Consider a real-valued function \( x(t) \) of time \( t \), and a smooth real-valued function \( f(x) \). Suppose that \( x(t) \) evolves according to (for \( t > t_0 \))

\[
\dot{x}(t) = f(x), \quad x(t_0) = x_0, \quad \text{for } t > t_0
\]

1-dimensional, first-order, autonomous (no explicit time dependence) dynamical system.

**Motivation:** consider the damped system \( m\dot{x} + b\dot{x} = F(x) \) (inertia, damping, applied force).

Suppose that the inertia is small relative to the damping, i.e. \( |m| \ll |b| \).

Then we consider the \( \omega \)-damped limit \( \omega \dot{x} = F(x) \) or \( \dot{x} = f(x) \), where \( f(x) = F(x)/\omega \).

**Note:** if \( f(x) \) is a linear function of \( x \), e.g. \( f(x) = \alpha x + \beta \),

then we can solve \( \dot{x} = f(x) \), i.e. \( \dot{x} = \alpha x = \beta \) using several methods. Unfortunately, \( f(x) \) is rarely linear in nature and in general, we cannot write down an analytic solution.

However, we may still learn information about the system using geometric arguments.

**Example:** \( f(x) = \sin x \), so \( \{ \dot{x}(t) = \sin(x(t)), \dot{x}(0) = x_0 \} \).

By separating variables,

we obtain the solution \( t = \log \left| \frac{\csc(x_0) + \cot(x_0)}{\csc(x) + \cot(x)} \right| \).

But we want (1) \( x(t) \), not \( t(x) \), and (2) an expression we can interpret!

**LDD method:** draw a picture!
Vector field:

- $\sin x > 0 \iff \dot{x} > 0$
- $\sin x < 0 \iff \dot{x} < 0$

$x$ increasing (wave right)

So any $x_0 \in (0, 2\pi)$ will be attracted to the point $x = \pi$ as $t \to \infty$.

- We see that $x \in \{-2\pi, -\pi, \pi, 2\pi, \ldots\}$ are attracting fixed points.
- In contrast, $x \in \{0, \pm 2\pi, \pm 4\pi, \ldots\}$ are repellent fixed points.

We refer to attracting fixed points as stable, and repellent fixed points as unstable.

Note: If the system is initialized exactly at an unstable fixed point, then it will remain there for all time, however any small disturbances will attract repulsion.

Can we sketch $x(t)$ for $x_0 = \frac{\pi}{4}$?

- As $t \to \infty$, $x(t)$ approaches $x = \pi$.

- We need to do more work to obtain quantitative information (e.g., values on the $t$-axis), or we may solve the system numerically (see later). But pictographical solutions are useful!

How does the trajectory $x(t)$ change when the initial condition $x_0$ is varied?
We may represent sample solutions pictographically using a phase portrait.

"Local stability" - which fixed point is approached as \( t \to \infty \) depends on the initial conditions.

Note: the trajectories cannot cross each other!

Why? If two curves cross, then the \( x \)-values are equal, but the \( \dot{x} \) values are different. Yet, we have \( \dot{x} = f(x) \), hence the contradiction!

**Examples:**

1. \( \dot{x} = f(x) = x^2 - 1 \),
   \[ = (x-1)(x+1) \]
   Hence, fixed points are \( x_\pm = \pm 1 \).

   \( x_+ = -1 \): stable fixed point
   \( x_- = +1 \): unstable fixed point

2. \( \dot{x} = x - \cos x = f(x) \)
   Finding a fixed point \( x_\pm = \cos x_\pm \) cannot be done analytically, yet we may still rationalize the stability of \( x_\pm \), even if its value is unknown!

   Rather than plotting \( f(x) \), we plot \( y = x \) and \( y = \cos x \) on the same axes:

   For \( x > x_\pm \), we have \( x > \cos x \), so \( \dot{x} > 0 \)
   For \( x < x_\pm \), we have \( x < \cos x \), so \( \dot{x} < 0 \)

   So, \( x_\pm \) is an unstable fixed point.
Application to population models: logistic growth.

- Let \( N(t) \) be the population of an organism at time \( t > 0 \).
- For small populations, we assume the population growth rate is proportional to the population size, so \( \dot{N} = rN, \ r > 0 \).
- But when the population becomes large, the effects of over-crowding and limited resources means that the growth rate decreases and becomes negative for \( N > K \), where \( K \) is the carrying capacity.

Hence, we consider the model:

\[
\dot{N} = f(N) = rN \left( 1 - \frac{N}{K} \right)
\]

- So for \( N(0) > 0 \), the population always approaches the carrying capacity \( K \) for large time.
- At \( N(0) = 0 \), there is no population and so no growth/death.
- \( N \) is maximal at \( N = K/2 \).
Linear stability analysis.
- Benefits over graphical methods: we may derive the growth/decay rates when the trajectory is "close" to a fixed point.

**Idea:** Consider $\dot{x} = f(x)$, with fixed point $x_* = \text{s.t. } f(x_*) = 0$.

Consider $x(t) = x_* + \eta(t)$, where $\eta(t)$ is a small perturbation ($|\eta| \ll 1$)

So
$$\frac{d}{dt} (x_* + \eta(t)) = f(x_* + \eta(t)) \approx f(x_*) + \eta f'(x_*) + O(\eta^2) \quad \text{Taylor expansion}$$

So
$$\dot{\eta} \approx \eta f'(x_*) + O(\eta^2) \quad \text{neglect higher-order terms as assumed small}$$

So $\eta$ approximately evolves according to the system
$$\dot{\eta} = r\eta, \quad \text{where } r = f'(x_*)$$

so
$$\eta(t) = \eta(0)e^{rt}$$

**[Initial perturbation $\eta(0)$ is also assumed small].**

- For $r < 0$, $\eta \to 0$ as $t \to \infty \Rightarrow x \to x_*$ as $t \to \infty$
  - So $x_*$ is a **stable fixed point**.

- For $r > 0$, $\eta$ grows, so $x_*$ is an **unstable fixed point**.
  
  **Note:** When $\eta$ becomes large, the Taylor expansion $\eta = r\eta$ becomes invalid as higher-order terms become important. But the behavior near the fixed point is still correct.

- For $r = 0$, the $O(\eta^2)$ terms are not negligible and so we must do **non-linear analysis** instead.
Note: \( r_1 = \sqrt{|f'(x_1)|} \) gives the characteristic timescale of growth \((r > 0)\) or decay \((r < 0)\).

**Summary:** the sign of \( f'(x_1) \) determines the linear stability!

**Example (logistic growth),**

\[
\begin{align*}
  f(N) &= rN \left(1 - \frac{N}{K}\right), \quad N_0 = 0 \text{ and } N_* = K, \\
  f'(N) &= r \left(1 - \frac{2N}{K}\right)
\end{align*}
\]

\[
\therefore f'(0) = r > 0 \implies N_* = 0 \quad \text{unstable}\]

and \( f'(K) = -r < 0 \implies N_* = K \quad \text{stable},\)

In both cases, the characteristic timescale is \( 1/r \).

**Example - when \( f'(x_1) = 0 \)** [when linear stability analysis is invalid]

- \( f(x) = -x^3, x_1 = 0 \) stable
- \( f(x) = x^3, x_1 = 0 \) unstable

**Potentials** (gradient-driven motion \(\rightarrow\) over-damped motion)

We define a potential \( V(x) \) so that \( f(x) = -V'(x) \) and \( x = -V'(x) \)

[Note that shifting \( V \) up or down by a constant does not affect the dynamics!]

\(\rightarrow\) picture: the particle always moves downhill in the potential

\[
\frac{dV}{dt} = \frac{dV}{dx} \frac{dx}{dt} = V'(x) \left[-V'(x)\right] = -[V'(x)]^2 \leq 0
\]

So the potential height under the particle never increases, and remains a constant only at a fixed point.
Example: \( f(x) = \sin(x), \quad V(x) = \cos(2x) \)

So the particle "falls down" the potential to stable fixed points at \( x_+ \in \{ \pm \pi, \pm 3\pi, \ldots \} \), while the points \( x_+ \in \{ 0, \pm 2\pi, \pm 4\pi, \ldots \} \) at the top of the potential are unstable.

Note: Due to our over-damped assumption (from which inertia was neglected), the particle cannot oscillate whilst approaching a fixed point. \( \blacklozenge \) First-order systems preclude oscillations about a fixed point \( \blacklozenge \).

Example: \( V(x) = -\frac{1}{2}x^2 + \frac{1}{4}x^4 + \text{constant} = \frac{1}{2}x^2(1 - \frac{x^2}{2}) \)

Existence and Uniqueness:
- Unlike linear systems, the existence and uniqueness of non-linear systems is not guaranteed!

Example: \( \dot{x} = x^{1/3} \), \( x(0) = 0 \):

1. \( x=0 \) is a solution for all \( t>0 \)
2. Solving via separation of variables, we obtain that \( x(t) = \left(\frac{2}{3}t\right)^{3/2} \) is also a solution.

In fact, there are infinitely many solutions \( x \).

Note: \( f'(\pi) \) diverges as \( x \to 0 \) in this case.
Theorem: Consider the initial value problem \( x' = f(x), \ x(0) = x_0 \). Suppose that \( f(x) \) and \( f'(x) \) are continuous on the open interval \( I \), where \( x_0 \in I \). Then the initial value problem has a solution on some interval \( I(0, T) \) and the solution is unique.

In our above example, \( f'(x) \) has an asymptote (so discontinuity) at \( x = 0 \) (= \( x_0 \)), so the above conditions are violated! Note, existence is only guaranteed up to a finite time!

Example: \( y' |t| = -\frac{1}{2y} \), \( y(0) = y_0 \neq 0 \)

\[ \Rightarrow y' + \frac{1}{2} = 0 \]
\[ \Rightarrow \frac{dy}{dt} + \frac{1}{2} = 0 \]
\[ \Rightarrow \frac{d(y^2)}{dt} + 1 = 0 \]
\[ \Rightarrow y^2 = \text{constant} - t \]
\[ \Rightarrow y^2 = y_0^2 - t \]
So \( y(t) = \text{sign}(y_0) \sqrt{y_0^2 - t} \).

Solution becomes invalid at \( t = y_0^2 \).

Numerical methods.

To get a more quantitative answer, we may solve the equation \( x' = f(x), \ x(0) = x_0 \) numerically.

By introducing a time step \( h > 0 \) and a uniform time mesh \( t_n = nh \) for \( n = 0, 1, 2, \ldots \), we let the numerical approximation be \( x_n \approx x(t_n) \).

Euler's method.

\[ x' = f(x) \]
Integrate \( \int_{t_n}^{t_{n+1}} f(x) \, dt \Rightarrow x_{n+1} - x_n = \int_{t_n}^{t_{n+1}} f(x) \, dt \approx h f(x_n) \)

\[ \Rightarrow x_{n+1} = x_n + h f(x_n) \]

Explicit Euler.

As \( h \to 0 \), the accuracy of the solution is \( O(h) \)
Trapezium rule: Approximate \( \int_{x_n}^{x_{n+1}} f(x) \, dx \approx \frac{h}{2} [f(x_n) + f(x_{n+1})] \)

\( \Rightarrow x_{n+1} = x_n + \frac{h}{2} [f(x_n) + f(x_{n+1})] \)

\( O(h^2) \) convergence, but method is implicit (Use Newton-Raphson to find \( x_{n+1} \))

**Improved Euler / RK2**

In the Trapezium rule, remove the implicit component with a suitable approximation, namely

\[
\begin{align*}
\underline{x_{n+1} = x_n + h f(x_n)} \\
\underline{x_{n+1} = x_n + h \left[ f(x_n) + \frac{1}{2} f(x_{n+1}) \right]} \\
\end{align*}
\]

\( O(h^2) \) convergence and explicit.

**Two-Stage Runge-Kutta method**

\[
\begin{align*}
\underline{x_{n+1} = x_n + h f(x_n)} \\
x_{n+1} = x_n + h \left[ f(x_n) + \frac{1}{2} f(x_{n+1}) \right] \\
\end{align*}
\]

**Runge-Kutta 4th-Order (with 4 stages)**

Define \( \begin{align*} k_1 &= h f(x_n) \\
k_2 &= h f(x_n + \frac{1}{2} k_1) \\
k_3 &= h f(x_n + \frac{1}{2} k_2) \\
k_4 &= h f(x_n + k_3) \end{align*} \)

Then \( x_{n+1} = x_n + \frac{1}{6} [k_1 + 2k_2 + 2k_3 + k_4] \)

\( O(h^4) \) convergence but more cost per timestep.