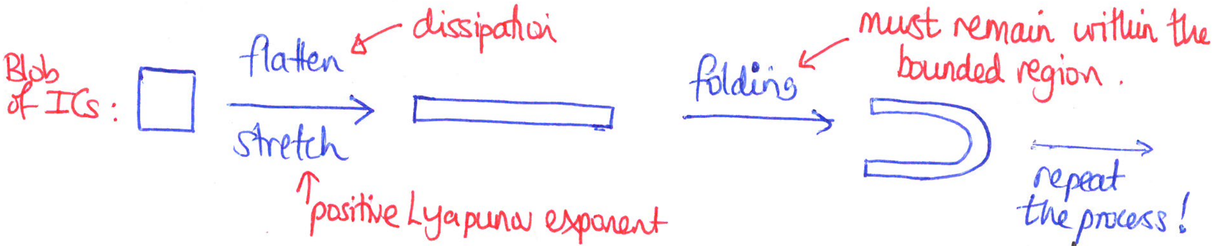
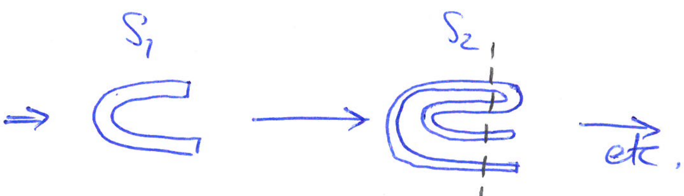
Strange Attractors.

- <u>Why</u> do systems become chaotic?

↳ basic mechanism: stretching and folding.



gives a layered structure ⟶ fractals.



⟶ Cantor-like set (S_∞)

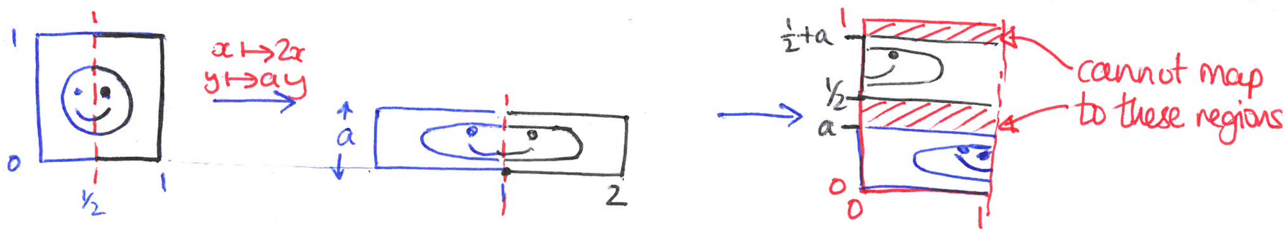
Example: Baker's map

maps $x, y \in [0,1]^2$ to itself:

(parameter $0 < a \leq 1/2$)

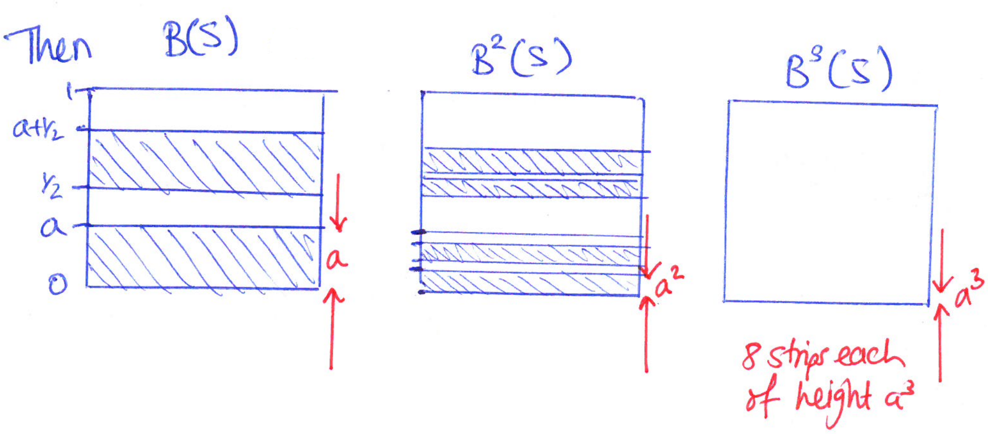
$$(x_{n+1}, y_{n+1}) = \begin{cases} (2x_n, ay_n) & 0 \leq x_n \leq 1/2 \\ (2x_n - 1, ay_n + 1/2) & \frac{1}{2} \leq x_n \leq 1 \end{cases}$$

$$\underbrace{(2x_n - 1, ay_n + 1/2)}_{= (2x_n, ay_n) + (-1, 1/2)}$$



Now define S as the unit square

Then



⟶ 2^n strips each of height a^n.

↓

limiting set $(n \to \infty)$ is a fractal $(A = B^\infty(S))$

[need some technical arguments to show that a limiting set exists]

What is the box dimension of A?

Note: $B^n(S)$ consists of 2^n strips each of height a^n and length 1

So it takes $\sim a^{-n}$ to cover each strip by boxes of length $\varepsilon = a^n$.

As there are 2^n strips en total, we have a total number of boxes

$$N \sim a^{-n} 2^n = (a/2)^{-n}$$

$$\Rightarrow \text{box dimension } d = \lim_{\varepsilon \to 0} \frac{\log N}{\log(1/\varepsilon)} = \lim_{n \to \infty} \frac{\log((a/2)^{-n})}{\log(a^{-n})}$$

$$= \frac{\log(a/2)}{\log(a)} = 1 + \frac{\log(1/2)}{\log(a)}$$

Sanity check: $d \to 2$ as $a \to 1/2$ as the attractor fills an increasingly large portion of S.

↳ Dissipation: Note: $\text{Area}[B(R)] < \text{area}(R)$ for any region R in the square
$(a < 1/2)$

↳ In fact: $\text{Area}[B(R)] = 2a \cdot \text{Area}(R)$ where $2a < 1$

$\Rightarrow$ area contraction (c.f. volume contraction in Lorenz)

↳ : Attractor A has zero area and the map cannot have any repelling fixed points as these expand area.

- Case $a = 1/2$ → area-preserving → dynamics never settle down to a lower-dimensional attractor (e.g. Hamiltonian Chaos)

associated with conservative systems

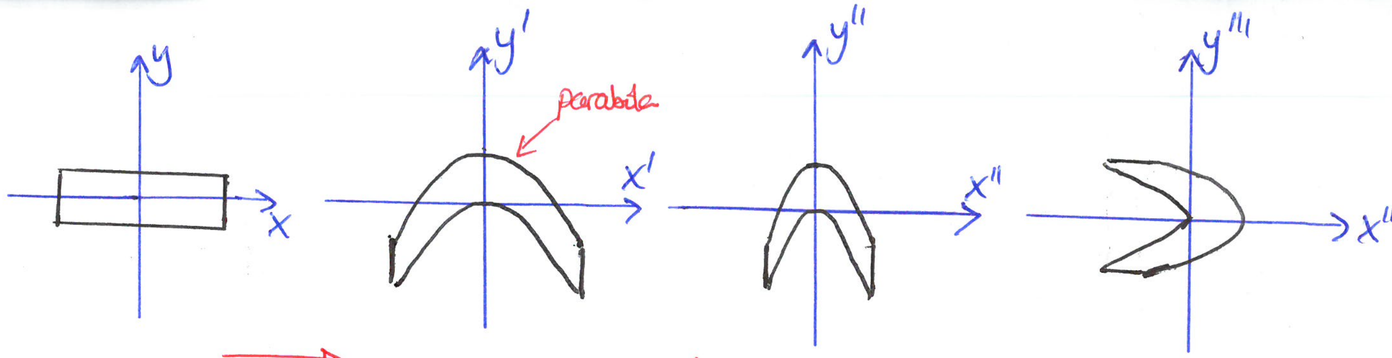
Hénon map

↳ a map that captures the essential features of the Lorenz attractor

$$\begin{cases} x_{n+1} = y_n + 1 - a x_n^2 \\ y_{n+1} = b x_n \end{cases}$$

- $-1 < b < 1$

(parameters a and b) $[a = 1.4, b = 0.3]$



T': $x' = x$
$y' = 1 + y - ax^2$
(stretch and fold)

T'': $x'' = bx'$
$y'' = y'$
(fold more)

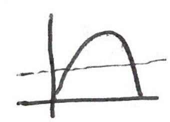
T''': $x''' = y''$
$y''' = x''$
(change orientation)

- $(x_n, y_n) = (x, y)$
- $(x_{n+1}, y_{n+1}) = (x''', y''')$

Properties.

① Invertible, i.e. a unique trajectory through each point in phase space [c.f. Lorenz]
↳ given (x_{n+1}, y_{n+1}), we can uniquely find (x_n, y_n)

For $b \neq 0$, $\begin{cases} \circ\ x_n = y_{n+1}/b \\ \circ\ y_n = x_{n+1} - 1 + a\left(\frac{y_{n+1}}{b}\right)^2 \end{cases}$

Note: Logistic map and other 1D unimodal maps are not invertible 

② Dissipative – contracts area at the same rate everywhere in phase space [c.f. Lorenz]
$(a < 1/2)$ ↳ for $-1 < b < 1$

③ For certain parameter regimes, there is a trapping region, i.e. $\exists R$ s.t. $T(R) \subset R$. [c.f. Lorenz]

④ Some trajectories can escape to infinity [c.f. Logistic, unlike Lorenz]
quadratic term! ↵

Proof 2: If determinant of Jacobian has modulus less than 1 then map is area-contracting (think of linearization about a fixed point)

$$J = \begin{pmatrix} -2ax & 1 \\ b & 0 \end{pmatrix} \quad \therefore \det J = -b \leftarrow \text{constant factor.}$$

Note: $|\det J|$ appears when doing change of variables in 2D integrals

So $\underline{|\det J| < 1}$ $\underline{\forall b \in (-1, 1)}$

Values of a and b?

- If b is too close to zero then contraction is too strong and fine structure will be invisible
- If b is too close to one then folding will not be strong enough → $b = 0.3$ works!

- If a is too small/large then trajectories escape to infinity
- For intermediate a, there exists a period-doubling cascade to chaos as a is increased
  - → $a = 1.4$ lies in the chaotic region.

Strange attractor ($a=1.4$, $b=0.3$)

→ self-similarity to arbitrarily small scales. ← Cantor-like in transverse direction
– smooth in the longitudinal direction.

– Why? The attractor is a closure of a branch of the unstable manifold of a saddle that sits on the edge of an attractor.

Rössler System (3D ODE system)

$$\begin{cases} \dot{x} = -y - z \\ \dot{y} = x + ay \\ \dot{z} = b + \underbrace{zx}_{\text{quadratic nonlinearity}} - zc \end{cases}$$

– chaos with $a = b = 0.2$, $c = 5.7$

– exhibits stretching and folding (see slides).

Attractor reconstruction

– how to infer the existence/form of a strange attractor based on limited experimental observations.

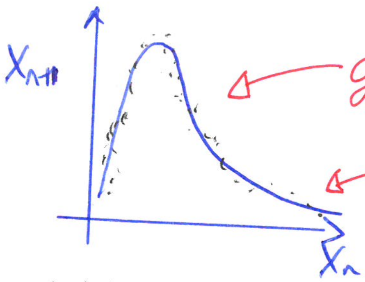
e.g. time series of one variable $B(t)$ in a high-dimensional system with imperfect experimental measurements/control.

– use time delays to generate phase space trajectories, $\underline{X}(t) = \begin{pmatrix} B(t) \\ B(t+T) \end{pmatrix}$

T is the delay time (to be chosen)

→ can also consider 3D phase portrait

- What is an approximate 1D map that governs the dynamics?
  - ↳ Consider intersection of phase-space trajectories with a Poincaré Section (a given line/plane in phase space)
  - Let X_n be the value of $B(t+T)$ at the n^{th} intersection with the Poincaré section



← get a unimodal map (plus experimental noise)

← suggests a simple map of the form $X_{n+1} = f(X_n)$

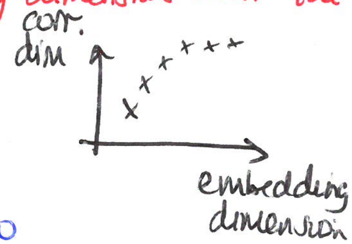
↳ exhibits PDC and U-sequence of periodic windows.

⊛ Note: method only works as attractor is nearly a 2D surface

Notes ① How to choose the number of time delays? [embedding dimension]

↳ need enough delays so that the underlying attractor can disentangle itself in phase space

Method – increase the embedding dimension until the correlation dimension saturates



↳ If the embedding dimension is too large then the data may be too sparse in phase space

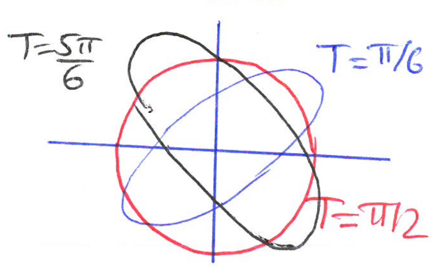
② What is the best delay time (T) to take?

Rule of Thumb → 10% – 50% of the mean orbital period around the attractor.

Example (delay time)

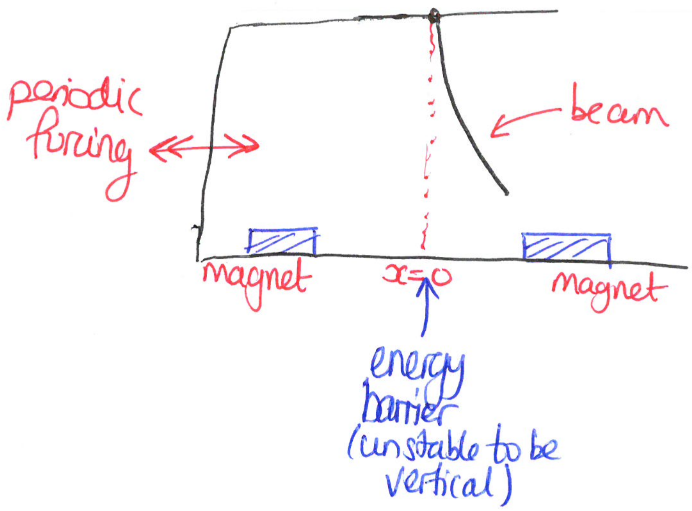
- Consider a limit-cycle attractor where we have an observation $x(t) = \sin(t)$

Define $\underline{x}(t) = \begin{pmatrix} x(t) \\ x(t+T) \end{pmatrix}$ ← need to determine best T.



- $T=\pi/2$ is optimal as the reconstructed attractor is as "open" as possible ∴ most robust to noise.

Forced Double-Well Oscillator (a nonautonomous system)



• If the forcing is large then the beam can wobble and switch between magnets,

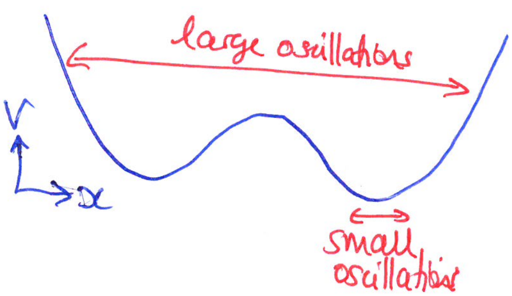
Model (dimensionless): displacement $x(t)$ evolves according to (in the moving frame)

$$\ddot{x} + b\dot{x} - x + x^3 = \boxed{F\cos\omega t}$$

— inertia force arising from change of frame,

damping, magnetic force, forcing, frequency.

Potential $V(x) = \frac{1}{4}x^4 - \frac{1}{2}x^2$



Note: trajectories can now cross in $(x,\dot{x})$-plane as really we have $(x,\dot{x},t)$ as the system.

Poincaré section.

↳ Plot $(x(t), \underbrace{y(t)}_{=\dot{x}(t)})$ whenever t is an integer multiple of $2\pi/\omega$

i.e. strobe system at the same phase in the drive cycle.

→ gives a fractal set!

Transient Chaos.

↳ temporary chaotic behaviour before the system settles down to one of several stable limit cycles.

– hard to predict which attractor will be approached ⇝ fractal basin boundaries of initial conditions.