- How do the dynamics of a system depend on the system's parameters? (ie. the coefficients of each term in the ODE).

- Qualitative changes occur when a parameter is changed past a critical value, known as a bifurcation.

- Bifurcations are related to the change in stability of different fixed points. Moreover, fixed points may appear/disappear at a bifurcation.

- e.g. beam-buckling, pushing a heavy object along a non-smooth surface.

Saddle-node bifurcation
- Other names: turning-point bifurcation
- Fold bifurcation
- Blue sky bifurcation

- The creation/annihilation of fixed points.

- This form of bifurcation appears naturally in many systems of varying dimension and complexity. We consider the simplest "prototypical" example (the normal form).

Saddle-node: \( \dot{x} = r + x^2 \), \( r \in \mathbb{R} \) is a parameter

\( r < 0 \)

1 unstable
1 stable.

\( r = 0 \)

1 half-stable (Fixed points merge!)

\( r > 0 \)

No fixed points! (Fixed points destroyed)

\( \circ \) A bifurcation occurs at \( r = 0 \) \( \circ \)
(qualitative change in dynamics)
What are the fixed points as a function of $r$? 

Denote $x_u$ and $x_s$ as unstable/stable fixed points.

Note: $0 = r + x^2$; for $r \leq 0$ we have $x_* = \pm \sqrt{-r}$.

So $x_u = +\sqrt{-r}$

$x_s = -\sqrt{-r}$.

For $r > 0$, we have no solutions.

We represent the fixed points and their stability in a bifurcation diagram.

--- unstable

--- stable

--- Flow direction (optional)

The "control" parameter.

[The flow direction arrows are not necessary to constitute a bifurcation diagram - the main focus should be the existence/stability of fixed points.]

--- A more general system: $\dot{x} = f(x, r)$

--- sufficiently near to the fixed point at $r = r_c$ the function $f(x; r_c)$ may be approximated by a parabola in $x$ [like Taylor expansion!]
Example: Consider \( \dot{x} = f(x, r) = r - x - e^{-x} \)

Show that this system exhibits a saddle-node bifurcation at \( r = r_c \), where \( r_c \) is to be determined.

Method: solving \( f(x, r) = 0 \) for some given \( r \) analytically is intractable, so we adopt a graphical approach: when do the curves \( y = r - x \) and \( y = e^{-x} \) intersect, and when do the two fixed points collide?

At \( r = r_c \), we require that \( y = r - x \) and \( y = e^{-x} \) intersect tangentially at \( x = x^*_c \) [\( r_c \) and \( x^*_c \) are unknown]

\[ r - x^*_c = e^{-x^*_c} \]

\[ \frac{d}{dx} e^{-x} \bigg|_{x=x^*_c} = \frac{d}{dx} (r - x) \bigg|_{x=x^*_c} \implies -e^{-x^*_c} = -1 \implies x^*_c = 0 \]

Substitute \( x^*_c = 0 \) into (1) \( \implies r_c = 1 \).

Transcritical bifurcation. Describes when a fixed point always exists for all parameter values, but the stability of that point may change.

Normal form:

Transcritical \( \dot{x} = rx - x^2 \), \( r \in \mathbb{R} \) is a parameter, [looks like the logistic equation]
Example: Consider $x = r \log x + x - 1$ for $x \approx 1$. Find $r = r_c$ at which a transcritical bifurcation occurs. Then recast the system in normal form (approximate!)

- Note: $x=1$ is always a fixed point.
- We define $u = x - 1$, where $|u| \ll 1$

$\Rightarrow u = r \log(u+1) + u$

$= r \left[ u - \frac{1}{2} u^2 + O(u^3) \right] + u$

$= (r+1)u - \frac{1}{2} ru^2 + O(u^3)$

At $r = -1$, the fixed point $u = 0$ is repelled (no semi-stable).

Hence, a transcritical bifurcation occurs at $r_c = -1$.

To reduce to normal form, we need to rescale the variables to set the $u^2$ coefficient to -1.

Let $u = x v$, where $x \in \mathbb{R}$ is to be determined.

Substitute into $\circ$ $\Rightarrow \dot{v} = (r+1)v - (\frac{1}{2} rv) v^2 + O(v^3)$

Set $x = \frac{2}{r}$

Define $\begin{cases} R = r+1 \\ X = v \end{cases}$

Neglect the cubic term $\Rightarrow \dot{x} \approx RX - X^2$ [approximate normal form].
Application to solid-state lasers. [A simplified model.]

External energy to excite atoms from their ground state

- Weak excitation: excited atoms oscillate independently of each other
  ⇒ randomly phased light waves (i.e., a lamp)
- Sufficiently large excitation: atoms oscillate in-phase ⇒ laser

Can we rationalise the threshold at which the laser appears?

- $n(t)$: number of photons
- $N(t)$: number of excited atoms
- $G > 0$: gain coefficient
- $k > 0$: rate of escape through endfaces of the laser
- $N_0$: fixed number of excited atoms
  (in the absence of laser action)
- $\alpha$: rate at which atoms drop to their ground states

\[
\begin{align*}
\dot{n}(t) &= \frac{GnN}{(GN-k)n} - kn = (GN-k)n \\
N(t) &= N_0 - \alpha n \\
\Rightarrow \dot{n} &= (GN_0 - k)n - \alpha Gn^2.
\end{align*}
\]

Hence, we have a transcritical bifurcation at $N_0 = k/G$ (laser threshold)
**Pitchfork bifurcation.**
- Typically arises in problems that possess symmetry.
- Fixed points appear/disappear in symmetric pairs.

**Supercritical pitchfork bifurcation.** \( x = r x - x^3, \ r \in \mathbb{R} \)

Note: the equations are invariant under the mapping \( x \mapsto -x \), consistent with symmetry.

\[
\begin{array}{c}
\text{\( r < 0 \)} \\
\text{\( r = 0 \)} \\
\text{\( r > 0 \)}
\end{array}
\]

- 1 stable \( x_* = 0 \)
- 1 stable (repeated) \( x_* = 0 \)
- 2 stable, 1 unstable \( x_* = 0 \)
\( x_\pm = \pm \sqrt{r} \)

**Bifurcation diagram:**

**Example:** \( \dot{x} = -x + \beta \tanh(x) \),
- Note: \( \frac{d}{dx} \tanh(x) = 1 - \tanh^2(x) \), \( \frac{d}{dx} \tanh(x) \bigg|_{x=0} = 1 \)

\( \beta < 1 \)
\( \beta = 1 \)
\( \beta > 1 \)
Bifurcation diagram. Note: \( \tanh(x) \sim x - \frac{x^3}{3} + O(x^5) \) for \( |x| \ll 1 \)

\[ f(x; \beta) \sim -x + \beta \left(x - \frac{x^3}{3}\right) + O(x^5) \]

\[ = x(\beta - 1) - \frac{\beta}{3} x^3 + O(x^5). \]

So typical pitchfork shape occurs only for \( x \) small.

Note: \( \tanh x \sim 1 \) as \( x \gg 1 \) \( \Rightarrow f(x; \beta) \sim \beta \text{sign}(x) - x \) for \( |x| \gg 1 \).

So \( x \sim \pm \beta \) for \( \beta \) large.

Alternative: plot \( \beta = \beta(x) = x / \tanh(x) \).

\[ \text{dependence on the parameter is simpler than the dependence on the variable.} \]

Subcritical pitchfork bifurcation: \( \dot{x} = r x + x^3 \), \( r \in \mathbb{R}^+ \).

Cubic term now acts to destabilize.

Bifurcation diagram:

Problem: cubic terms enhance the instability in this case and yield blow-up:

i.e., solution approaches infinity in finite time! \([\text{Try solving the case}!!]\)

This feature is physically unrealistic!!

We resolve this issue by including higher-order terms (preserving symmetry):

\[ \dot{x} = r x + x^3 - x^5, \quad r \in \mathbb{R}^+ \]

\^[We can rescale variables to ensure that coefficient of \( x^5 \) is \(-1\).]
Bifurcation diagram.

Hysteresis: non-reversibility of the system when \( r \) is increased/decreased.

For \( r \in (r_1, 0) \), the origin is locally (but not globally) stable - long-time behaviour depends on the initial conditions.

---

Application: An overdamped bead on a rotating hoop.

\[ \phi : \text{angle between bead and downward vertical direction, } \phi \in (-\pi, \pi) \]

\[ r = rsin\phi : \text{distance of bead from vertical axis.} \]

---

Tangential force balance:

\[ m\phi = -b\phi - mg\sin\phi + m\omega^2 rsin\phi \cos\phi \]

Over-damped limit (we will formally justify this approximation later!)

Neglecting inertial effects gives

\[ b\phi = mg\sin\phi \frac{r\omega^2 \cos\phi - 1}{g} \]

Define \( \gamma = \frac{r\omega^2}{g} \), [dimensionless acceleration]

Fixed points:

\[
\begin{align*}
\forall \gamma > 0 : & \quad \phi_* = 0, \quad \phi_* = \pi \quad \text{[bottom and top]} \\
\forall \gamma > 1 : & \quad \cos\phi_* = \frac{1}{\gamma} \quad \text{, i.e. } \phi_* = \arccos(\gamma^{-1})
\end{align*}
\]
Furthermore, linear stability analysis reveals that:

- \( \phi_* = 0 \): stable for \( \gamma < 1 \), unstable for \( \gamma > 1 \)
- \( \phi_* = \pi \): unstable for all \( \gamma \).

Exercise:
(Analytically or graphically)

**Bifurcation diagram:**

\( \gamma < 1 \)

\( \gamma > 1 \)

- Long-time asymptotic behaviour depends on the form of the perturbation (i.e., which fixed point is "selected")

Physical interpretation: For \( \gamma > 1 \), the centrifugal force increases as the bead moves from the bottom, acting to destabilize the \( \phi_* = 0 \) solution.

**Justification for dropping inertia term:** Dimensionless variables.

We aim to introduce a characteristic time \( T \) and dimensionless time \( \tau = t / T \) so that \( \frac{d\phi}{d\tau} \) and \( \frac{d^2\phi}{d\tau^2} \) are of the size \( O(1) \), we need to find \( T \).

Substitute into \( \Phi \) and apply the chain rule: \( \frac{d\phi}{d\tau} = \frac{dt}{d\tau} \frac{d\phi}{dt} = \frac{1}{T} \frac{d\phi}{dt} \), etc.

\[ \Rightarrow \frac{m\tau}{I^2} \frac{d^2\phi}{d\tau^2} = -\frac{b}{T} \frac{d\phi}{d\tau} + mg \sin \phi \left[ \gamma \cos \phi - 1 \right] \]

Where \( \gamma = rw^2 / g \)

Re-arrange:

\[ \frac{m\tau}{I^2} \frac{d^2\phi}{d\tau^2} + \frac{d\phi}{d\tau} + \frac{mgT}{b} \sin \phi \left[ 1 - \gamma \cos \phi \right] = 0 \]
We choose $T$ s.t. $\frac{mgT}{b} = 1 \Rightarrow T = \frac{b}{mg}$.

We define the dimensionless parameter $\varepsilon > 0$ as: $\varepsilon = \frac{mr}{bT} = \frac{mr mg}{b b} = \varepsilon\left(\frac{mg}{b}\right)^2$.

Then we have the dimensionless equation:

$$\varepsilon \frac{d^2\phi}{dt^2} + \frac{d\phi}{dt} + \sin\phi\left[1 - \varepsilon \cos\phi\right] = 0. \quad \text{(1)}$$

So we may neglect inertia when $\left|\varepsilon \frac{d^2\phi}{dt^2}\right| \ll 1$, which is satisfied if (but not only if) $\varepsilon \ll 1$ and $\frac{d^2\phi}{dt^2} = O(1)$.

So drag and centrifugal forces balance.

What about initial conditions? $-2^{nd}$-order $\Rightarrow$ 2 IC's; $-1^{st}$-order $\Rightarrow$ 1 IC. \text{ mis-match when setting } z = 0.

This apparent paradox may be resolved by realising that $\frac{d^2\phi}{dt^2}$ may be very large for small times. To investigate the initial transient, define the new dimensionless timescale $\sigma$ so that $T = \varepsilon \sigma$, where $\varepsilon \ll 1$ and $\sigma = O(1)$,

$$\Rightarrow \frac{d^2\phi}{d\sigma^2} + \frac{d\phi}{d\sigma} + \varepsilon \sin\phi\left[1 - \cos\phi\right] = 0.$$

Balance for $\sigma = O(1)$.

So over short timescales, inertia balances drag, whilst centrifugal forces are irrelevant. Hence, initially:

$$\left\{ \begin{array}{c} \phi(\sigma) = \phi(0) + (1 - e^{-\sigma}) \frac{d\phi}{d\sigma} \bigg|_{\sigma = 0} \\ \frac{d\phi}{d\sigma} = e^{-\sigma} \frac{d\phi}{d\sigma} \bigg|_{\sigma = 0}. \end{array} \right.$$ 

So any initial velocity quickly vanishes over an $O(\varepsilon)$ timescale, yielding a small change in displacement. After this initial transient, the over-damped limit becomes appropriate and our first-order analysis is valid.

\text{This kind of problem is a singular limit - see courses on fluid mechanics (boundary layers) or perturbation theory.}
Imperfect bifurcations and Catastrophes.

What happens when imperfections are present in a system, violating symmetry?

Prototypical example: \( \dot{x} = h + r \alpha - x^3 \), \( h, r \in \mathbb{R} \)

\( h \) is the imperfection parameter.

(\( h = 0 \): Supercritical pitch fork bifurcation)

First consider \( r \) fixed, \( h \) varying.

Fixed points:

- For \( r \leq 0 \):
  \[ y = x^3 - r \alpha \]
  \[ y = h. \]
  1 stable fixed point

- For \( r > 0 \):
  \[ y = x^3 - r \alpha \]
  \[ y = h. \]
  \[ h_c(r) \]
  \[ h_c(r) \]
  1 stable fixed point
  2 stable, one unstable
  \( \Rightarrow \) Saddle-node bifurcation at \( h = \pm h_c(r) \)

Consider \( r > 0 \): What is \( h_c(r) \)?

To find \( h_c(r) \), we need horizontal line tangent to cubic.

So \( x \) satisfy \( \frac{d}{dx} [r \alpha - x^3] = 0 \) \( \Rightarrow x = \pm \frac{r}{\sqrt[3]{3}} \).

Note: \( (x^3 - r \alpha) \bigg|_{x = x_\pm} = \pm \frac{2}{3} r \sqrt[3]{3} \).

\( h_c(r) = \frac{2r}{3 \sqrt[3]{3}} \) and \( h = \pm h_c(r) \)

Bifurcation diagram \( (r > 0) \)

Note: we have a hysteresis loop!
A "codimension-two" bifurcation occurs:

Now consider \( h \) fixed, \( r \) varying.

Sketches of bifurcation diagrams:

Note: For \( h \neq 0 \), one branch remains stable \( \forall r \), and is smooth rather than exhibiting a corner (like at the pitchfork bifurcation for \( h = 0 \)).

See Strogatz p.73 for 3-dimensional drawings of the parameter space and an explanation of cusp catastrophes.

An application to biology: Insect outbreak. (Spruce budworm)

**Dynamics:** Spruce budworm attack the leaves of the balsam fir tree. Upon outbreak, the budworms can kill/defoliate most of the fir trees in a forest in about four years.

**Observation:** The system exhibits two timescales:

1. Fast timescale: evolution of budworm population is \( O(\text{months}) \)
2. Slow timescale: growth of trees (10+ years) \( \Rightarrow O(\text{decades}) \)

\[ \rightarrow \] For the evolution of budworm, make a quasistatic approximation for the tree population, i.e., forest variables are constant.
The model: 

\[ N(t) = RN \left(1 - \frac{N}{K}\right) - \rho(N) \]

- **logistic growth**
- **death rate due to predation**

- **Candidate function for** \( \rho(N) = \frac{BN^2}{A^2 + N^2} \)**

[empirical fit of predation]

**Full model:**

\[ \dot{N} = RN \left(1 - \frac{N}{K}\right) - \frac{BN^2}{A^2 + N^2} \]

**Non-dimensionalisation**

To get an idea of large/small [so as to characterise an outbreak], we non-dimensionalise.

- \([N] = [K] = [A]\)
- \([R]\) = \(\text{time}\)
- \([B]\) = \(\text{time} \times \frac{1}{[N]}\)

We define the dimensionless population \(x = \frac{N}{A}\) \([\text{could also take } N/K = x]\)

Let \(t = TT\), where \(T\) is dimensionless time and \(T\) is a time-scale [to be defined]

Sub into \(\Theta\) \[ \frac{A}{T} \frac{dx}{dt} = RA x \left(1 - \frac{A}{K} x\right) - B \frac{x^2}{1+x^2} \]

\[ \text{choose } T \text{ to balance these terms, i.e. } \frac{A}{T} = B \Rightarrow T = \frac{A}{B} \]

Also, let \(r = RT\), \(k = \frac{K}{A}\).

\[ \text{Dimensionless model} \]

\[ \frac{dx}{dt} = r x \left(1 - \frac{x}{k}\right) - \frac{x^2}{1+x^2} \]

\(r, k > 0\)

**Fixed points:** By linearising 1 about \(x = 0\), one can see that the fixed point \(x^* = 0\) is always unstable.

The remaining fixed points satisfy \(r \left(1 - \frac{x}{k}\right) = \frac{x}{1+x^2}\).
Fixed points occur at the intersections of \( y = r(1 - \frac{x}{k}) \) and \( y = \frac{x}{1 + x^2} \).

Note: As \( r \) and \( k \) vary, the curve \( y = \frac{x}{1 + x^2} \) remains fixed, making things easier!

Possible to have 1, 2, or 3 intersections depending on \( r \) and \( k \).

Case of 3 (non-zero) fixed points [denoted \( a, b, c \)]

- Following the saddle-node bifurcation, only \( a \) remains.

FACT: Stability alternates between fixed points.

\( a^* = 0 \) is always unstable.

Note: Outbreak can also occur due to a saddle-node bifurcation.
Calculating the bifurcation curves.

- What does \((r,k)\) parameter space look like?

Condition for a saddle-node bifurcation: \(y = r(1-x/k)\) and \(y = \frac{x}{1+x^2}\) intersect tangentially.

\[\therefore \text{ we require } 1 \quad r\left(1 - \frac{x}{k}\right) = \frac{x}{1+x^2}\]
and \(2 \quad \frac{d}{dx}\left[r\left(1 - \frac{x}{k}\right)\right] = \frac{d}{dx}\left[\frac{x}{1+x^2}\right] \Rightarrow -\frac{r}{k} = \frac{1-x^2}{(1+x^2)^2}\)

- Use 2 to eliminate \(r/k\) from 1

\[\Rightarrow r = \frac{2x^3}{(1+x^2)^2}.\]

and hence \(k = \frac{2x^3}{x^2-1}\).

For \(k > 0\) we need \(x > 1\) (as \(x > 0\)).

We may treat \(x > 1\) as a parameter (confusing terminology?)
and plot \((k, r) = (k(x), r(x))\)

Note: As \(x \to \infty\), \[\left\{ \begin{array}{l} r(x) \sim 2/x \\ k(x) \sim 2x \end{array} \right. \quad \quad \quad \quad r(x), k(x) \sim 4, \quad \text{so } r \sim \frac{4}{k}.

Note: as \(x \to 1\), \[\left\{ \begin{array}{l} r \sim 2/4 = 1/2 \\ k \sim \frac{1}{x-1} \to \infty \end{array} \right\} \quad \quad \quad \therefore r \to 1/2

\[\text{labels correspond to only stable fixed points that exist}.\]

See Strogatz p.80 for a discussion of how the effect of slowly varying \(r\) and \(k\) may affect the dynamics, and outbreak. 