Lectures 3–5 : Bifurcations

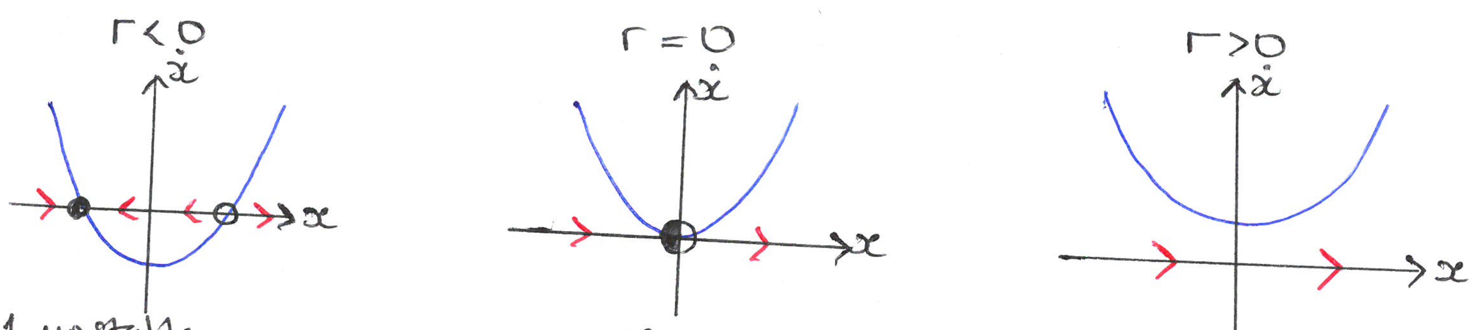
- How do the dynamics of a system depend on the system's parameters? (ie. the coefficients of each term in the ODE).
  - ↳ qualitative changes occur when a parameter is changed past a critical value, known as a bifurcation.
- Bifurcations are related to the change in stability of different fixed points. Moreover, fixed points may appear/disappear at a bifurcation.
- e.g. beam-buckling, pushing a heavy object along a non-smooth surface.

Saddle-node bifurcation.

[other names: – turning-point bifurcation
– fold bifurcation
– blue sky bifurcation]

- The creation/annihilation of fixed points.
- This form of bifurcation appears naturally in many systems of varying dimension and complexity. We consider the simplest "prototypical" example (the normal form)

$$\boxed{\text{Saddle-node: } \quad \dot{x} = r + x^2, \quad r \in \mathbb{R} \text{ is a parameter}}$$



1 unstable
1 stable.

1 half-stable
(fixed points merge!)

No fixed points!
(fixed points destroyed)

⊛ A bifurcation occurs at $r=0$ ⊛
(qualitative change in dynamics)

What are the fixed points as a function of r?

- denote x_u and x_s as unstable/stable fixed points

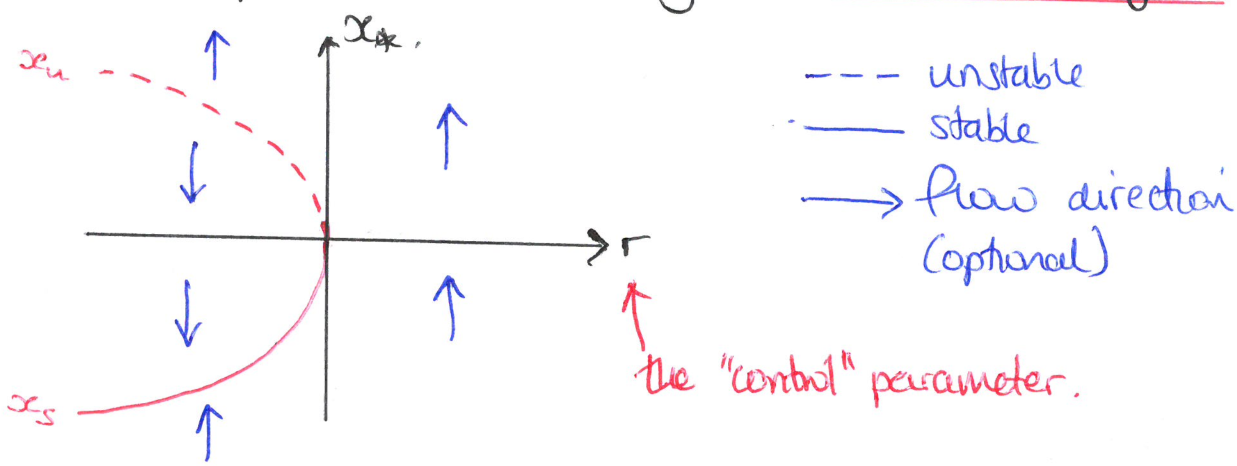
– Note: $0 = r + x_*^2$ $\therefore$ for $r \leq 0$ we have $x_* = \pm\sqrt{\underbrace{-r}_{>0}}$

So $x_u = +\sqrt{-r}$

$x_s = -\sqrt{-r}$

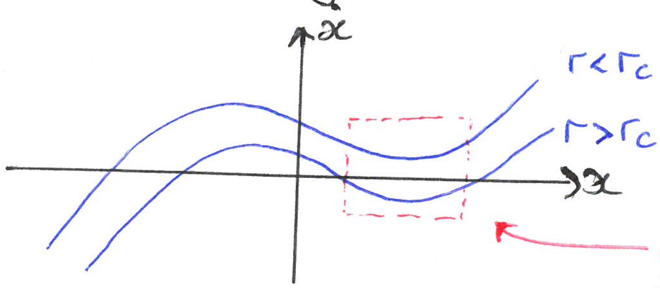
For $r > 0$, we have no solutions.

We represent the fixed points and their stability in a bifurcation diagram.



[the flow direction arrows are not necessary to constitute a bifurcation diagram – the main focus should be the existence/stability of fixed points]

A more general system: $\dot{x} = f(x; r)$ ← parameter



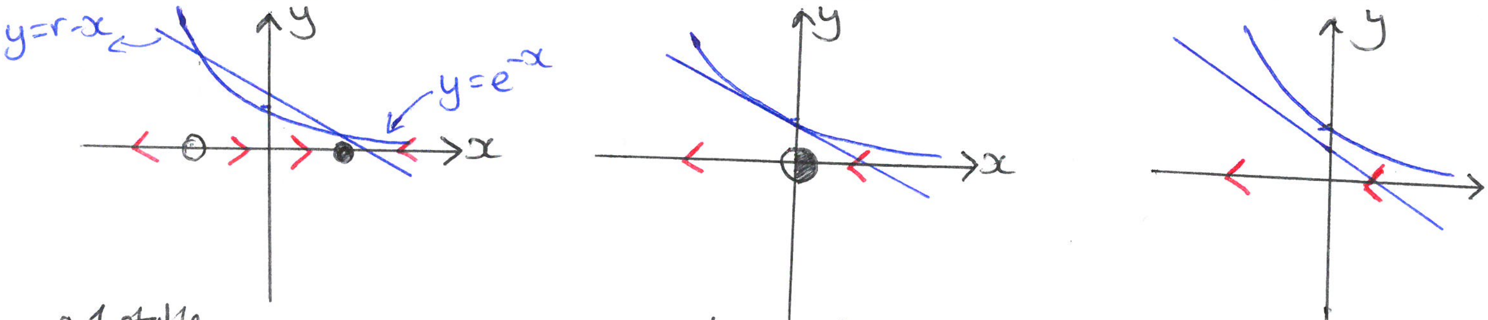
sufficiently near to the fixed point at $r = r_c$ the function $f(x; r_c)$ may be approximated by a parabola in x [use Taylor expansions!]

Example: Consider $\dot{x} = f(x;r) = r - x - e^{-x}$

Show that this system exhibits a saddle-node bifurcation at $r = r_c$, where r_c is to be determined.

– method: solving $f(x;r) = 0$ for some given r analytically is intractable, so we adopt a graphical approach: when do the curves $y = r - x$ and $y = e^{-x}$ intersect, and when do the two fixed points collide?

$r > r_c$ $\qquad$ $r = r_c$ $\qquad$ $r < r_c$



- 1 stable
- 1 unstable

- 1 semistable

- No fixed points

→ At $r = r_c$, we require that $y = r - x$ and $y = e^{-x}$ intersect tangentially at $x = x_*$ $\quad$ [r_c and x_* are unknowns]

$\therefore$ We require ① $\qquad r_c - x_* = e^{-x_*}$

② $\qquad \left.\frac{d}{dx} e^{-x}\right|_{x=x_*} = \left.\frac{d}{dx}(r_c - x)\right|_{x=x_*} \iff -e^{-x_*} = -1 \Rightarrow x_* = 0$

Substitute $x_* = 0$ into ① $\Rightarrow$ $r_c = 1$.

Transcritical bifurcation: →Describes when a fixed point always exists for all parameter values, but the stability of that point may change.

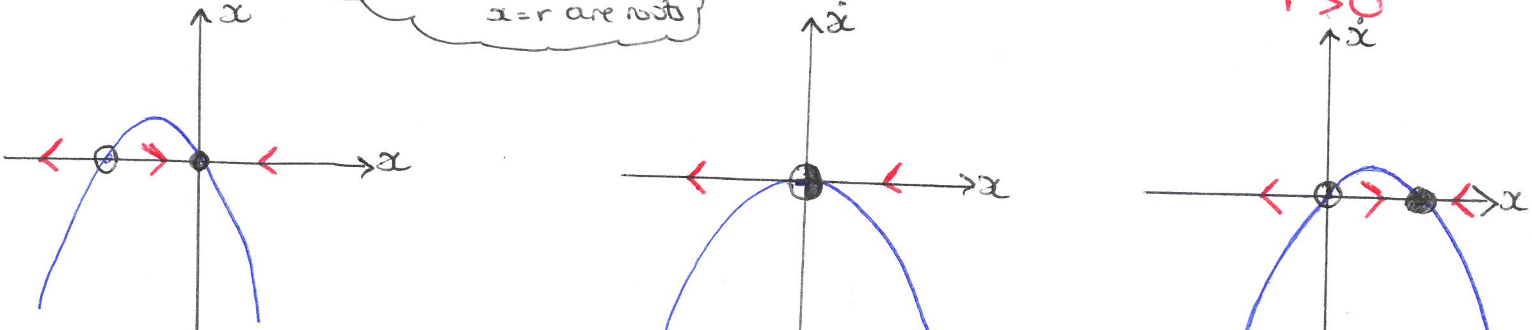
Normal form:

$$\boxed{\text{Transcritical} \qquad \dot{x} = rx - x^2, \qquad r \in \mathbb{R} \text{ is a parameter}}$$

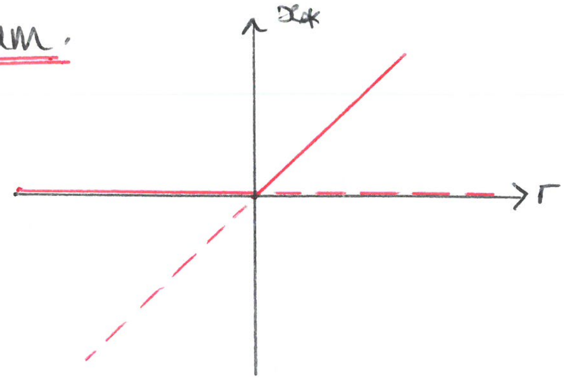
[looks like the logistic equation]

Note: $rx - x^2 = x(r - x)$
$\therefore x = 0$
$x = r$ are roots

$r < 0$ $\qquad$ $r = 0$ $\qquad$ $r > 0$



Bifurcation Diagram.



[an exchange of stability occurs!]

Note: There are always two fixed points (except when the coalesce at $r=0$)

Example: Consider $\dot{x} = r\log x + x - 1$ for $x \approx 1$.
Find $r=r_c$ at which a transcritical bifurcation occurs.
Then recast the system in normal form (approximate!)

- Note: $x=1$ is always a fixed point.
- We define $u = x-1$, where $|u| \ll 1$

$$\Rightarrow \quad \dot{u} = r\log(u+1) + u$$
$$= r\left[u - \tfrac{1}{2}u^2 + O(u^3)\right] + u$$
$$= (r+1)u - \tfrac{1}{2}ru^2 + O(u^3) \qquad (*)$$

At $r=-1$, the fixed point $u=0$ is repeated (is semi-stable)
Hence, a transcritical bifurcation occurs at $\underline{r_c = -1}$.

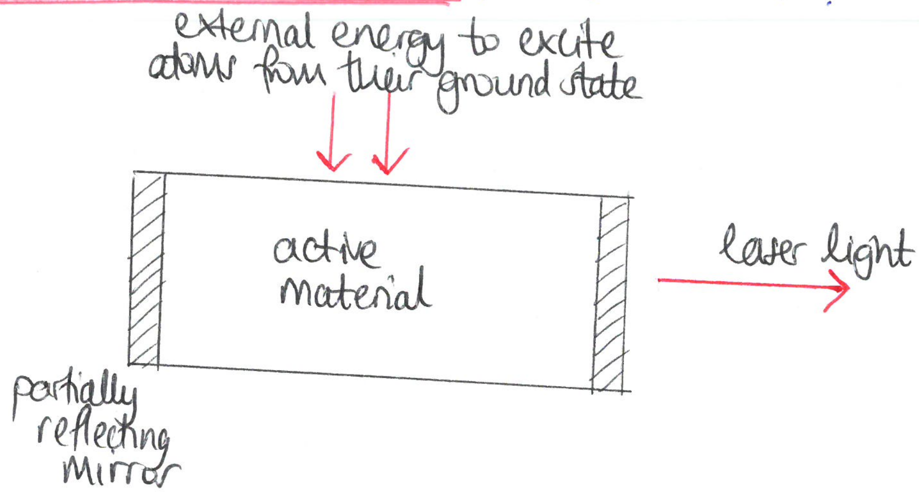
To reduce to normal form, we need to rescale the ~~xx~~ variables to set the u^2 coefficient to -1.
Let $u = \alpha v$, where $\alpha \in \mathbb{R}$ is to be determined.
Substitute into $(*)$ $\Rightarrow$ $\dot{v} = (r+1)v - \underbrace{\left(\tfrac{1}{2}r\alpha\right)}_{\text{set } \alpha = 2/r} v^2 + O(v^3)$

Define $\begin{cases} R = r+1 \\ X = v \end{cases}$

↳ Neglect the cubic term $\Rightarrow$ $\underline{\dot{X} \approx RX - X^2}$. [approximate normal form].

Application to solid-state lasers. [a simplified model]

[Not covered in lectures] ⑤



- **Weak excitation:** excited atoms oscillate independently of each other $\Rightarrow$ randomly phased light waves (i.e. a lamp)
- **Sufficiently large excitation:** atoms oscillate in-phase $\Rightarrow$ laser.

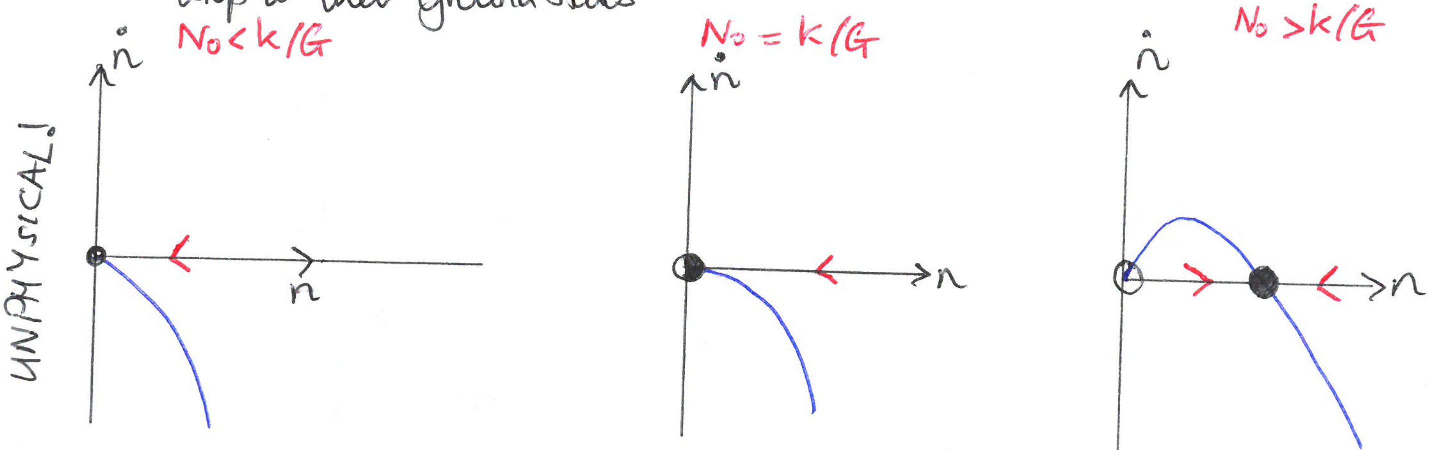
Can we rationalise the threshold at which the laser appears?

- $n(t)$: number of photons
- $N(t)$: number of excited atoms
- $G>0$: gain coefficient
- $k>0$: rate of escape through the endfaces of the laser
- N_0 : fixed number of excited atoms (in the absence of laser action)
- α : rate at which atoms drop to their ground states (stimulated emission)

random encounters more likely; escape of photons

$$\left\{\begin{aligned} \dot{n}(t) &= \underbrace{GnN}_{\text{gain}} - \underbrace{kn}_{\text{loss}} = (GN-k)n \\ N(t) &= N_0 - \alpha n \end{aligned}\right.$$

$$\Rightarrow \boxed{\dot{n} = (GN_0 - k)n - \alpha G n^2.}$$



Hence, we have a transcritical bifurcation at $N_0 = k/G$ (laser threshold)

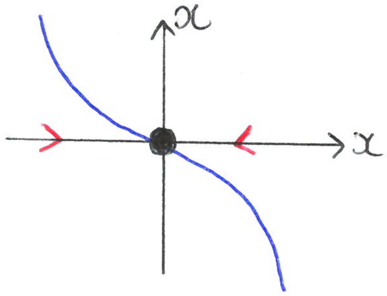
Pitchfork bifurcation.

- Typically arises in problems that possess symmetry.
  $\therefore$ fixed points appear/disappear in symmetric pairs.

Supercritical pitchfork bifurcation. $\dot{x} = rx - x^3, \quad r \in \mathbb{R}$

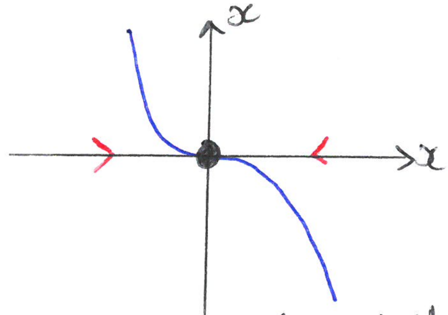
Note: the equations are invariant under the mapping $x \mapsto -x$, consistent with symmetry.

$r<0$



- 1 stable
  $x_* = 0$

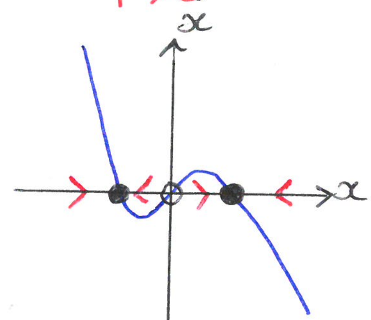
$r=0$



- 1 stable (repeated)
  $x_* = 0$

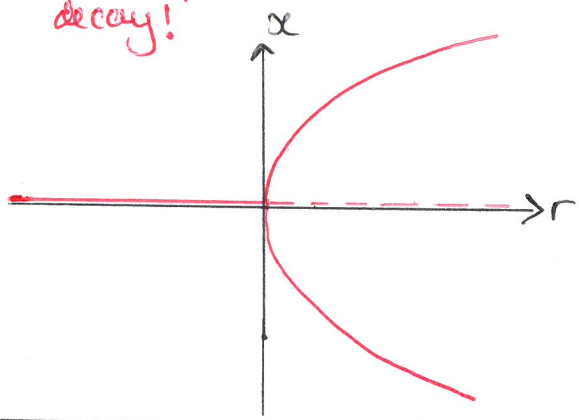
linear stability analysis invalid here
– get algebraic rather than exponential decay!

$r>0$



- 2 stable, 1 unstable
  $x_* = 0$
  $x_\pm = \pm\sqrt{r}$.

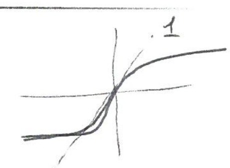
Bifurcation diagram:



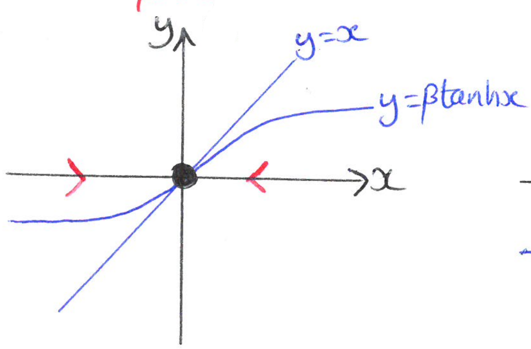
(looks like a pitchfork!)

Example: $\dot{x} = -x + \beta \tanh(x)$.

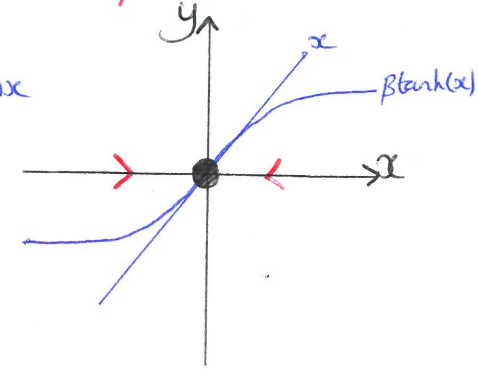
Note: $\frac{d}{dx}\tanh(x) = 1 - \tanh^2 x \quad \therefore \frac{d}{dx}\tanh(x)\Big|_{x=0} = 1.$



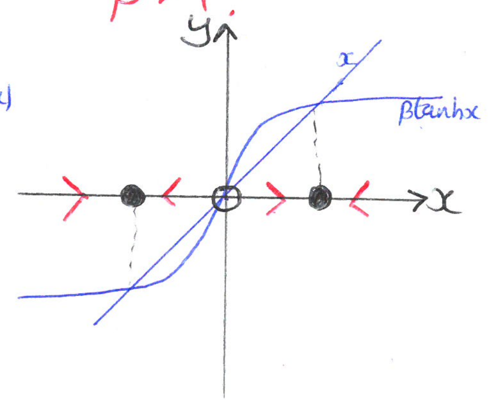
$\beta<1$



$\beta=1$



$\beta>1$



Bifurcation diagram. Note: $\tanh(x) \sim x - \frac{x^3}{3} + O(x^5)$ for $|x| \ll 1$

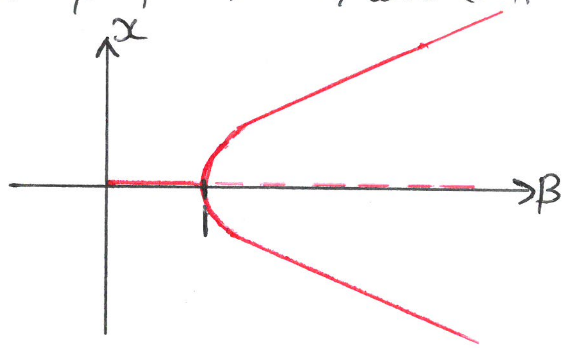
$$\therefore f(x;\beta) \sim -x + \beta\left(x - \frac{x^3}{3}\right) + O(x^5)$$
$$= x(\beta - 1) - \frac{\beta}{3}x^3 + O(x^5).$$

So typical pitchfork shape occurs only for x small.

Note: $\tanh x \sim 1$ as $x \gg 1 \Rightarrow f(x;\beta) \sim \beta\,\mathrm{sign}(x) - x$,
[$\tanh x \sim -1$ for x negative] for $|x| \gg 1$.

So $x \sim \pm\beta$ for β large.

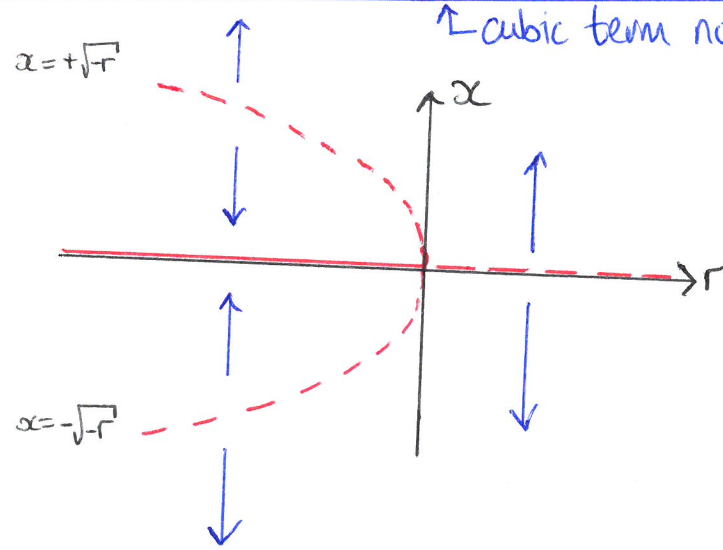
Alternative: plot $\beta = \beta(x) = x/\tanh(x)$. ← dependence on the parameter is simpler than the dependence on the variable.



$\boxed{\text{Subcritical pitchfork bifurcation: } \dot{x} = rx + x^3, \quad r \in \mathbb{R}.}$

↑ cubic term now acts to destabilize.

Bifurcation diagram:



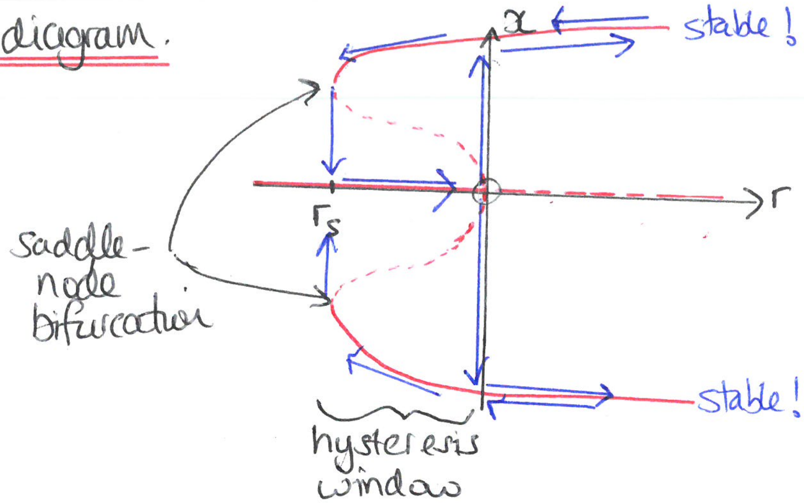
Problem: cubic terms enhance the instability in this case and yield blow-up: i.e. solution approaches infinity in finite time! [Try solving the ODE!!]
This feature is physically unrealistic!!

↳ We resolve this issue by including higher-order terms (preserving symmetry):

$$\boxed{\dot{x} = rx + x^3 - x^5, \quad r \in \mathbb{R}}$$

↑ we can rescale variables to ensure that coefficient of x^5 is -1.

Bifurcation diagram.

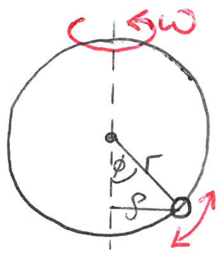


Hysteresis: non-reversibility of the system when r is increased/decreased.

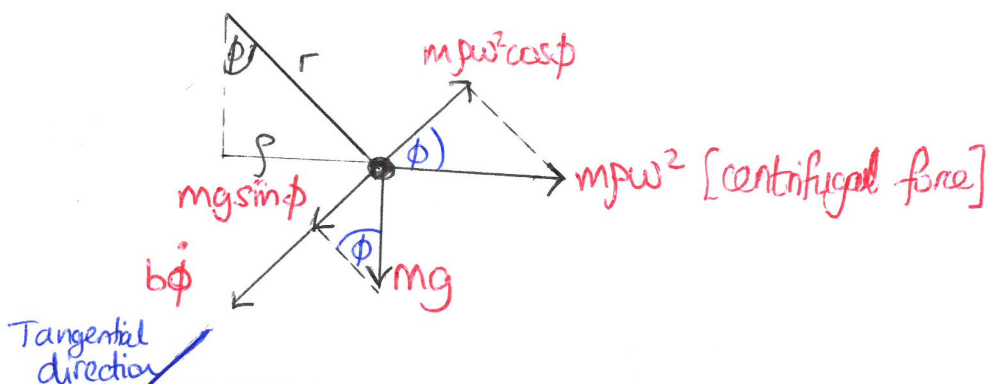
For $r \in (r_s, 0)$; the origin is locally (but not globally) stable – long-time behaviour depends on the initial conditions.

- **Note**; when r is varied, we may have jumps in the asymptotic behaviour depending on which ~~the~~ solution branch the system currently occupies.

Application: An overdamped bead on a rotating hoop.



- ϕ: angle between bead and downward vertical direction, $\phi \in \underbrace{(-\pi, \pi]}_{\text{by periodicity}}$
- $\rho = r\sin\phi$: distance of bead from vertical axis.



- bead mass m
- hoop radius r
- rotation rate ω.

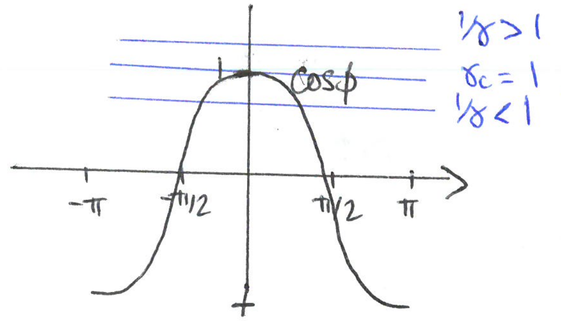
Tangential force balance: $\boxed{mr\ddot\phi = -b\dot\phi - mg\sin\phi + m\omega^2 r\sin\phi\cos\phi}$ (*)

Over-damped limit [we will formally justify this approximation later!]

Neglecting inertial effects gives
$$\boxed{b\dot\phi = mg\sin\phi\left[\frac{r\omega^2}{g}\cos\phi - 1\right]}$$

Define $\gamma = r\omega^2/g$. [dimensionless acceleration]

Fixed points:
$$\begin{cases} \forall\gamma > 0: & \phi_* = 0,\ \phi_* = \pi \quad \text{[bottom and top]} \\ \forall\gamma > 1: & \cos\phi_* = 1/\gamma, \text{ i.e. } \phi^* = \arccos(\gamma^{-1}). \end{cases}$$



So as γ increases through $\gamma_c = 1$, two additional solutions appear at $\phi_* = 0$

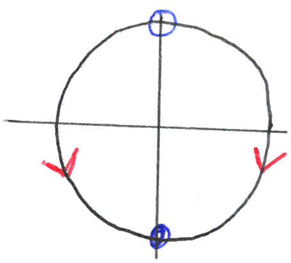
As $\gamma \to \infty$, $\phi_* \to \pm\pi/2$.

Furthermore, linear stability analysis reveals that

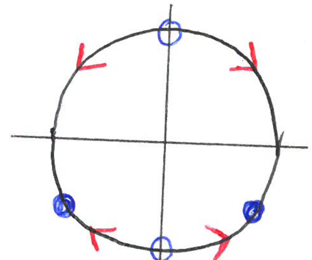
$\phi_* = 0$: stable for $\gamma < 1$, unstable for $\gamma > 1$

$\phi_* = \pi$: unstable for all γ.

Exercise: (analytically or graphically)



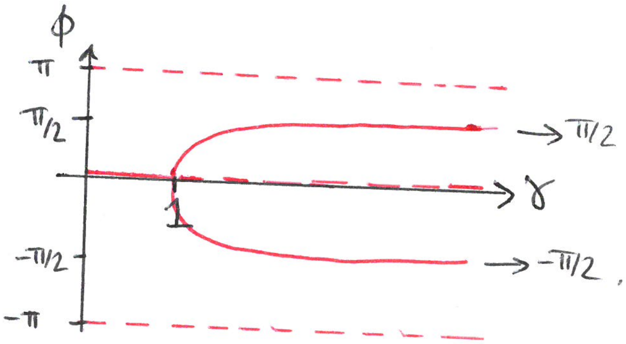
$\gamma < 1$



$\gamma > 1$

[Additional solutions appear for sufficiently fast spinning].

Bifurcation diagram:



– long-time asymptotic behaviour depends on the form of the perturbation (i.e. which fixed point is "selected").

Physical interpretation: For $\gamma > 1$, the centrifugal force increases as the bead moves from the bottom, acting to destabilise the $\phi_* = 0$ solution.

Justification for dropping inertia term: dimensionless variables.

We aim to introduce a characteristic time T and dimensionless time $\tau = t/T$ so that $\frac{d\phi}{d\tau}$ and $\frac{d^2\phi}{d\tau^2}$ are of the size $O(1)$. We need to find T.

Substitute into ⊛ and apply the chain rule: $\frac{d\phi}{dt} = \frac{d\tau}{dt}\frac{d\phi}{d\tau} = \frac{1}{T}\frac{d\phi}{d\tau}$, etc.

$$\Rightarrow \quad \frac{mr}{T^2}\frac{d^2\phi}{d\tau^2} = -\frac{b}{T}\frac{d\phi}{d\tau} + mg\sin\phi\left[\gamma\cos\phi - 1\right] \quad , \text{ where } \gamma = r\omega^2/g$$

$$\text{re-arrange} \Rightarrow \quad \underbrace{\frac{mr}{Tb}}\frac{d^2\phi}{d\tau^2} + \frac{d\phi}{d\tau} + \underbrace{\frac{mgT}{b}}\sin\phi\left[1 - \gamma\cos\phi\right] = 0$$

We choose T s.t. $\dfrac{mgT}{b} = 1 \Rightarrow \underline{T = \dfrac{b}{mg}}$.

We define the dimensionless parameter $\varepsilon > 0$ as: $\underline{\varepsilon = \dfrac{mr}{bT} = \dfrac{mr}{b}\dfrac{mg}{b} = \dfrac{r}{g}\left(\dfrac{mg}{b}\right)^2}$

Then we have the dimensionless equation:

$$\underline{\varepsilon \frac{d^2\phi}{d\tau^2} + \frac{d\phi}{d\tau} + \sin\phi\left[1 - \gamma\cos\phi\right] = 0} \quad (\dagger)$$

So we may neglect inertia when $\left|\varepsilon \dfrac{d^2\phi}{d\tau^2}\right| \ll 1$, which is satisfied if (but not only if) $\varepsilon \ll 1$ and $\dfrac{d^2\phi}{d\tau^2} = O(1)$. Note: $\underline{\varepsilon \ll 1 \Leftrightarrow b^2 \gg m^2 g r}$.

– so drag and centrifugal forces balance

What about initial conditions? – 2nd-order ⇒ 2 IC's; – 1st-order ⇒ 1 IC } mis-match when setting $\varepsilon = 0$.

This apparent paradox may be resolved by realising that $\dfrac{d^2\phi}{d\tau^2}$ may be very large for small times. To investigate the initial transient, define the new dimensionless timescale σ so that $\tau = \varepsilon\sigma$, where $\varepsilon \ll 1$ and $\sigma = O(1)$,

$$\Rightarrow \underline{\frac{d^2\phi}{d\sigma^2} + \frac{d\phi}{d\sigma} + \cancel{\varepsilon\sin\phi\left[1 - \cos\phi\right]} = 0}.$$

balance for $\sigma = O(1)$.

So over short timescales, inertia balances drag, whilst centrifugal forces are irrelevant. Hence, initially:

$$\begin{cases} \phi(\sigma) = \phi(0) + (1 - e^{-\sigma}) \left.\dfrac{d\phi}{d\sigma}\right|_{\sigma=0}. \\ \dfrac{d\phi}{d\sigma} = e^{-\sigma}\left.\dfrac{d\phi}{d\sigma}\right|_{\sigma=0}. \end{cases}$$

So any initial velocity quickly vanishes over an $O(\varepsilon)$ timescale, yielding a small change in displacement. After this initial transient, the over-damped limit becomes appropriate and our first-order analysis is valid.

⊛ This kind of problem is a singular limit – see courses on fluid mechanics (boundary layers) or perturbation theory ⊛

Imperfect bifurcations and Catastrophes.

- What happens when imperfections are present in a system, violating symmetry?

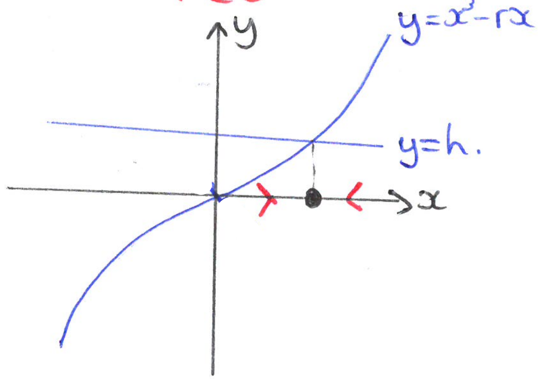
$$\boxed{\text{Prototypical example: } \quad \dot{x} = h + rx - x^3 \quad , h, r \in \mathbb{R}}$$

↑ h is the imperfection parameter
($h=0$: supercritical pitchfork bifurcation)

First consider r fixed, h varying. Note: $\dot{x} = h - (x^3 - rx)$

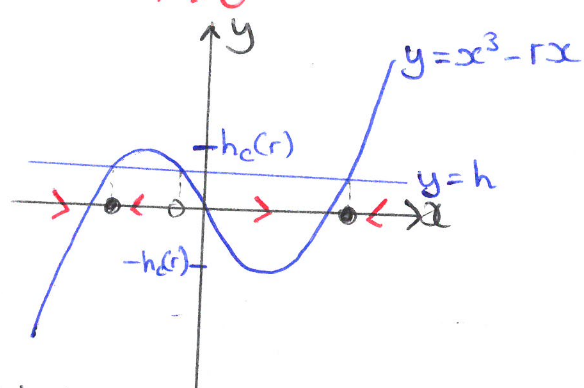
Fixed points:

$r \leq 0$



1 stable fixed point

$r > 0$

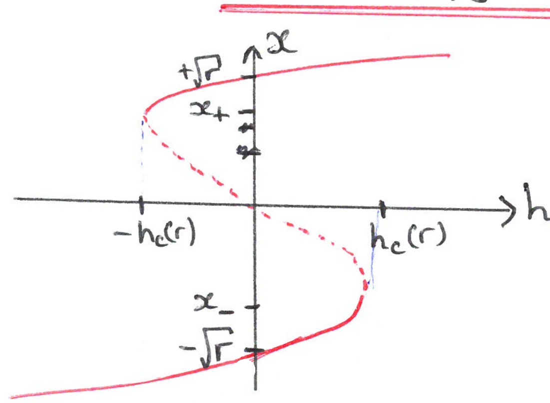


$|h| > h_c(r)$: 1 stable fixed point
$|h| < h_c(r)$: 2 stable, one unstable
↳ saddle-node bifurcation at $h = \pm h_c(r)$

- Consider $r > 0$: What is $h_c(r)$? [need horizontal line tangent to cubic]
So x to satisfy $\frac{d}{dx}\left[rx - x^3\right] = 0 \Rightarrow x_\pm = \pm\sqrt{\frac{r}{3}}$.

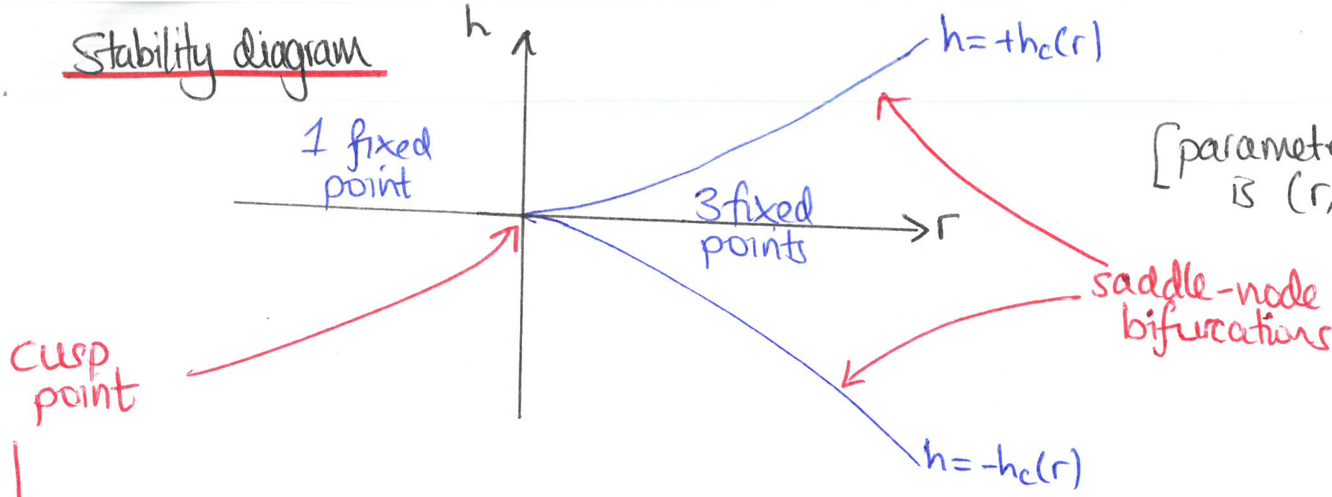
Note: $(x^3 - rx)\big|_{x = x_\pm} = \pm \frac{2}{3} r \sqrt{\frac{r}{3}}$ $\therefore$ $h_c(r) = \frac{2r}{3}\sqrt{\frac{r}{3}}$ and $h = \pm h_c(r)$

Bifurcation diagram
($r > 0$)



Note: we have a hysteresis loop!

Stability diagram

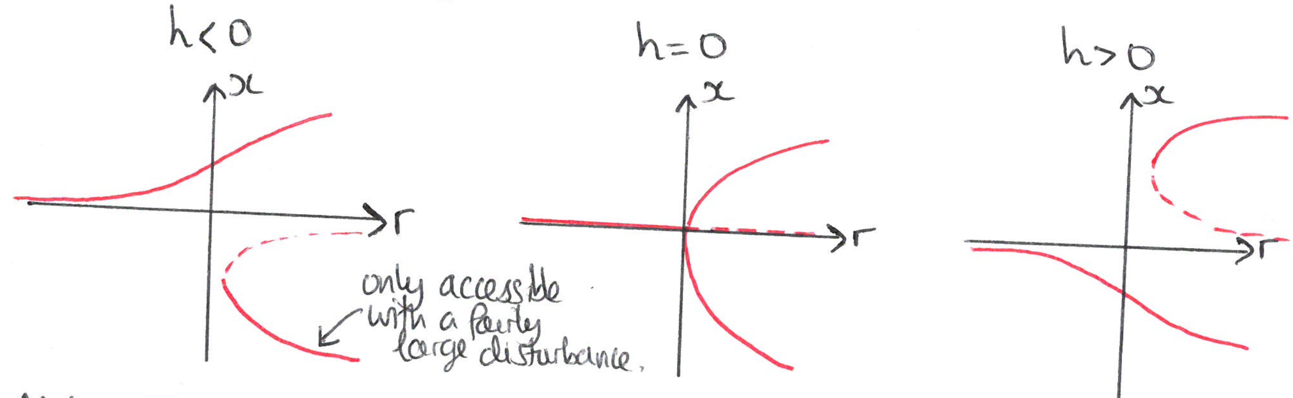


[parameter space is (r,h)-plane]

↳ a "codimension-two" bifurcation occurs
i.e. need to tune two parameters, h and r, to achieve this bifurcation.

Now consider h fixed, r varying.

Sketches of bifurcation diagrams:



Note: For $h \neq 0$, one branch remains stable $\forall r$, and is smooth rather than exhibiting a corner (like at the pitchfork bifurcation for $h=0$).

⊛ see Strogatz p. 73 for 3-dimensional drawings of the parameter space and an explanation of cusp catastrophes ⊛.

An application to biology: Insect outbreak. (Spruce budworm)

Dynamics: spruce budworm attack the leaves of the balsam fir tree.
Upon outbreak, the budworms can kill/defoliate most of the fir trees in a forest in about four years.

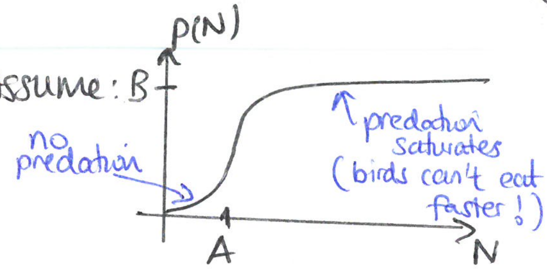
Observation: The system exhibits two timescales:
① Fast timescale: evolution of budworm population is O(months)
② Slow timescale: growth of trees (10+ years) ⇒ O(decades)

⇒ For the evolution of budworm, make a quasistatic approximation for the tree population, i.e. forest variables are constant.

The model: $N(t)$: budworm population at time t.

$$\dot{N} = \underbrace{RN\left(1-\frac{N}{K}\right)}_{\text{logistic growth}} - \underbrace{p(N)}_{\text{death rate due to predation.}}$$

Assume:



- Candidate function for $p(N) = \dfrac{BN^2}{A^2+N^2}$ [empirical fit of predation]

Full model: $$\boxed{\dot{N} = RN\left(1-\frac{N}{K}\right) - \frac{BN^2}{A^2+N^2}} \quad (*)$$

Non-dimensionalisation.

To get an idea of large/small [so as to characterise an outbreak], we non-dimensionalise.

Note:
- $[N]=[K]=[A]$
- $[R] = 1/\text{time}$
- $[B] = \dfrac{1}{\text{time}} \times \dfrac{1}{[N]}$

We define the dimensionless population $x = \dfrac{N}{A}$ [could also take $\dfrac{N}{K}=x$]

Let $t = \tau T$, where τ is dimensionless time and T is a time-scale [to be defined]

Sub into $(*) \Rightarrow$ $$\frac{A}{T}\frac{dx}{d\tau} = RAx\left(1-\frac{A}{K}x\right) - B\frac{x^2}{1+x^2}.$$

choose T to balance these terms, ie. $\dfrac{A}{T} = B \Rightarrow \boxed{T = \dfrac{A}{B}}$

Also, let $r = RT$, $k = \dfrac{K}{A}$.

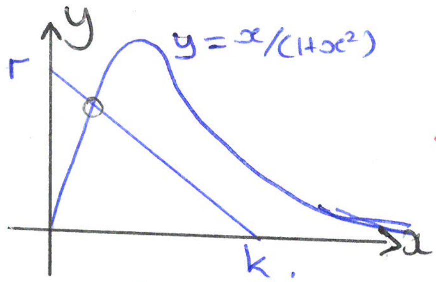
$\rightsquigarrow$ **Dimensionless model** $$\boxed{\frac{dx}{d\tau} = rx\left(1-\frac{x}{k}\right) - \frac{x^2}{1+x^2}} \quad (\dagger) \quad (r,k>0).$$

Fixed points: By linearising $(\dagger)$ about $x=0$, one can see that the fixed point $x_*=0$ is always unstable.

The remaining fixed points satisfy $$r\left(1-\frac{x}{k}\right) = \frac{x}{1+x^2}.$$

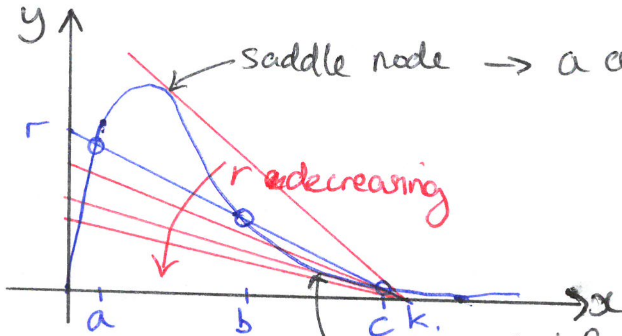
Fixed points occur at the intersections of $y = r\left(1-\frac{x}{k}\right)$ and $y = \frac{x}{1+x^2}$.

Note: As r and k vary, the curve $y = \frac{x}{1+x^2}$ remains fixed, making things easier!



possible to have 1, 2, or 3 intersections depending on r and k.

Case of 3 (non-zero) fixed points [denoted a,b,c]



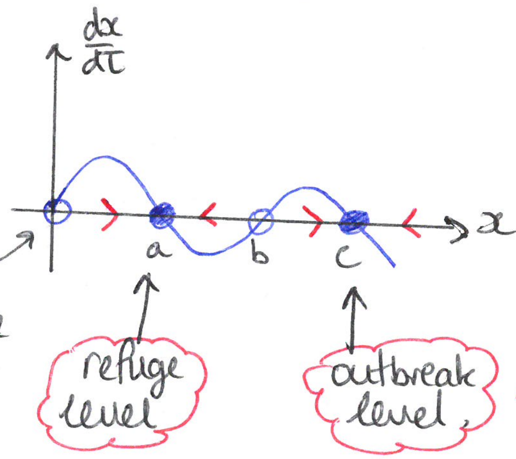
saddle node → a and b collide and annihilate.

saddle-node bifurcation when line and curve are tangential [so b&c collide and annihilate]

– following the saddle-node bifurcation, only a remains.

k, r fixed

$x_* = 0$ is always unstable



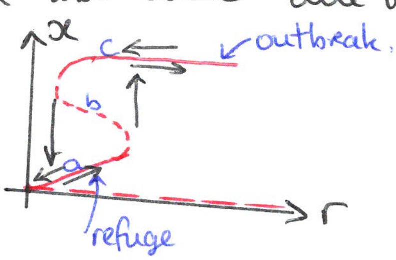
FACT: Stability alternates between fixed points.

outbreak only occurs if $x(0) = x_0 > b$

$\therefore$ b is a threshold.

Note: Outbreak can also occur due to a saddle-node bifurcation

k fixed



Calculating the bifurcation curves.

– What does (r,k) parameter space look like?

Condition for a saddle-node bifurcation: $y = r(1 - x/k)$ and $y = \frac{x}{1+x^2}$ intersect tangentially.

$\therefore$ we require ① $r\left(1 - \frac{x}{k}\right) = \frac{x}{1+x^2}$

and ② $\frac{d}{dx}\left[r\left(1-\frac{x}{k}\right)\right] = \frac{d}{dx}\left[\frac{x}{1+x^2}\right] \Rightarrow -\frac{r}{k} = \frac{1-x^2}{(1+x^2)^2}$.

– Use ② to eliminate r/k from ①

$\Rightarrow \quad r = \frac{2x^3}{(1+x^2)^2}$. and hence $k = \frac{2x^3}{x^2-1}$.

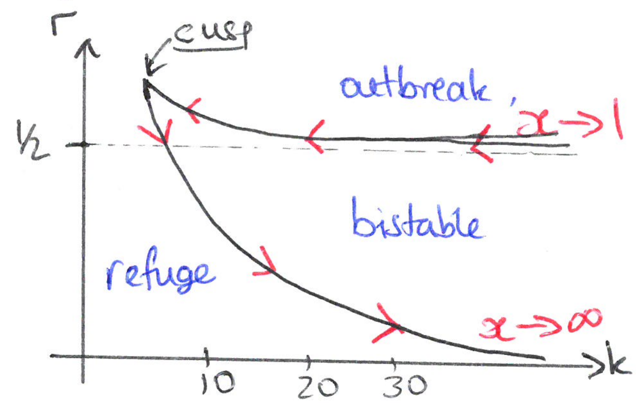
↳ For $k>0$ we need $x>1$ (as $x>0$)

We may treat $x>1$ as a parameter (confusing terminology!) and plot $(k,r) = (k(x), r(x))$

Note: As $x \to \infty$, $\begin{cases} r(x) \sim 2/x \\ k(x) \sim 2x \end{cases}$ $\therefore$ $r(x)\,k(x) \sim 4$. So $r \sim 4/k$.

Note: as $x \to 1$, $\begin{cases} r \sim 2/4 = 1/2 \\ k \sim 1/(x-1) \to \infty \end{cases}$ $\therefore$ $r \to 1/2$

labels correspond to only stable fixed points that exist.



⊛ see Strogatz p.80 for a discussion of how the effect of slowly varying r and k may affect the dynamics, and outbreak ⊛.