE}{dt} = 2xx' + 2yy' = 2x^2y - 2x^2y = 0\)

Also, we have a line of points \((x, y) = (0, \alpha)\ \forall \alpha \in \mathbb{R}\) along which the system is in equilibrium.

Note: \((x, y) = (0, 0)\) is a minimum of \(E\) and is a fixed point. (But non-isolated).

[trajectories follow circular arcs, but are not closed]
Reversible systems: "time-reversal symmetry"

Consider \( m\ddot{x} = F(x) \) \( \Rightarrow \begin{cases} \dot{x} = y \\ \dot{y} = \frac{1}{m} F(x) \end{cases} \)

If we map \( t \mapsto -t \) (use chain rule) and \( y \mapsto -y \) then both equations remain the same.

So if \((x(t), y(t))\) is a solution then so is \((x(-t), -y(-t))\)

\( \Rightarrow \) each trajectory has a twin!

Flip about \( y = 0 \) and change direction of arrows.

More generally: \( \begin{cases} \dot{x} = f(x, y) \\ \dot{y} = g(x, y) \end{cases} \) is reversible if the system is invariant under the mapping \( t \mapsto -t, y \mapsto -y, \)

\( \{ \) is odd in \( y \) \( f(x, -y) = -f(x, y) \)

and \( g \) is even in \( y \)

\( g(x, -y) = g(x, y) \)

Like conservative systems, centers are robust in reversible systems.

Suppose \( x = 0 \) is a linear center of a reversible system. Then close to the origin, all trajectories are closed curves.

Consider a trajectory that starts on positive \( x \)-axis sufficiently near the origin. By local swirling of the vector field (anti-clockwise), we expect the trajectory to reach the negative \( x \)-axis at some later time. By reversibility, the twin trajectory yields a closed orbit.

Example: \( \begin{cases} \dot{x} = y - y^3 \\ \dot{y} = -x - y^2 \end{cases} \) reversible! \( \Rightarrow \) Fixed points \((0, 0), (-1, \pm 1)\)

Note: Jacobian \( J(x, y) = \begin{pmatrix} 0 & 1 - 3y^2 \\ -1 & -2y \end{pmatrix} \)

\( \Rightarrow \) at \((0, 0)\), \( A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \) \( \Rightarrow \) \( \det(A) = 0 \)

\( \Rightarrow \) the system is reversible, the origin is a nonlinear center.

At \((-1, \pm 1)\), \( A = \begin{pmatrix} 0 & -2 \\ -1 & \mp 2 \end{pmatrix} \) \( \Rightarrow \) \( \det(A) = -2 < 0 \) \( \Rightarrow \) saddle points at \((-1, \pm 1)\).
Note: the saddle points are joined by a pair of trajectories
1) "heteroclinic trajectories" or "saddle connections"
[reminiscent of heteroclinic orbits for conservative systems].

Example: \[
\begin{align*}
\dot{x} &= -2\cos x - \cos y \\
\dot{y} &= -2\cos y - \cos x
\end{align*}
\] \rightarrow\text{reversible system. [and 2\pi-periodic in } x \text{ and } y]\)

Fixed points satisfy \(\cos x_\pm = \cos y_\pm = 0\)

\(\therefore\) fixed points \((\pm \frac{\pi}{2}, \pm \frac{\pi}{2})\), \((-\frac{\pi}{2}, \pm \frac{\pi}{2})\).

Jacobian \(J(x,y) = \begin{pmatrix} 2\sin x & \sin y \\ \sin x & 2\sin y \end{pmatrix}\)

We find that:
\[
\begin{align*}
\circ (\pm \frac{\pi}{2}, \frac{\pi}{2}) & \text{ unstable node} \\
\circ (\pm \frac{\pi}{2}, -\frac{\pi}{2}) & \text{ stable node} \\
\circ (-\frac{\pi}{2}, \frac{\pi}{2}) & \text{ saddle} \\
\circ (\frac{\pi}{2}, -\frac{\pi}{2}) & \text{ saddle}
\end{align*}
\]
**Nonlinear pendulum.**

\[ \frac{d^2 \theta}{dt^2} + g \sin \theta = 0 \]

Let \( \tau = \sqrt{\frac{g}{L}} t \) be a dimensionless timescale \( \Rightarrow \frac{d^2 \theta}{d\tau^2} + \sin \theta = 0 \)

So \( \begin{cases} \frac{d\theta}{d\tau} = \omega \\ \frac{d\omega}{d\tau} = -\sin \theta \end{cases} \)

**Angular velocity.**

Fixed points \((\theta, \omega)\) are \((0, 0)\) and \((\pi, 0)\) \(\text{[and } 2\pi \text{ rotations in } \theta]\)

Jacobian \(J(\theta, \omega) = \begin{pmatrix} 0 & 1 \\ -\cos \theta & 0 \end{pmatrix} \)

So at \((0, 0)\): \(A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \Rightarrow \text{Tr}(A) = 0, \det(A) = 1 \)

:\text{linear center at origin.}

\(\Downarrow\) Is the origin also a nonlinear center?

Yes \(\{\)

1. We have a time-reversible system
2. We have a conservative system due to the potential \(V(\theta) = -\cos \theta\)

\(\Rightarrow \frac{d\theta}{dt} = -V'(\theta)\). \(\because E = \frac{1}{2} \dot{\theta}^2 - \cos \theta\) is a conserved quantity.

Furthermore \(E(\theta, \omega) = \frac{1}{2} \omega^2 - \cos \theta\) has a minimum at \((\theta, \omega) = (0, 0)\).

\(\Downarrow\) Furthermore, for \(|\theta| \ll 1\) (and \(|\omega| \ll 1\)), we have \(E(\theta, \omega) \approx \frac{1}{2} (\theta^2 + \omega^2) - 1 + O(\theta^4)\)

\(\therefore\) nearly circular orbits close to the origin.

At \((\pi, 0)\): \(A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\), \(\det A < 0 \Rightarrow \text{saddle!} \)

- Eigenvalues are \(\lambda_1 = 1, \lambda_2 = +1\) with eigenvectors \(v_1 = (-1), v_2 = (1)\)
Phase portrait

$E > 1$ higher-energy rotations

$critical\ case\ E = 1$ [heteroclinic trajectories]

$E = 1$ lower-energy oscillations. \quad (E < 1)

$E = -1$: lowest energy state at $(\theta, \omega) = (0, 0)$ \quad [pendulum hanging downwards]

$E = 1$: heteroclinic trajectories.

The effect of damping:

$\ddot{\theta} + b \dot{\theta} + \sin \theta = 0$, \quad $b > 0$ is the damping parameter.

The system continually loses energy, i.e. $E = \frac{1}{2} \dot{\theta}^2 - \cos \theta$ decreases over time \quad [trajectories]

$\frac{dE}{dt} = \frac{d}{dt} \left( \frac{1}{2} \dot{\theta}^2 - \cos \theta \right) = \dot{\theta} \ddot{\theta} + \sin \theta = -b \dot{\theta}^2 < 0$

Linear stability analysis shows that for $b > 0$, $(\Theta_*, \omega_*) = (0, 0)$ becomes a stable spiral, whilst $(\Theta_*, \omega_*) = (0, \pi)$ remains a saddle.

If $E(0) < 1$, the oscillations are damped

If $E(0) > 1$, the pendulum whips around, until $E(t) = 1$, at which point no more rotations occur and there are damped oscillations about the origin.
Index Theory

- Linear stability analysis only gives uninformation locally to a fixed point.
- Index theory gives us global information.
  - does a closed trajectory encircle a fixed point?
  - If yes, what kind of fixed points?
- What types of trajectory can coalesce at bifurcation?
- Can rule out the existence of closed orbits.

The index of a closed curve.

\[ I_c \]

**An integer that describes the net winding of a vector field on a closed curve \( C \).**

- not necessarily a trajectory
- \( C \) does not intersect itself

- At \( x \) moves counterclockwise around \( C \), \( f(x) \) changes continuously, so \( \phi \) varies continuously also.
- After one loop, \( x \) returns to its original direction but \( \phi \) has changed by an integer multiple of \( 2\pi \)

**Index of \( C \)**

Define \( I_c = \frac{1}{2\pi} \oint_C \phi \)

\( \Rightarrow I_c \) is the net number of counterclockwise revolutions

as \( x \) moves once counterclockwise around \( C \).

**Example method**

Translate each vector

\[ I_c = \text{net number of counterclockwise revolutions made by the numbered vectors} \]

So \( I_c = +1 \)

Example:

\[ I_c = -1. \]
Example. \[
\begin{align*}
\dot{x} &= x^2 y \\
\dot{y} &= x^2 - y^2
\end{align*}
\] C is the unit circle \(x^2 + y^2 = 1\).

Properties.

1. If \(C\) can be continuously deformed to \(C'\) without passing through a fixed point then \(I_C = I_{C'}\).
   - Proof: Deforming \(C\) continuously means that \(\phi\) varies continuously, and hence \(I_C\) varies continuously. But \(I_C\) only takes integer values, and can only change by jumping. Therefore \(I_C\) is constant.  

2. If \(C\) doesn't include any fixed points then \(I_C = 0\).
   - Proof: We can continuously deform \(C\) to any \(\phi\)-circle without changing \(I_C\). But \(\phi\) is \(\approx\) constant on this circle by continuity. Smoothness of \(f(x) \Rightarrow \int f(\phi) = 0 \Rightarrow I_C = 0\). 

3. If we map \(t \mapsto -t\) then the arrows change direction but the index is unchanged.
   - Proof: \(\phi \mapsto -\phi + \pi\). \(\int f(\phi)\) remains constant.

4. Suppose the closed curve \(C\) is actually a trajectory of the system, i.e. \(C\) is a closed orbit. Then \(I_C = \pm 1\).

The vector field is everywhere tangent to \(C\).

As \(x\) winds around \(C\) once, the tangent vector also rotates once in the same sense.
Index of a point: Suppose $x_*$ is an isolated fixed point. The index $I_C$ of $x_*$ is defined as $I_C$, where $C$ is any closed curve that encloses $x_*$ and no other fixed points. By Property (1), $I_C$ is independent of $C$ and is only a property of $x_*$. 

Example: Stable node

![Diagram of a stable node]

(example before) $\Rightarrow I = +1$

Example: Unstable node

![Diagram of an unstable node]

Set $t \to -t$ to get stable nodal $\Rightarrow I = +1$

Example: Saddle

![Diagram of a saddle]

(similar to previous example) $\Rightarrow I = -1$.

Theorem: If $C$ is a closed curve that surrounds $n$ isolated fixed points $x_1, \ldots, x_n$ then $I_C = I_1 + \ldots + I_n$, where $I_j$ is the index of $x_j$ for $j = 1, \ldots, n$.

Idea of proof: Deform contour $C$ to a new contour $T$: $\Rightarrow I_C = I_T$

- $x_j$ is a small circle about $x_j$ with index $I_j$.
- Each $x_j$ is connected by a two-way bridge.
- To contributions to $I_T$ cancel out as bridges become narrower, $\Rightarrow$ only need contribution from circles.

$I_C = I_T = \sum_{k=1}^{N} I_k$.

Corollary: A closed orbit must enclose fixed points whose indices sum to $+1$.

Proof: Let $C$ denote the closed orbit. By property 4; $I_C = +1$.

By Thm, $\sum_{k=1}^{N} I_k = I_C = +1$.

So there is always a fixed point inside a closed orbit.

- If there is only one fixed point inside a closed orbit, the fixed point cannot be a saddle.
Rabbits vs. sheep example: Impossibility of closed orbits.

\[
\begin{align*}
  \dot{x} &= x(3-x-2y) \\
  \dot{y} &= y(2-x-y)
\end{align*}
\]
\(x, y \geq 0\).

Recall the fixed points:

\[
\begin{align*}
  (0,0) &\text{: unstable node (index = +1)} \\
  (3,0) \text{ and (0,2)} &\text{: stable nodes (index = +1)} \\
  (1,1) &\text{: saddle point (index = -1)}
\end{align*}
\]

Candidate closed orbits:

- **C₁**: Impossible as no fixed points enclosed.
- **C₂**: Violates condition that interior induces sum \(\text{sum} = 0\).
- **C₃**: Orbit crosses the y-axis, which has straight-line trajectories. But trajectories cannot cross.

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- Other candidate closed orbits are rejected by similar arguments.