Lectures 7&8 – Linear systems

– To make analytical progress, we have often neglected inertial effects and considered the over-damped limit, reducing a second-order ODE to a first-order ODE. How can we analyse the dynamics when inertia is important?

Consider the harmonic oscillator (*) $m\ddot{x} + kx = 0$, where $x(t)$ is displacement.

We may defined the velocity $v(t) = \dot{x}(t)$, yielding the first-order system of equations

$$(\dagger)\quad \begin{cases} \dot{x} = v \\ \dot{v} = -\frac{k}{m}x \end{cases} \qquad [\text{from } (*) \text{ with } \ddot{x} = \dot{v}]$$

($\dagger$) is one example of a two-dimensional linear system

$$\begin{cases} \dot{x} = ax + by \\ \dot{y} = cx + dy \end{cases}$$

We now develop the machinery to study these 2D systems. In particular, we define

$$\underset{\sim}{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \Rightarrow \quad \dot{\underset{\sim}{x}} = A\underset{\sim}{x} \quad (✡)$$

The extra dimension allows for a much richer set of dynamics than one-dimension, despite $\dot{\underset{\sim}{x}} = A\underset{\sim}{x}$ being linear. We also find that understanding linear systems is crucial to characterising fixed points in nonlinear 2D systems.

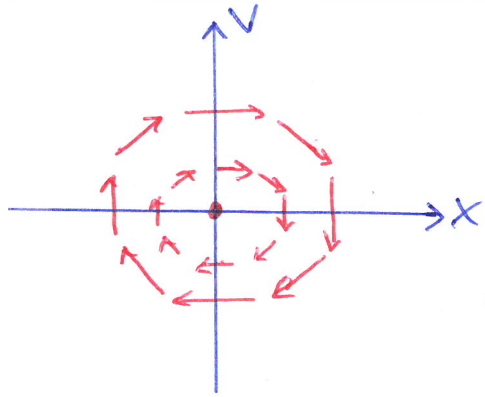
Why is (✡) linear? If $\dot{\underset{\sim}{x}} = A\underset{\sim}{x}$ and $\dot{\underset{\sim}{y}} = A\underset{\sim}{y}$, then $\underset{\sim}{z}(t) = \alpha\underset{\sim}{x}(t) + \beta\underset{\sim}{y}(t)$ satisfies $\dot{\underset{\sim}{z}} = A\underset{\sim}{z}$, due to properties of matrices and differentiation.

Example (the harmonic oscillator, continued)

Displacement $x(t)$
Velocity $v(t)$
Frequency $\omega = \sqrt{k/m}$

$$\boxed{\begin{aligned} \dot{x} &= v \\ \dot{v} &= -\omega^2 x \end{aligned}}$$

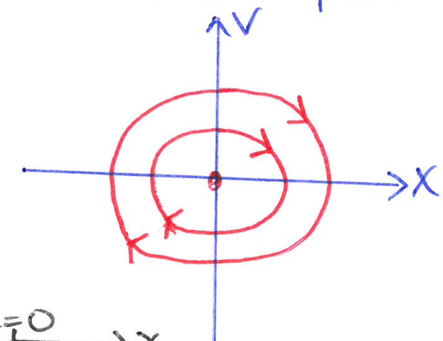
· Vector field $(\dot{x}, \dot{v}) = (v, -\omega^2 x)$ at each point (x,v) in the phase plane:



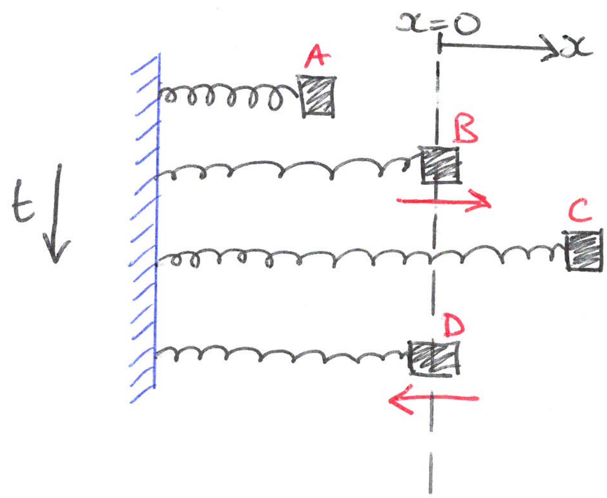
- arrow direction is parallel to the vector $(v, -\omega^2 x)$
- arrow length determines the magnitude of $(v, -\omega^2 x)$
- The origin $(x,v) = (0,0)$ is a fixed point.

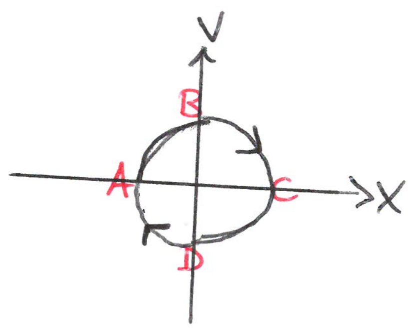
Phase portrait.

From the vector field, we can infer (and later prove!) that a trajectory will form a closed (ellipsoidal) orbit about the origin, where the motion is clockwise in phase space



- Fixed point corresponds to the spring being initialised at equilibrium.
- Larger orbits correspond to larger initial displacements/velocities of the spring.





Note: The effect of inertia is that the mass overshoots the rest position $(x=0)$. We will see that this "overshooting" is a typical property of two-dimensional systems.

$$\frac{y(t)}{y(0)} = \left[\frac{x(t)}{x(0)}\right]^{-1/a} \Leftrightarrow \left[\frac{y(t)}{y(0)}\right]^{-a} = \frac{x(t)}{x(0)}.$$

Example

$$\dot{\underset{\sim}{x}} = A\underset{\sim}{x}, \quad \underset{\sim}{x} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad A = \begin{pmatrix} a & 0 \\ 0 & -1 \end{pmatrix}$$

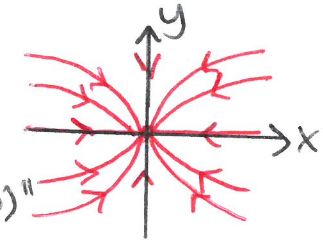
How do the trajectories depend on the parameter $a \in \mathbb{R}$?

Note: $\begin{cases} \dot{x} = ax \\ \dot{y} = -y \end{cases} \Rightarrow \left.\begin{array}{l} x(t) = x(0)e^{at} \\ y(t) = y(0)e^{-t}. \end{array}\right\}$ Fixed point $\underline{x} = \underline{x}_* = \underline{0}$.

↳ So $y \to 0$ as $t \to \infty$ (independent of a)

- **Case $a < -1$:**
  - $x \to 0$ faster than $y \to 0$ as $t \to \infty$.
  - x dominates y as $t \to -\infty$

"STABLE NODE at $(x,y) = (0,0)$"

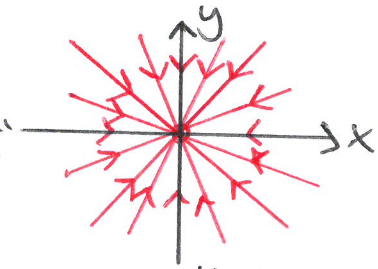


- **Case $a = -1$:** So for non-zero initial conditions,

$$\frac{x(t)}{x(0)} = \frac{y(t)}{y(0)} \Rightarrow \text{rays!}$$

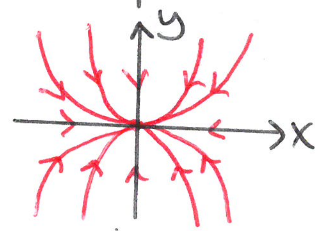
Both $x \to 0$, $y \to 0$ as $t \to \infty$

"STAR at $\underline{x} = \underline{0}$"



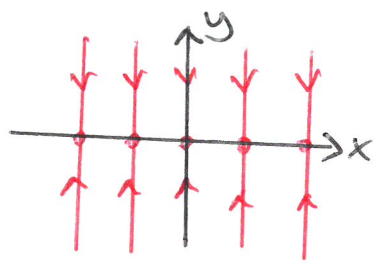
- **Case $-1 < a < 0$:** Similar to the case $a < -1$, but the roles of x and y swap!

"STABLE NODE"



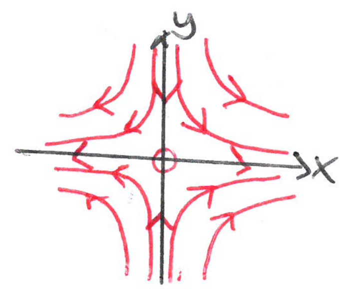
- **Case $a = 0$:** $x(t) = x(0)$, i.e. constant

"LINE OF FIXED POINTS"



- **Case $a > 0$:** Now $x(t) \to \infty$ as $t \to \infty$ and $x(t) \to 0$ as $t \to -\infty$

SADDLE (UNSTABLE)

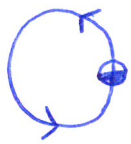


Note: the lines $x = 0$ and $y = 0$ bound regions (i.e. each quadrant) in which a trajectory never leaves (assuming it is in that region at some time!). We call the borders to these regions separatrices (or separatrix, singular!)

Classification of linear systems.

Let $\underline{x}_*$ be a fixed point and let $\underline{x}(t)$ be a trajectory.

- If $\underline{x}(t) \to \underline{x}_*$ as $t \to \infty$ for all $\underline{x}(0)$ sufficiently close to $\underline{x}_*$ then we say that $\underline{x}_*$ is an attractor.
- If $\underline{x}(t) \to \underline{x}_*$ as $t \to \infty$ for any $\underline{x}(0)$ then $\underline{x}_*$ is globally attracting.
- If $\underline{x}(t)$ has $\underline{x}(0)$ sufficiently close to $\underline{x}_*$ and $\underline{x}(t)$ remains close to $\underline{x}_*$ for all t, then $\underline{x}_*$ is Lyapunov stable. (Note: all $t > 0$, not just $t \to \infty$)
- If $\underline{x}_*$ is Lyapunov stable but not attracting then $\underline{x}_*$ is neutrally stable.
  e.g. harmonic oscillator with very small oscillations.
- Example: fixed point attracting but not Lyapunov stable
  $\dot{\theta} = 1 - \cos\theta$



- Lyapunov $\not\Rightarrow$ attracting
- attracting $\not\Rightarrow$ Lyapunov

- If $\underline{x}_*$ is Lyapunov stable and attracting, then $\underline{x}_*$ is stable, or asymptotically stable.
- If $\underline{x}_*$ is neither attracting or Lyapunov stable, then $\underline{x}_*$ is unstable.

Notation $\begin{cases} \circ & \text{unstable fixed point} \\ \bullet & \text{Lyapunov stable fixed point.} \end{cases}$

Solution to $\dot{\underline{x}} = A\underline{x}$

Inspired by solutions to the equivalent scalar system ($\dot{x} = ax$), we try $\underline{x}(t) = \underline{v}e^{\lambda t}$, where $\underline{v}$ is a non-zero vector and λ a scalar (possibly complex).

Substitute into $\dot{\underline{x}} = A\underline{x} \Rightarrow \underline{\underline{\lambda \underline{v} = A\underline{v}}}$. Eigenvalue problem $\begin{cases} \underline{v} \neq \underline{0} : & \text{Eigenvector} \\ \lambda \in \mathbb{C} : & \text{Eigenvalue.} \end{cases}$

By linearity, general solution is $\underline{x}(t) = c_1\underline{v}_1 e^{\lambda_1 t} + c_2 \underline{v}_2 e^{\lambda_2 t}$, where scalars c_1 and c_2 are determined by the initial conditions. [$\underline{v}_1$ & $\underline{v}_2$ are linearly independent if $\lambda_1 \neq \lambda_2$]

↳ The eigenvalues are eigenvectors are crucial in determining the behaviour of the system!

Computing eigenvalues (λ):

If $\underline{v} \neq \underline{0}$ then $A\underline{v} = \lambda \underline{v} \iff (A - \lambda I)\underline{v} = \underline{0}$

$\iff \det(A - \lambda I) = 0$ ← gives characteristic polynomial

For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\det(A - \lambda I) = \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = (a-\lambda)(d-\lambda) - bc$

$$= ad - \lambda(a+d) + \lambda^2 - bc$$

$$= \lambda^2 - \lambda\underbrace{(a+d)}_{=Tr(A)} + \underbrace{(ad-bc)}_{=\det(A)}$$

So $\underline{\lambda^2 - T\lambda + D = 0}$, where $\begin{cases} T = Tr(A) \\ D = \det(A) \end{cases}$.

↑ characteristic polynomial

So we have two eigenvalues: $\begin{cases} \lambda_1 = \frac{1}{2}\left(T + \sqrt{T^2 - 4D}\right) \\ \lambda_2 = \frac{1}{2}\left(T - \sqrt{T^2 - 4D}\right) \end{cases}$.

Example: $\dot{\underline{x}} = A\underline{x}$, $\underline{x}(0) = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$, $A = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}$.

$T = Tr(A) = -1$, $D = \det(A) = -6$ $\Rightarrow$ eigenvalues λ satisfy: $\lambda^2 + \lambda - 6 = 0$

$\Rightarrow \underline{\lambda_1 = 2, \lambda_2 = -3}$.

To find (non-zero) eigenvectors $\underline{v}_1$ and $\underline{v}_2$, note that

- $(A - \lambda_1 I)\underline{v}_1 = \begin{pmatrix} -1 & 1 \\ 4 & -4 \end{pmatrix}\underline{v}_1 = \underline{0} \Leftarrow \underline{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ (or any non-zero multiple!)
- $(A - \lambda_2 I)\underline{v}_2 = \begin{pmatrix} 4 & 1 \\ 4 & 1 \end{pmatrix}\underline{v}_2 = \underline{0} \Leftarrow \underline{v}_2 = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$ (or any non-zero multiple)

General solution: $\underline{x}(t) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{-3t}$

$\underline{x}(0) = \begin{pmatrix} 2 \\ -3 \end{pmatrix} \Rightarrow \underbrace{\begin{pmatrix} 1 & 1 \\ 1 & -4 \end{pmatrix}}_{:=B} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$

$:= B$, but $B^{-1} = -\frac{1}{5}\begin{pmatrix} -4 & -1 \\ -1 & 1 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 4 & 1 \\ 1 & -1 \end{pmatrix}$

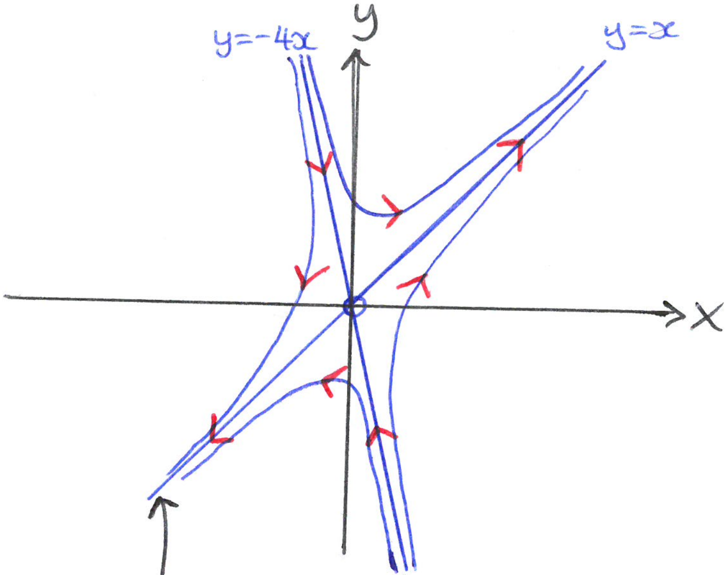
So $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 4 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \iff \underline{c_1 = 1, c_2 = 1}$.

So $\underline{\underline{x}(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + \begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{-3t}}$

Phase portrait

$\underline{x}(t) = c_1 \underline{x}_1(t) + c_2 \underline{x}_2(t)$, where $\underline{x}_1(t) = \begin{pmatrix}1\\1\end{pmatrix} e^{2t}$ ← dominates at $t \to \infty$

$\underline{x} = \begin{pmatrix}x\\y\end{pmatrix}$ $\qquad \underline{x}_2(t) = \begin{pmatrix}1\\-4\end{pmatrix} e^{-3t}$ ← dominates as $t \to -\infty$



y=x is the unstable manifold

y=-4x is the stable manifold.

The origin is a <u>saddle point</u>

Stable manifold:
↳ set of initial conditions s.t. $\underline{x}(t) \to \underline{x}_*$ as $t \to \infty$

Unstable manifold:
↳ set of IC's s.t. $\underline{x}(t) \to \underline{x}_*$ as $t \to -\infty$.

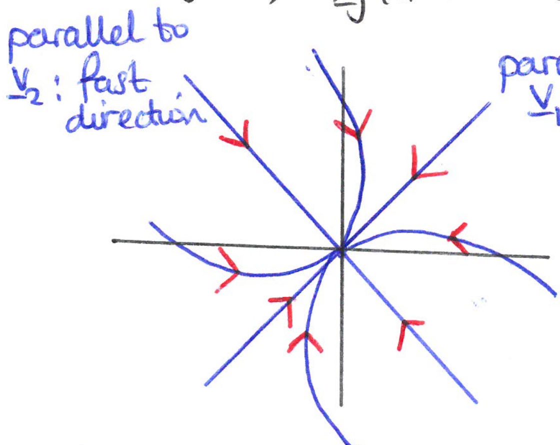
(*) All trajectories tend to the stable manifold as $t \to -\infty$, and approach the unstable manifold as $t \to +\infty$. [counter-intuitively!] (*)

Consider the general trajectory $\underline{x}(t) = c_1 \underline{v}_1 e^{\lambda_1 t} + c_2 \underline{v}_2 e^{\lambda_2 t}$, where $(\lambda_j, \underline{v}_j)$ are eigenpairs of the matrix A (where $\dot{\underline{x}} = A\underline{x}$).

$\boxed{\text{Define } \underline{x}_j(t) = \underline{v}_j e^{\lambda_j t}}$

Case $\lambda_1, \lambda_2 \in \mathbb{R}$ with $\lambda_2 < \lambda_1 < 0$

- As $\lambda_j \in \mathbb{R}$, $\underline{x}_j(t) \to 0$ as $t \to \infty$ as a stable node.



- As $t \to \infty$, trajectories approach the origin tangent to the slow direction.
- As $t \to -\infty$, trajectories become parallel to the fast direction.

Case $0 < \lambda_1 < \lambda_2$

Same picture, but reverse the direction of the arrows!

Recall that $\lambda_{1,2} = \frac{1}{2}(T \pm \sqrt{T^2 - 4D})$.

Consider the case where $T^2 < 4D$, ie. $\lambda_{1,2} \in \mathbb{C}$.

Define $\alpha = \frac{1}{2}T$, $\omega = \frac{1}{2}\sqrt{|T^2 - 4D|}$ $\Rightarrow$ $\underline{\underline{\lambda_{1,2} = \alpha + i\omega}}$.

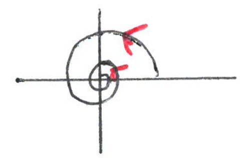
real

determines growth

determines oscillation frequency {period $= 2\pi/\omega$

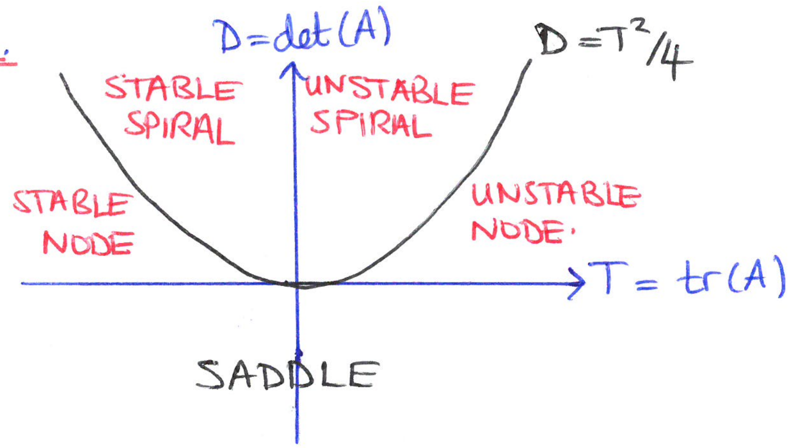
So by Euler's formula, $\underline{x}(t)$ has solutions that depends on t via
$\underline{e^{\alpha t}\cos(\omega t) \text{ and } e^{\alpha t}\sin(\omega t)}$.

$\therefore$ {
- decaying oscillations for $\alpha = Re(\lambda) < 0 \Leftrightarrow T < 0$
- growing oscillations for $\alpha > 0 \Leftrightarrow T > 0$
- circle for $\alpha = 0$.



(*) To compute rotation direction, compute a few vectors in the vector field and the direction should become obvious (*)

Regime diagram.



- $D < 0$: $\Rightarrow \lambda_+ > 0, \ \lambda_- < 0$
- $T < 0 \Rightarrow Re(\lambda_\pm) < 0 \ \therefore$ decay towards the origin as $t \to \infty$
- $T > 0 \Rightarrow Re(\lambda_\pm) > 0 \ \therefore$ origin unstable
- $D > T^2/4 \Leftrightarrow T^2 - 4D < 0 \ \therefore$ spiral
- $D < T^2/4 \Leftrightarrow T^2 - 4D > 0 \ \therefore$ node

<u>Boundary case $T = 0, D > 0$:</u> no growth / decay $\therefore$ <u>center</u>.

<u>Boundary $D = 0$:</u> $\lambda_+ = T, \lambda_- = 0 \Leftarrow$ no decay in $\underline{v}_-$ direction $\therefore$ a line of fixed points $\rightsquigarrow$ <u>"non-isolated fixed points"</u>.

e.g. $\dot{\underline{x}} = \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix}\underline{x}$ $\therefore$ The line $\underline{x}_* = \begin{pmatrix} 0 \\ y \end{pmatrix}$ is a fixed point $\forall y \in \mathbb{R}$.

- **Boundary $D = T^2/4$:** $\lambda_\pm = \frac{T}{2}$ (both equal!) , $\lambda = T/2$

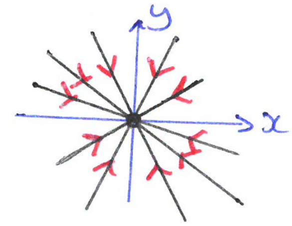
  - **Subcase 1:** Two linearly independent eigenvectors $\underline{v}_1, \underline{v}_2$

    So $\underline{x}(t) = c_1 \underline{v}_1 e^{\lambda t} + c_2 \underline{v}_2 e^{\lambda t}$

    $= e^{\lambda t}[c_1 \underline{v}_1 + c_2 \underline{v}_2] = e^{\lambda t}\underline{v}$ for any $\underline{v} \in \mathbb{R}^2$.

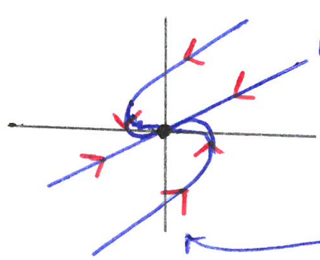
    So all lines are solutions: "star"

    ↳ stable with $T<0$ and unstable with $T>0$



  - **Subcase 2:** Only one linearly independent eigenvector $\underline{v}$ i.e. $\text{rank}(A) = 1$.

    - Need Exponential Response function → solutions of form $e^{\lambda t}$ and $te^{\lambda t}$. [theory for generalized eigenvectors]



← parallel to $\underline{v}$

✱ Almost a spiral but not quite! ✱

← becomes parallel to $\underline{v}$ as $t \to -\infty$.

Case study: Romeo and Juliet.

$$\begin{cases} R(t) = \text{Romeo's love/hate of Juliet} \\ J(t) = \text{Juliet's love/hate of Romeo.} \end{cases}$$

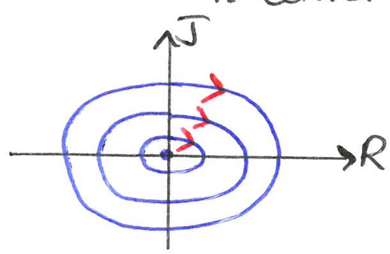
Example 1.

$(a,b>0)$ $\dot{R} = aJ$, $\dot{J} = -bR$

Interpretation: Juliet is put off by Romeo's advances, yet Romeo likes Juliet more when she likes him.

So $\frac{d}{dt}\begin{pmatrix} R \\ J \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & a \\ -b & 0 \end{pmatrix}}_{A} \begin{pmatrix} R \\ J \end{pmatrix} \Rightarrow Tr(A) = 0, \quad \det(A) = ab > 0$

∴ center!



→ A never-ending cycle of love and hate :(

(but they simultaneously love eachother ¼ of the time).

Example 2.

$$\dot{R} = -aR + bJ \qquad a, b > 0$$
$$\dot{J} = bR - aJ$$

Interpretation: Cautious lovers – they try to hold themselves back (determined by a) but they like the advances of the other (b)

So $\frac{d}{dt}\begin{pmatrix} R \\ J \end{pmatrix} = \overbrace{\begin{pmatrix} -a & b \\ b & -a \end{pmatrix}}^{A} \begin{pmatrix} R \\ J \end{pmatrix}$

$\therefore \mathrm{Tr}(A) = -2a < 0$

$\det(A) = a^2 - b^2$.

So $T^2 - 4D = 4a^2 - 4(a^2 - b^2) = 4b^2 > 0$

Eigenvalues $\lambda_\pm = -a \pm b$.

Eigenvectors:

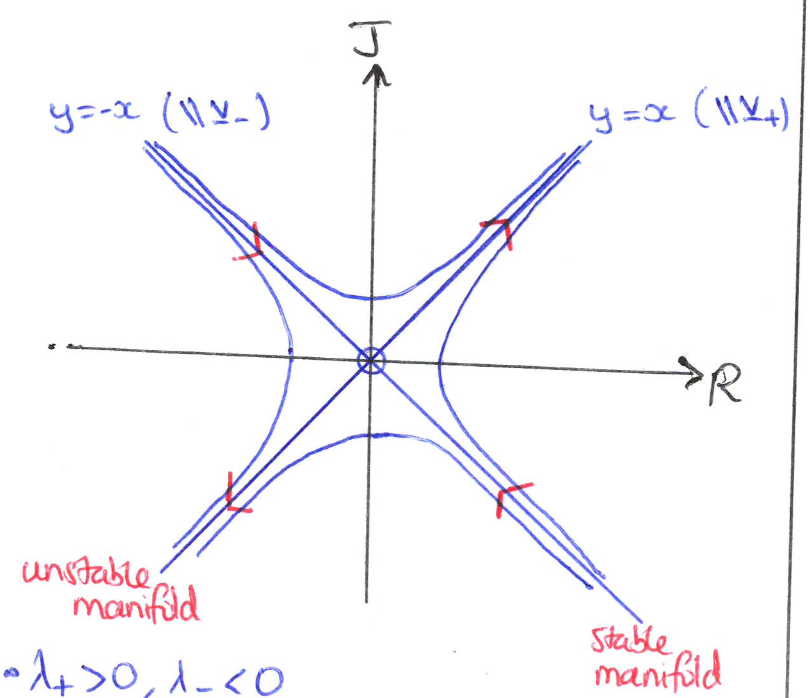
- $(A - \lambda_+ I)\underline{v}_+ = \underline{0}$

$\Leftrightarrow \begin{pmatrix} -b & b \\ b & -b \end{pmatrix} \underline{v}_+ = \underline{0} \Leftarrow \underline{v}_+ = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

- $(A - \lambda_- I)\underline{v}_- = \underline{0}$

$\Leftrightarrow \begin{pmatrix} b & b \\ b & b \end{pmatrix} \underline{v}_- = \underline{0} \Leftarrow \underline{v}_- = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

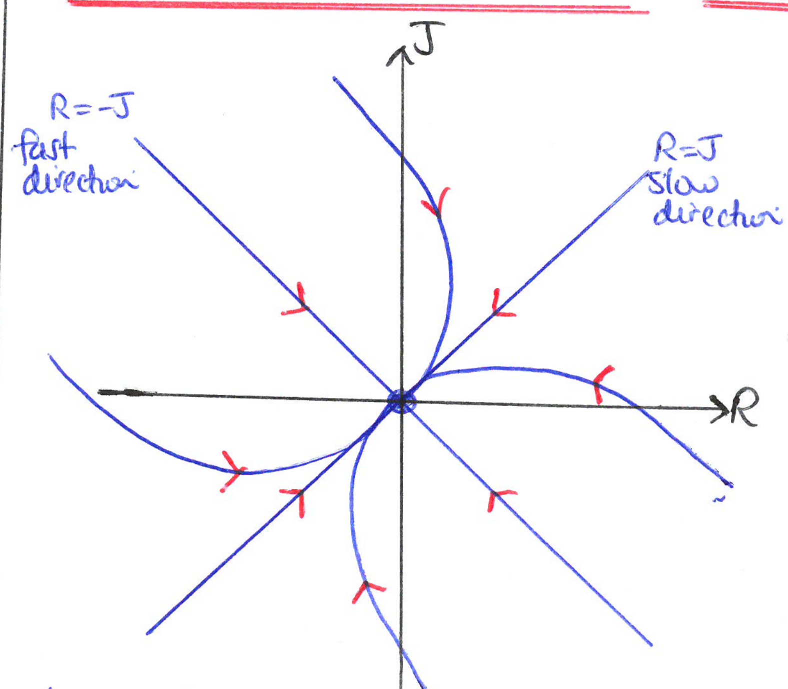
Case $a^2 - b^2 < 0$ $\Leftrightarrow 0 < a < b$ $\rightsquigarrow$ SADDLE



- $\lambda_+ > 0$, $\lambda_- < 0$

Depending on initial conditions, either both love eachother, or both hate eachother!

Case $0 < b < a \Leftrightarrow b^2 - a^2 < 0$: STABLE NODE



- $\lambda_+ < 0$, $\lambda_- < 0$, $\lambda_- < \lambda_+ < 0$
- Too much caution leads to apathy and their feelings fizzle out!

Example 3.

$$\dot{R} = R - J$$
$$\dot{J} = R + J$$

→ Romeo and Juliet both spurred on by their own feelings, but Romeo is put off by Juliet's advances. In contrast, Juliet likes Romeo's advances!

$$\frac{d}{dt}\begin{pmatrix} R \\ J \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}}_{A}\begin{pmatrix} R \\ J \end{pmatrix} \Rightarrow \left.\begin{array}{l} Tr(A) = 2 > 0 \\ \det(A) = 2 > 0 \end{array}\right\} \text{UNSTABLE SPIRAL.}$$

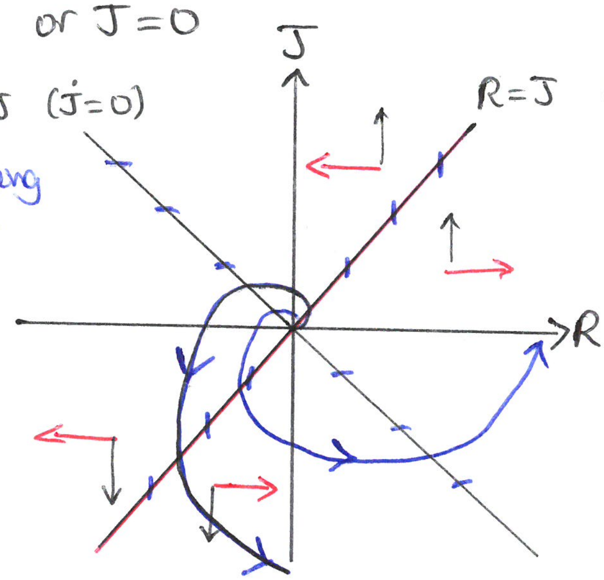
We define the <u>nullclines</u> as the curves (or lines in this case!) along which $\dot{R} = 0$ or $\dot{J} = 0$

$R = -J$ ($\dot{J} = 0$)

So J = constant along this line.

$R = J$ ($\dot{R} = 0$)

∴ R = constant along this line!



- $R > J$: $\dot{R} > 0$
- $R > -J$: $\dot{J} > 0$

Example 4. Do opposites attract?

$$a, b > 0: \begin{cases} \dot{R} = aR + bJ \\ \dot{J} = -bR - aJ. \end{cases} \qquad A = \begin{pmatrix} a & b \\ -b & -a \end{pmatrix} \qquad \begin{array}{l} Tr(A) = 0 \\ \det(B) = b^2 - a^2. \end{array}$$

So $\lambda_\pm = \frac{\pm 1}{2}\sqrt{-4(b^2 - a^2)} = \pm\sqrt{a^2 - b^2}$ ∴ $a > b \Rightarrow \lambda_\pm$ real ⇝ <u>saddle</u>.

$a < b \Rightarrow \lambda_\pm$ imaginary ⇝ <u>center</u>.

- If $a > b$, then depending on the initial conditions, either they eventually both love eachother or both hate eachother
- If $a < b$ then their love circles forever
- If $a = b$ then we have non-isolated fixed points along the line $R = -J$.