Lectures 7 & 8 - Linear Systems

To make analytical progress, we have often neglected inertial effects and considered the over-damped limit, reducing a second-order ODE to a first-order ODE. How can we analyse the dynamics when inertia is important?

Consider the harmonic oscillator $m\ddot{x} + kx = 0$, where $x(t)$ is displacement.

We may define the velocity $v(t) = \dot{x}(t)$, yielding the first-order system of equations:

$$
\begin{cases}
\dot{x} = v \\
\dot{v} = -\frac{k}{m} x
\end{cases}
$$

$\odot$ is one example of a two-dimensional linear system

$$
\begin{cases}
\dot{x} = ax + by \\
\dot{y} = cx + dy
\end{cases}
$$

We now develop the machinery to study these 2D systems. In particular, we define

$$
x(t) = (x(t), y(t)), \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow \dot{x} = Ax \odot
$$

The extra dimension allows for a much richer set of dynamics than one-dimensional, despite $\dot{x} = Ax$ being linear. We also find that understanding linear systems is crucial to characterising fixed points in nonlinear 2D systems.

Why is $\odot$ linear? If $\dot{x} = Ax$ and $\dot{y} = Ay$, then $\dot{z} = Az$ satisfies $\ddot{z} = A\ddot{z}$, due to properties of matrices and differentiation.

**Example (the harmonic oscillator, continued)**

- Displacement $x(t)$
- Velocity $v(t)$
- Frequency $\omega = \sqrt{\frac{k}{m}}$

\[\begin{align*}
\dot{x} &= v \\
\dot{v} &= -\omega^2 x
\end{align*}\]
Vector field \((\mathbf{x}, \mathbf{v}) = (v, -w^2 x)\) at each point in the phase plane:

- Arrow direction is parallel to the vector \((v, -w^2 x)\)
- Arrow length determines the magnitude of \((v, -w^2 x)\)
- The origin \((x, v) = (0, 0)\) is a fixed point.

Phase portrait:

From the vector field, we can infer (and later prove!) that a trajectory will form a closed (elliptical) orbit about the origin, where the motion is clockwise in phase space.

- Fixed point corresponds to the spring being initialised at equilibrium.
- Larger orbits correspond to larger initial displacements/velocities of the spring.

Note: The effect of inertia is that the mass overshoots the rest position \((x = 0)\). We will see that this "overshooting" is a typical property of two-dimensional systems.
Example
\[ \dot{x} = Ax, \quad x = (x, y), \quad A = \begin{pmatrix} a & 0 \\ 0 & -1 \end{pmatrix} \]

How do the trajectories depend on the parameter \( a \)?

Note:
\[
\begin{align*}
\begin{cases} 
\dot{x} = ax \\
y = -y
\end{cases} & \Rightarrow 
\begin{cases}
x(t) = x(0)e^{at} \\
y(t) = y(0)e^{-t}
\end{cases}
\end{align*}
\]
Fixed point \( x = x^* = 0 \).

\( \Rightarrow \) So \( y \to 0 \) as \( t \to \infty \) (independent of \( a \)).

- **Case \( a < -1 \):** \( x \to 0 \) faster than \( y \to 0 \) as \( t \to \infty \).
  - \( x \) dominates \( y \) as \( t \to -\infty \)

- **Case \( a = -1 \):** So for non-zero initial conditions,
  \[ \frac{x(t)}{x(0)} = \frac{y(t)}{y(0)} \Rightarrow \text{rays!} \]
  Both \( x \to 0, y \to 0 \) as \( t \to \infty \)

- **Case \( -1 < a < 0 \):** Similar to the case \( a < -1 \), but the roles of \( x \) and \( y \) swap!

- **Case \( a = 0 \):** \( x(t) = x(0) \), i.e., constant
  "Line of fixed points"

- **Case \( a > 0 \):** Now \( x(t) \to \infty \) as \( t \to \infty \)
  and \( x(t) \to 0 \) as \( t \to -\infty \)

**Note:**

The lines \( x = 0 \) and \( y = 0 \) bound regions (i.e., each quadrant) in which a trajectory never leaves (assuming it is in that region at some time!). We call the borders to these regions separatrices (or separatrix, singular!)
Classification of linear systems:

Let \( x_0 \) be a fixed point and let \( x(t) \) be a trajectory.

- If \( x(t) \to x_0^* \) as \( t\to\infty \) for all \( x(0) \) sufficiently close to \( x_0^* \) then we say that \( x_0^* \) is an **attractor**.

- If \( x(t) \to x_0^* \) as \( t\to\infty \) for any \( x(0) \) then \( x_0^* \) is **globally attracting**.

- If \( x(t) \) has \( x(0) \) sufficiently close to \( x_0^* \) and \( x(t) \) remains close to \( x_0^* \) for all \( t \) then \( x_0^* \) is **Lyapunov stable**. (Note: all \( t>0 \), not just \( t\to\infty \))

- If \( x_0^* \) is **Lyapunov stable but not attracting**, then \( x_0^* \) is **neutral stable**, e.g. harmonic oscillator with very small oscillations.

- **Example**: fixed point attracting but not Lyapunov stable

\[
\theta = 1 - \cos \omega t
\]

- If \( x_0^* \) is **Lyapunov stable and attracting**, then \( x_0^* \) is **asymptotically stable**.

- If \( x_0^* \) is neither attracting or Lyapunov stable, then \( x_0^* \) is **unstable**.

**Notation**

\[
\begin{align*}
\circ & \text{ unstable fixed point} \\
\bullet & \text{ Lyapunov stable fixed point}
\end{align*}
\]

**Solution to \( \dot{x} = Ax \)**

Inspired by solutions to the equivalent scalar system (\( \dot{x} = \lambda x \)), we try \( x(t) = v e^{\lambda t} \), where \( v \) is a non-zero vector and \( \lambda \) a scalar (possibly complex).

Substitute into \( \dot{x} = Ax \) to get \( \lambda v = Av \). Eigenvalue problem \( \lambda v = Av \), Eigenvector \( \lambda \in \mathbb{C} \), Eigenvalue.

By linearity, general solution is \( x(t) = c_1 v_1 e^{\lambda_1 t} + c_2 v_2 e^{\lambda_2 t} \), where scalars \( c_1 \) and \( c_2 \) are determined by the initial conditions, \( \{v_1, v_2\} \) are linearly independent.

- The eigenvalues are eigenvectors are crucial in determining the behavior of the system!
Computing eigenvalues ($\lambda$):

If $x \neq 0$ then $Av = \lambda v \implies (A - \lambda I)v = 0 \implies \det(A - \lambda I) = 0 \leftarrow$ gives characteristic polynomial

For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\det(A - \lambda I) = \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = (a-\lambda)(d-\lambda) - bd$

$= ad - \lambda(a+d) + \lambda^2 - bd$

$= \lambda^2 - \lambda(a+d) + (ad-bd)$

$= \text{Tr}(A)^2 - \det(A)$

So $\lambda^2 - \lambda T + D = 0$, where $T = \text{Tr}(A)$, $D = \det(A)$.

characteristic polynomial

So we have two eigenvalues: $\begin{cases} \lambda_1 = \frac{1}{2} \left( T + \sqrt{T^2 - 4D} \right) \\ \lambda_2 = \frac{1}{2} \left( T - \sqrt{T^2 - 4D} \right) \end{cases}$

Example: \[ \dot{x} = Ax, \quad x(0) = \begin{pmatrix} 2 \\ -3 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}. \]

$T = \text{Tr}(A) = -1$, $D = \det(A) = -6 \implies$ eigenvalues $\lambda$ satisfy: $\lambda^2 + \lambda - 6 = 0$ \[ \implies \lambda_1 = 2, \quad \lambda_2 = -3. \]

Thus the eigenvalues are non-zero.

To find eigenvectors $v_1$ and $v_2$, note that

$\begin{pmatrix} A - \lambda_1 I \end{pmatrix} v_1 = \begin{pmatrix} -1 & 1 \\ 4 & -4 \end{pmatrix} v_1 = 0 \iff v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ (or any non-zero multiple!)

$\begin{pmatrix} A - \lambda_2 I \end{pmatrix} v_2 = \begin{pmatrix} 4 & 1 \\ 4 & 1 \end{pmatrix} v_2 = 0 \iff v_2 = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$ (or any non-zero multiple)

General solution: \[ x(t) = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} -1 \\ -4 \end{pmatrix} e^{-3t} \]

$x(0) = \begin{pmatrix} 2 \\ -3 \end{pmatrix} \implies \begin{pmatrix} 1 & 1 \\ 4 & -4 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$

$= B$, but $B^{-1} = \frac{1}{5} \begin{pmatrix} 4 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 4 & 1 \\ 1 & -1 \end{pmatrix}$

So $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 4 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \implies c_1 = 1$, $c_2 = 1$.

So $x(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + \begin{pmatrix} -1 \\ -4 \end{pmatrix} e^{-3t}$. 
Phase portrait
\[ x(t) = c_1 x_1(t) + c_2 x_2(t), \quad \text{where} \quad x_1(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t} \quad \text{--- dominates at} \quad t \to \infty \\
\quad x_2(t) = \begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{-3t} \quad \text{--- dominates as} \quad t \to -\infty \]

The origin is a saddle point.

Stable manifold:
set of initial conditions s.t.
\[ x(t) \to x_* \quad \text{as} \quad t \to \infty \]

Unstable manifold:
set of IC's s.t.
\[ x(t) \to x^- \quad \text{as} \quad t \to -\infty \]

All trajectories tend to the stable manifold as \( t \to -\infty \), and approach the unstable manifold as \( t \to \infty \). [counter-intuitively!]

Consider the general trajectory \( x(t) = c_1 x_1 e^{\lambda_1 t} + c_2 x_2 e^{\lambda_2 t} \).

Define \( x_j(t) = x_j e^{\lambda_j t} \).

Case \( \lambda_1, \lambda_2 \in \mathbb{R} \) with \( \lambda_2 < \lambda_1 < 0 \)

- As \( \lambda_2 \in \mathbb{R}_- \), \( x_j(t) \to 0 \) as \( t \to \infty \) as a stable node.

\( x_1 \) : fast direction, \( x_2 \) : slow direction

- As \( t \to \infty \), trajectories approach the origin tangent to the slow direction.

- As \( t \to -\infty \), trajectories become parallel to the fast direction.

Case \( 0 < \lambda_1 < \lambda_2 \)

Same picture, but reverse the direction of the arrows!
Recall that $\lambda_{1,2} = \frac{1}{2} (T \pm \sqrt{T^2 - 4D})$.

Consider the case where $T^2 < 4D$, i.e., $\lambda_{1,2} \in \mathbb{C}$.

Define $\alpha = \frac{1}{2} T$, $\omega = \frac{1}{2} \sqrt{T^2 - 4D}$.

$\lambda_{1,2} = \alpha + i\omega$.

So by Euler's formula, $x(t)$ has solutions that depend on $t$ via

$e^{\alpha t} \cos(\omega t)$ and $e^{\alpha t} \sin(\omega t)$.

\[ \begin{cases} 
\text{o decaying oscillations for } \alpha = \Re(\alpha) < 0 \iff T < 0 \\
\text{o growing oscillations for } \alpha > 0 \iff T > 0 \\
\text{o circle for } \alpha = 0.
\end{cases} \]

To compute rotation direction, compute a few vectors in the vector field and the direction should become obvious.

Regime diagram.

- $D = \det(A)$
- $D = T^2/4$

- $D < 0 \implies \lambda_+ > 0, \lambda_- < 0$
- $T < 0 \implies \Re(\lambda_+) < 0$, decay towards the origin $t \to \infty$
- $T > 0 \implies \Re(\lambda_+) > 0$, origin unstable
- $D > T^2/4 \implies T^2 - 4D < 0$, spiral
- $D < T^2/4 \implies T^2 - 4D > 0$, node

Boundary case $T = 0, D > 0$: no growth/decay, center.

Boundary $D = 0$: $\lambda_T = T, \lambda_0 = 0 \iff$ no decay in $y$ direction, a line of fixed points $\iff$ non-isolated fixed points.

E.g., $\dot{x} = \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix} x$; The line $x = \begin{pmatrix} 0 \\ y \end{pmatrix}$ is a fixed point $\forall y \in \mathbb{R}$. 
*Boundary $D = T^2/4$: \( \lambda = \frac{T}{2} \) (both equal!), \( \lambda = T^{1/2} \)

- Subcase 1: Two linearly independent eigenvectors \( v_1, v_2 \)
  
  \[
  x(t) = c_1 v_1 e^{\lambda_1 t} + c_2 v_2 e^{\lambda_2 t}
  \]

  
  \[
  = e^{\lambda t} [c_1 v_1 + c_2 v_2] = e^{\lambda t} v
  \]

  for any \( v \in \mathbb{R}^2 \).

  So all lines are solutions: "star"

  - \( D \) stable with \( T < 0 \) and unstable with \( T > 0 \)

- Subcase 2: Only one linearly independent eigenvector \( v \)
  
  \( \text{ie. rank}(A) = 1 \).

  - Need Exponential Response function \( \rightarrow \) solutions of form \( e^{\lambda t} \) and \( x(t) \).

  \( \text{[theory for generalized eigenvectors]} \)

  \( \rightarrow \) almost a spiral but not quite!

  \( \rightarrow \) becomes parallel to \( x \) as \( t \to -\infty \).

---

**Case study: Romeo and Juliet.**

\[
\begin{cases}
R(t) &= \text{Romeo's love/hate of Juliet} \\
J(t) &= \text{Juliet's love/hate of Romeo}.
\end{cases}
\]

**Example 1.**

\( y > 0 \)

\[
\begin{align*}
\dot{R} &= aJ, \\
\dot{J} &= -bR
\end{align*}
\]

**Interpretation:** Juliet is put off by Romeo's advances, yet Romeo likes Juliet more when she likes him.

So \( \frac{d}{dt} \begin{pmatrix} R \\ J \end{pmatrix} = \begin{pmatrix} 0 & a \\ -b & 0 \end{pmatrix} \begin{pmatrix} R \\ J \end{pmatrix} \) \( \Rightarrow \text{Tr}(A) = 0 \), \( \det(A) = ab > 0 \)

\( \therefore \) center!

\[ \rightarrow \]

\( \rightarrow \) A never-ending cycle of love and hate

(but they simultaneously love each other \( \frac{1}{4} \) of the time).
Example 2.
\[ \dot{R} = -aR + bJ \quad a, b > 0 \]
\[ J = bR - aJ \]

So \( \frac{d}{dt} \langle R \rangle = \begin{pmatrix} -a & b \\ b & -a \end{pmatrix} \begin{pmatrix} R \\ J \end{pmatrix} \)

1. \( \text{Tr}(A) = -2a = 0 \)
   \( \det(A) = a^2 - b^2 \)
   So \( T^2 - 4D = 4a^2 - 4(a^2 - b^2) = 4b^2 > 0 \)

Eigenvalues \( \lambda_\pm = -a \pm b \).

Eigen vectors:
\( (A - \lambda_+ I) v_+ = 0 \)
\[ \begin{pmatrix} -b & b \\ b & -b \end{pmatrix} \begin{pmatrix} v_+ \\ v_+ \end{pmatrix} = 0 \]
\( v_+ = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \)

\( v_- = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \)

\( 0 < a < b \)

Case \( a^2 - b^2 < 0 \) \( \quad \Leftrightarrow \text{SADDLE} \)

\( \begin{cases} y = -x \\ x = (1 + x) v_+ \end{cases} \)

\( \begin{cases} y = x \\ x = (1 - x) v_- \end{cases} \)

\( \text{unstable manifold} \)
\( \text{stable manifold} \)

\( \lambda_+ > 0, \lambda_- < 0 \)

Depending on initial conditions, either both love each other, or both hate each other!

Interpretation: Cautious lovers - they try to hold themselves back (determined by a), but they like the advances of the other (b).

Case \( 0 < b < a \) \( \Rightarrow \; b^2 - a^2 < 0 \) \( \quad \Leftrightarrow \text{NODE} \)

\( \lambda_+ < 0, \lambda_- < 0 \), \( \lambda_- < \lambda_+ < 0 \)

Too much caution leads to apathy and their feelings fade out!
Example 3.

\[ \dot{R} = R - J \]
\[ \dot{J} = R + J \]

and Juliet both their Romeo is spurred on by own feelings, but is put off by Juliet's advances. In contrast, Juliet likes Romeo's advances!

\[ \frac{d}{dt} \begin{pmatrix} R \\ J \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} R \\ J \end{pmatrix} \]

\[ \Rightarrow \text{Tr}(A) = 2 > 0 \]
\[ \text{det}(A) = 2 > 0 \]

\[ \text{UNSTABLE SPIRAL.} \]

We define the nullclines as the curves (or lines in this case!) along which \( R = 0 \) or \( J = 0 \)

\[ R = -J \quad (J = 0) \]

So \( J = \text{constant along this line} \)

\[ R = J \quad (\dot{R} = 0) \]

\( \therefore R = \text{constant along this line!} \)

\( \cdot \quad R > J; \quad \dot{R} > 0 \)
\( \cdot \quad R < J; \quad \dot{J} > 0 \)

Example 4. Do opposites attract?

\( a, b > 0: \)

\[ \begin{cases} \dot{R} = aR + bJ \\ \dot{J} = -bR - aJ \end{cases} \]

\( A = \begin{pmatrix} a & b \\ -b & -a \end{pmatrix} \)

\( \text{Tr}(A) = 0 \)
\( \text{det}(A) = b^2 - a^2 \)

So \( \lambda = \pm \sqrt{4(b^2-a^2)} = \pm \sqrt{a^2 - b^2} \)

\( \cdot \quad a > b \Rightarrow \lambda_+ \text{ real } \rightarrow \text{saddle.} \)
\( a < b \Rightarrow \lambda_- \text{ imaginary } \rightarrow \text{center.} \)

\( \cdot \quad \text{If } a > b, \text{ then depending on the initial conditions, either they eventually both love each other or both hate each other} \)
\( \cdot \quad \text{If } a < b \text{ then their love circles forever} \)
\( \cdot \quad \text{If } a = b \text{ then we have non-isolated fixed points along the line } R = -J. \)