# Fractals

↳ complex geometric shapes with fine structure at arbitrarily small scales [some exhibit self-similarity]

Aim: to understand/describe strange attractors, e.g. fractal dimension.

## Countable and uncountable sets

- sets $X$ and $Y$ are said to have the same cardinality (number of elements) if there is an invertible mapping that pairs each element of $X$ with precisely one element of $Y$ → one-to-one correspondence

- If a set $X$ can be put into a one-to-one correspondence with the natural numbers $\mathbb{N} = \{1, 2, 3, \dots\}$ then $X$ is said to be countable. Otherwise, $X$ is uncountable.

  ↳ in other words, can we list the elements of $X$ as $X = \{x_1, x_2, x_3, \dots\}$.

Example: The set of even numbers $E = \{2, 4, 6, \dots\}$ is countable as we can define an invertible mapping for each $n \in \mathbb{N}$ with $2n \in E$.

Example: The integers are countable as we can express them as the list

$$\{0, 1, -1, 2, -2, 3, -3, \dots\}.$$

Example: To show that the positive rationals are countable, we make a table such that the $(p,q)$ entry is $p/q$

i.e.

$$\begin{array}{ccccc}
\frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots \\
\frac{2}{1} & \frac{2}{2} & \frac{2}{3} & \frac{2}{4} & \cdots \\
\frac{3}{1} & \frac{3}{2} & \frac{3}{3} & \frac{3}{4} & \cdots \\
\vdots & \vdots & \vdots & \vdots &
\end{array}$$

We can weave through all the elements of the table, meaning that each positive rational appears after a finite number of steps, i.e. we have a list.

**Example:** Show that the set of real numbers between 0 and 1 is uncountable.

↳ Proof by contradiction: Suppose the set were countable with elements $\{x_1, x_2, \dots\}$.

In decimal form:

$$x_1 = 0.x_{11}x_{12}x_{13}\dots$$
$$x_2 = 0.x_{21}x_{22}x_{23}\dots$$
$$x_3 = 0.x_{31}x_{32}x_{33}\dots$$

Aim: construct $r \in [0,1]$ that is not on the list $\therefore$ the list is incomplete!

Define $r = 0.\bar{x}_{11}\bar{x}_{22}\bar{x}_{33}\dots$
where $\bar{x}_{nn}$ is any digit other than $x_{nn}$.

- Hence, $r \neq x_1$ as first digit differs

  $\dots$ $r \neq x_n$ as $n^{th}$ digit differs!

$\therefore$ $r$ not on the list # $\Rightarrow$ uncountable set.

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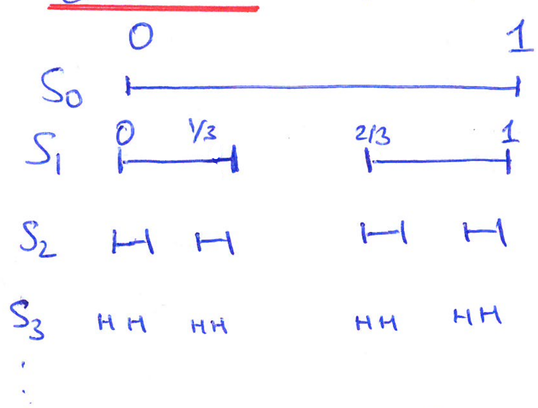
**Cantor Set - a fractal!**




← delete the middle third from each interval.

$S_\infty = C$. ← Cantor set. ← an infinite number of infinitesimal pieces separated by gaps of various sizes.

**Fractal properties of C.**

① structure at arbitrarily small scales - as we zoom in on $C$, we see more and more structure.

② self-similarity - it contains smaller copies of itself at all scales
i.e. the left portion of $S_{n+1}$ (or right portion) looks like $S_n$, but scaled by a factor of 3.

③ non-integer dimension! (see later)

# Non-fractal properties of C.

① <u>C has measure zero</u>, i.e. it can be covered by intervals whose total length is arbitrarily small.

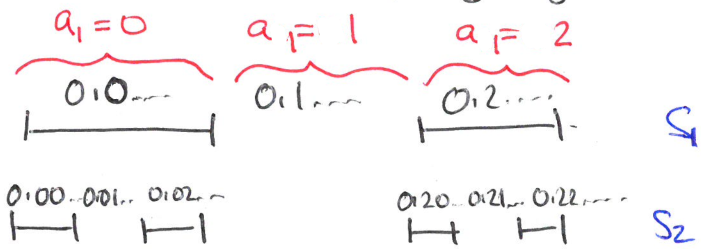
↳ Why? C can be covered by each of the sets $S_n$, where $S_n$ has length $L_n = \left(\frac{2}{3}\right)^n$

But $L_n \to 0$ as $n \to \infty$ ∴ C has total length zero.

② <u>C is uncountable.</u>

First note that C contains all points $c \in [0,1]$ that have no ones in their base 3 expansion.

Note: $x \in [0,1]$ in base 3 is $x = \frac{a_1}{3} + \frac{a_2}{3^2} + \frac{a_3}{3^3} + \cdots = 0.a_1a_2a_3\ldots$ where $a_j \in \{0,1,2\}$.




Note: the $a_n = 1$ entries are deleted at each stage

Note: what about endpoints like $0.1 = 1/3$ ?

Write $0.1 = 0.0222\ldots$ ← no ones appear in base 3 expansion

↳ Application: apply the diagonal argument to a list $\{c_1, c_2, \ldots\}$ of all points in C → contradiction. Write $c_i$ with digits $c_{ij}$.

Define $\bar{c} = 0.\bar{c}_{11}\bar{c}_{22}\bar{c}_{33}\ldots$

↳ where $\bar{c}_{nn} = \begin{cases} 0 & \text{if } c_{nn} = 2 \\ 2 & \text{if } c_{nn} = 0 \end{cases}$

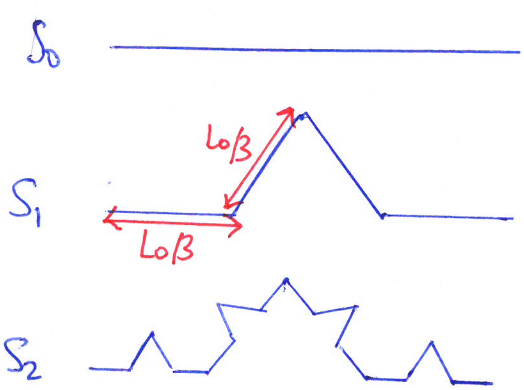
Hence, $\bar{c}$ is in C as it is written in terms of zeros and twos, but $\bar{c}$ is not on the list #

<u>Hence, C is uncountable</u> □

## Dimension of self-similar fractals. (see later for non-self-similar case)

↳ the definition for familiar geometric objects breaks down
- we cannot easily define a fractal dimension as the minimum number of coordinates needed to describe every point in the set

### Example: von Koch curve.




length $L_0$

each edge segment has length $L_0/3$

each length is $L_0/3^2$.

To get $S_n$, replace the middle third of each line segment of $S_{n-1}$ by an equilateral triangle

↓

$S_\infty = K$ : von Koch curve.

↳ Arclength: $l_1 = \frac{4}{3} l_0$, etc $\Rightarrow l_n = \left(\frac{4}{3}\right)^n l_0 \to \infty$ as $n \to \infty$. $\Rightarrow$ Infinite Arclength

$\Rightarrow$ length between any two points is infinite! Hence, K appears more than one-dimensional

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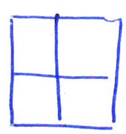
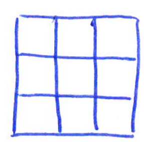
### Similarity dimension

- Consider a square of length $l$
  ↳ to reconstruct the square out of $m$ squares each of length $l/r$ (where $r$ is an integer), we need $m = r^2$

  e.g.



$r=2$
$m=4$



$r=3$
$m=9$

Note: $m = r^d$ where $d=2$

- For a cube, we need $m = r^3$ cubes of length $l/r$ to reconstruct the original cube $(d=3)$

- To generalize this definition to non-integers, we consider self-similar set that is composed of $m$ copies of itself scaled down by a factor of $r$

  Then the similarity dimension $d$ satisfies $m = r^d$, i.e. $d = \dfrac{\log m}{\log r}$

## Example: Cantor set.

$C$ is composed of two copies of itself, each scaled down by a factor of 3

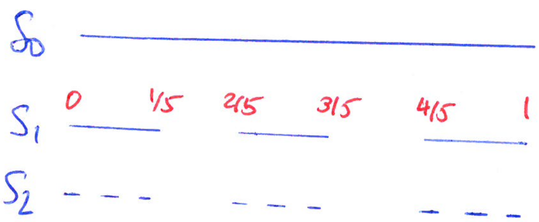
$\therefore m=2,\ r=3 \Rightarrow d = \dfrac{\log 2}{\log 3} = 0.63\ldots$

## Example: von Koch curve.

$K$ is composed of four pieces of itself, each scaled down by a factor of 3

$\therefore m=4,\ r=3 \quad \Rightarrow d = \dfrac{\log 4}{\log 3} = 1.26\ldots$ [Note; $d>1$, as anticipated]

## Example: even-fifths Cantor set.




$S_2$ is $m=3$ copies of $S_1$, each scaled by $r=5$ $\Rightarrow d = \dfrac{\log 3}{\log 5} = 0.68\ldots$

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## Cantor set - abstraction!

A closed set $S$ is called a topological Cantor set if it satisfies

① $S$ is "totally-disconnected" i.e. all points in $S$ are separated from each other. i.e. ($S$ contains no intervals)

② $S$ has no "isolated points": for any $p \in S$ and $\forall \varepsilon > 0$, $\exists q \in S$ $(q \neq p)$ such that $q$ lies within a distance $\varepsilon$ of $p$.

↳ Paradox: points are spread apart but also packed together!

↳ Note: these properties are topological rather than geometric, so no mention of self-similarity.

→ Topological Cantor sets often appear as strange attractors!

## Box dimension → for non-self-similar fractals.

↳ measure the set in a way that ignores irregularities of size less than $\varepsilon > 0$, and then consider the dependence as $\varepsilon \to 0$.

## Definition of Box dimension.

Let $S$ be a subset of a $D$-dimensional Euclidean space, and let $N(\varepsilon)$ be the minimum number of $D$-dimensional cubes of size $\varepsilon$ needed to cover $S$.

Define the box dimension $d$ so that $N(\varepsilon) \propto \varepsilon^{-d}$ as $\varepsilon \to 0$, (power law)

i.e. $d = \lim_{\varepsilon \to 0} \frac{\log N(\varepsilon)}{\log(1/\varepsilon)}$ [if the limit exists]

e.g. for a smooth curve of length $L$; $N(\varepsilon) \propto L/\varepsilon$ $(d=1)$

e.g. for a planar region of area $A$ bounded by a smooth curve; $N(\varepsilon) \propto A/\varepsilon^2$ (d=2

## Example: Cantor set.

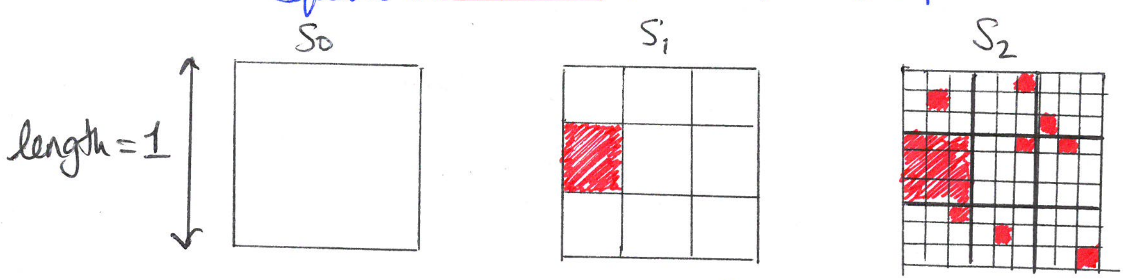
Note: $C$ is covered by each set $S_n$, where $S_n$ has $2^n$ intervals each of length $3^{-n}$

$\therefore$ if $\varepsilon = 3^{-n}$, we need $N = 2^n$ intervals to cover the Cantor set

$$\Rightarrow d = \lim_{\varepsilon \to 0} \frac{\log N}{\log(1/\varepsilon)} = \lim_{n \to \infty} \frac{\log(2^n)}{\log(3^n)} = \frac{\cancel{n}\log 2}{\cancel{n}\log 3} = \frac{\log 2}{\log 3}$$ [agrees with similarity dimension]

## Example: a non-self-similar fractal.

Construction: Divide a square region into 9 equal squares and discard one of the squares at random. Then repeat the process on each of the remaining squares.




Note: $\cdot S_1$ covered by $N = 8$ squares of side $\varepsilon = 1/3$ } no wastage!

$\cdot S_2$ ——— $N = 8^2$ ——— $\varepsilon = 1/3^2$

$\cdot S_n$ ——— $N = 8^n$ ——— $\varepsilon = 3^{-n}$

$$\Rightarrow d = \lim_{\varepsilon \to 0} \frac{\log N(\varepsilon)}{\log(1/\varepsilon)} = \lim_{n \to \infty} \frac{\log(8^n)}{\log(3^n)} = \frac{\log 8}{\log 3}$$

## Downsides of Box Dimension.

- difficult to find a minimal cover
- computation requires too much storage space
- some mathematical irregularities, e.g. set of rational numbers between 0 and 1 has a box dimension of 1, yet only has countably many points.

Alternative: Hausdorff dimension – coverings of small sets with varying size, not equal size.
↳ nicer mathematically, but even harder to compute numerically.

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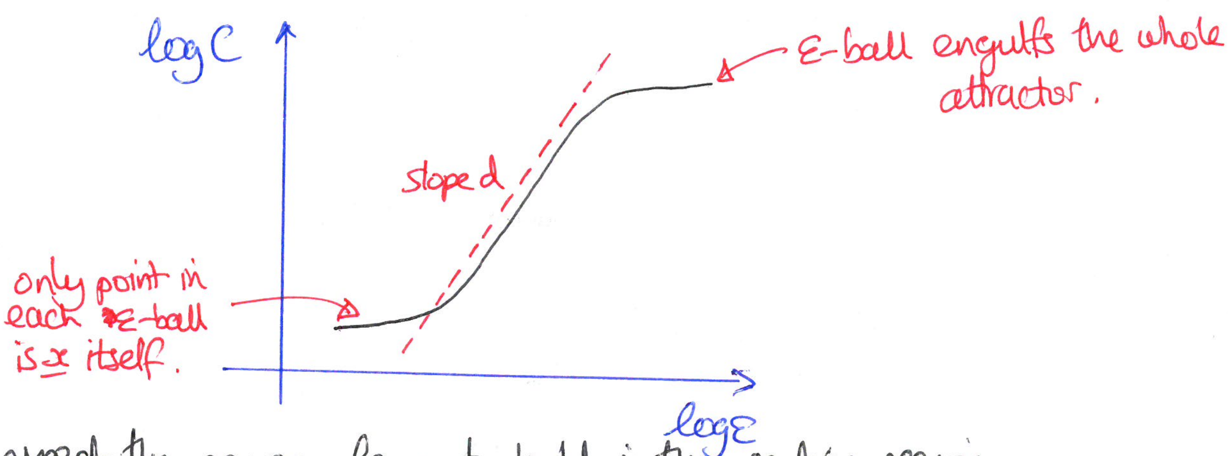
## Pointwise and Correlation Dimensions → how to characterize strange attractors!

Consider a chaotic system that settles down to a strange attractor in phase space.
↳ if the strange attractor exhibits fractal structure, what is the fractal dimension?

Method:
- Fix a point $\underline{x}$ on the attractor $A$
- Let $N_{\underline{x}}(\varepsilon)$ denote the number of points $\underline{x}_i$ on $A$ inside a ball of radius $\varepsilon$ about $\underline{x}$.
  the points $\underline{x}_i$ are generated from a simulation, discarding initial transients
- As most of the points in the ball come from earlier/later portions of the trajectory, $N_{\underline{x}}(\varepsilon)$ measures how frequently a typical trajectory visits an $\varepsilon$ neighbourhood of $\underline{x}$.
- As $\varepsilon$ increases, $N_{\underline{x}}(\varepsilon)$ grows as a power law $N_{\underline{x}}(\varepsilon) \propto \varepsilon^d$, where $d$ is the pointwise dimension at $\underline{x}$.
- To account for variations in $\underline{x}$, we average $N_{\underline{x}}(\varepsilon)$ over many $\underline{x}$, where the resulting quantity $C(\varepsilon) \propto \varepsilon^d$ and $d$ is the correlation dimension.




∴ we only expect the power law to hold in the scaling region

[minimum separation of points in $A$] $\ll \varepsilon \ll$ [diameter of $A$]

## Correlation vs. Box dimension.

- → correlation dimension takes account of the density of points of the attractor
  → box dimension weighs all boxes equally, no matter how many points they contain.
- In general, $d_{\text{correlation}} \leq d_{\text{box}}$, but they are usually close.

**Example** (Lorenz attractor – standard parameter values)

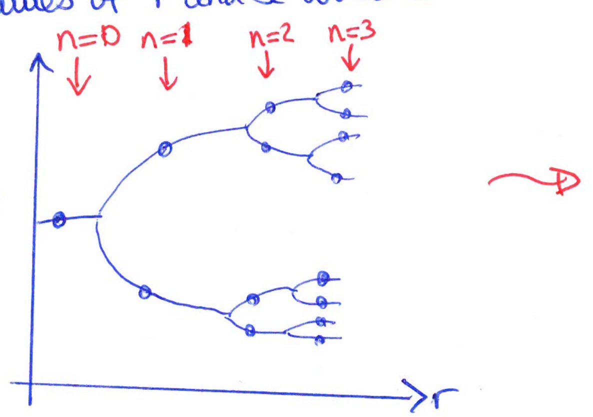
$d_{corr} = 2.05 \pm 0.01$ ← non-integer!

↳ estimated to within 5% using only a few thousand points ∴ rapid convergence!

**Example** (logistic map).

$x_{n+1} = r x_n (1 - x_n)$, $r = r_\infty$ (onset of chaos)

Consider values of $r$ and $x$ at which each $2^n$-cycle is superstable




$n \to \infty$ → topological Cantor set

↓

not self-similar

Find that $d_{corr} = 0.500 \pm 0.005$ at $r = r_\infty$.

Note: $d_{box} \approx 0.538 \geq d_{corr}$, as expected.

Note: At the onset of chaos, one might obtain residual correlations along ~~the~~ a single trajectory

## Multifractals.

↳ when the dimension varies across the attractor.

- For a multifractal $A$, let $S_\alpha$ be the subset of $A$ consisting of all points with pointwise dimension $\alpha$. → $S_\alpha$ is itself a fractal with fractal dimension $f(\alpha)$.

- an interwoven set of fractals on different dimensions $\alpha$, where $f(\alpha)$ determines their weight.
- useful for characterizing systems at the onset of chaos.