Flows on the line - lectures 1&2

- Consider a real-valued function $x(t)$ of time t, and a smooth real-valued function $f(x)$. Suppose that $x(t)$ evolves according to (for $t > t_0$)

$$\dot{x}(t) = f(x), \qquad x(t_0) = x_0, \qquad \left[\dot{x} = \frac{dx}{dt}\right]$$

  - 1-dimensional, first-order, autonomous (no explicit time dependence) dynamical system.

↳ motivation: consider the damped system $\underbrace{m\ddot{x}}_{\text{inertia}} + \underbrace{b\dot{x}}_{\text{damping}} = \underbrace{F(x)}_{\text{applied force}}$

Suppose that the inertia is small relative to the damping i.e. $|m\ddot{x}| \ll |b\dot{x}|$.

Then we consider the over-damped limit $b\dot{x} = F(x)$, or $\dot{x} = f(x)$, where $f(x) = F(x)/b$. [Justified later in the course]

Note: if $f(x)$ is a linear function of x, e.g. $f(x) = \alpha x + \beta$, then we can solve $\dot{x} = f(x)$, ie. $\dot{x} - \alpha x = \beta$ using several methods. Unfortunately, $f(x)$ is rarely linear in nature and in general, we cannot write down an analytic solution. However, we may still learn information about the system using geometric arguments.

Example: $f(x) = \sin x$, so $\left\{\begin{array}{l}\dot{x}(t) = \sin(x) \\ x(0) = x_0\end{array}\right\}$. By separating variables, we obtain the solution $t = \log\left|\frac{\operatorname{cosec}(x_0) + \cot(x_0)}{\operatorname{cosec}(x) + \cot(x)}\right|$.

natural logarithm

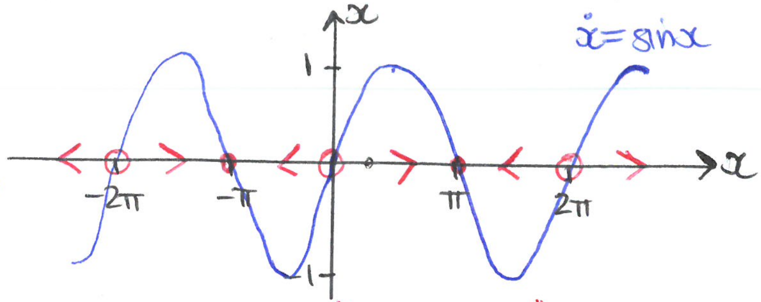
But we want (1) $x(t)$, not $t(x)$, and (2) an expression we can interpret!

↳ method: draw a picture!

Vector field.

- $\sin x > 0 \iff \dot{x} > 0$
  $\iff$ x increasing (move right)

- $\sin x < 0 \iff \dot{x} < 0$
  $\iff x$ decreasing



- arrows denote direction of evolution over time
- ● stable fixed point
- ○ unstable fixed point.

So any $x_0 \in (0, 2\pi)$ will be attracted to the point $x = \pi$ as $t \to \infty$

Fixed point: A value of $x(t) = x_*$ so that $\dot{x} = 0$, i.e. $f(x_*) = 0 \Rightarrow$ no motion. (equilibrium)

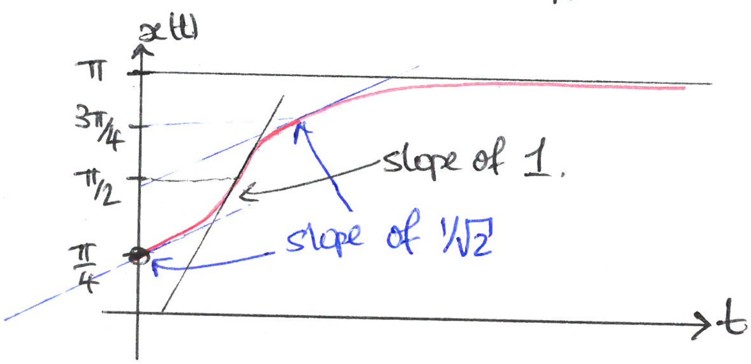
- we see that $x \in \{\ldots, -3\pi, -\pi, \pi, 3\pi, \ldots\}$ are attracting fixed points
- In contrast, $x \in \{0, \pm 2\pi, \pm 4\pi, \ldots\}$ are repellent fixed points

- We refer to attracting fixed points as stable, and repellent fixed points as unstable.

Note: If the system is initialised exactly at an unstable fixed point then it will remain there for all time, however any small disturbances will activate repulsion.

↳ can we sketch $x(t)$ for $x_0 = \pi/4$?

- As $t \to \infty$, $x(t)$ approaches $x_* = \pi$.



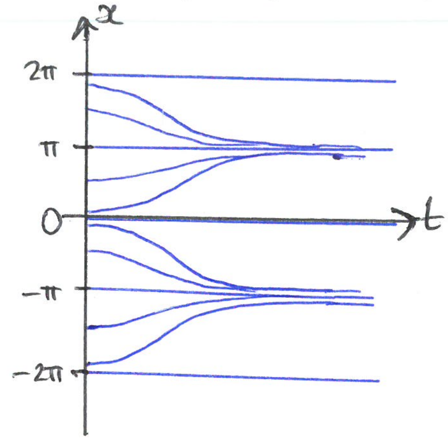
Note: • $\sin(\pi/4) = 1/\sqrt{2} = \sin\left(\frac{3\pi}{4}\right)$

• $\sin(\pi/2) = 1$

- we need to do more work to obtain quantitative ~~xxx~~ information (e.g. values on the t-axis), or we may solve the system numerically (see later). But pictographical solutions are useful!

→ how does the trajectory $x(t)$ change when the initial condition x_0 is varied?

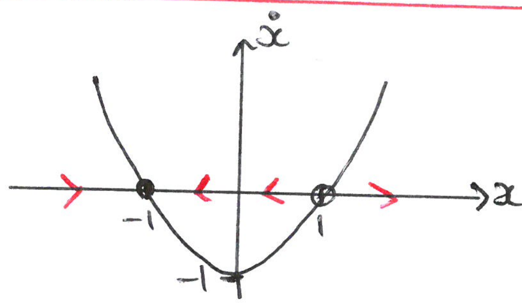
We may represent sample solutions pictographically using a phase portrait.



"local stability" – which fixed point is approached as $t \to \infty$ depends on the initial condition.

Note: the trajectories cannot cross eachother! Why? If two curves cross then the x-values are equal, but the $\dot{x}$ values are different. Yet, we have $\dot{x} = f(x)$, hence the contradiction!

Examples. ① $\dot{x} = f(x) = x^2 - 1$
$$= (x-1)(x+1).$$
Hence, fixed points are $x_* = \pm 1$.



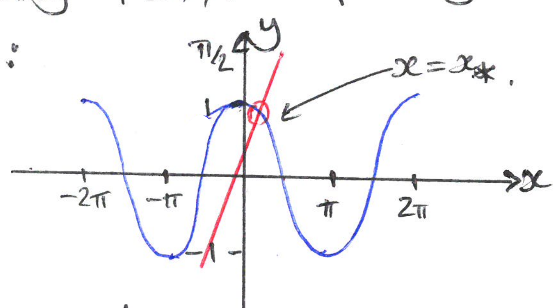
$x_* = -1$: stable fixed point

$x_* = +1$: unstable fixed point.

② $\dot{x} = x - \cos x = f(x)$

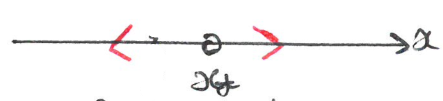
Finding a fixed point $x_* = \cos x_*$ cannot be done analytically, yet we may still rationalise the stability of x_*, even if its value is unknown!

- Rather than plotting $f(x)$, we plot $y = x$ and $y = \cos x$ on the same axes:



For $x > x_*$, we have $x > \cos x$, so $\dot{x} > 0$

For $x < x_*$, we have $x < \cos x$, so $\dot{x} < 0$



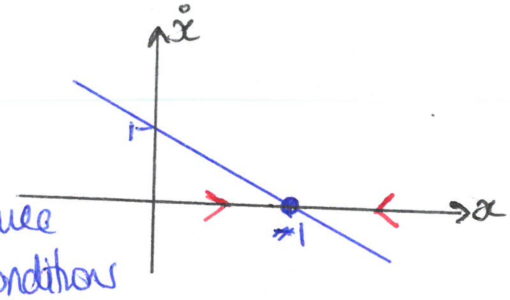
So x_* is an unstable fixed point.

3. $\dot{x} = 1 - x = f(x)$

Fixed point at $x_* = 1$.

↳ stable!

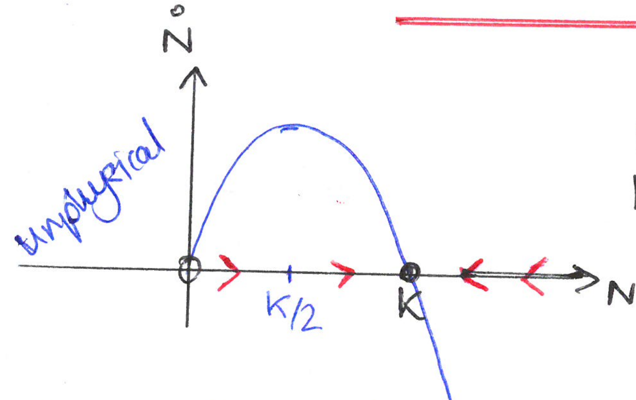
- in fact, in this example we have global stability: all initial conditions yield trajectories that will eventually approach $x = x_*$.



Application to population models: logistic growth.

- Let $N(t)$ be the population of an organism at time $t > 0$.
- For small populations, we assume the population growth rate is proportional to the population size, so $\dot{N} = rN$, $r > 0$
- But when the population becomes large, the effects of over-crowding and limited resources mean that the growth rate decreases and becomes negative for $N > K$, where K is the carrying capacity

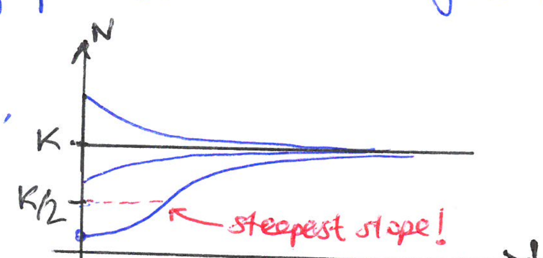
Hence, we consider the model: $\dot{N} = f(N) = rN\left(1 - \frac{N}{K}\right)$



$N_* = 0$: unstable fixed point

$N_* = K$: stable fixed point.

- So for $N(0) > 0$, the population always approaches the carrying capacity K for large time.
- At $N(0) = 0$, there is no population and so no growth/death.
- $\dot{N}$ is maximal at $N = K/2$.



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Linear stability analysis.

- benefits over graphical methods: we may derive the growth/decay rates when the trajectory is "close" to a fixed point.

Idea: Consider $\dot{x} = f(x)$, with fixed point x_* s.t. $f(x_*) = 0$.

Consider $x(t) = x_* + \eta(t)$, where $\eta(t)$ is a small perturbation ($|\eta| \ll 1$)

So $\frac{d}{dt}(\overset{\text{constant}}{x_*} + \eta(t)) = f(x_* + \eta(t))$

$$\approx \underbrace{f(x_*)}_{0} + \eta f'(x_*) + O(\eta^2) \quad \leftarrow \text{Taylor expansion}$$

So $\dot{\eta} \approx \eta \underbrace{f'(x_*)}_{\text{constant.}} + \cancel{O(\eta^2)}$ neglect higher-order terms as assumed small

So η approximately evolves according to the system

$$\dot{\eta} = r\eta, \quad \text{where } r = f'(x_*).$$

$\therefore \eta(t) = \eta(0)e^{rt}$ [initial perturbation $\eta(0)$ is also assumed small].

- For $r<0$, $\eta \to 0$ as $t \to \infty$ $\Rightarrow x \to x_*$ as $t \to \infty$
  So x_* is a stable fixed point.

- For $r>0$, η grows, so x_* is an unstable fixed point.
  ↳ Note: When η becomes large, the equation $\dot{\eta} = r\eta$ becomes invalid as higher-order terms in the Taylor expansion become important.
  But the behaviour near the fixed point is still correct!

- For $r = 0$, the $O(\eta^2)$ are not negligible and so we must do non-linear analysis instead!!

Note: $1/|r| = 1/|f'(x_*)|$ gives the characteristic timescale of growth ($r>0$) or decay ($r<0$).

Summary: the sign of $f'(x_*)$ determines the linear stability!

Example (logistic growth),

$f(N) = rN(1 - N/K)$, $N_* = 0$ and $N_* = K$.

$f'(N) = r\left(1 - \frac{2N}{K}\right)$

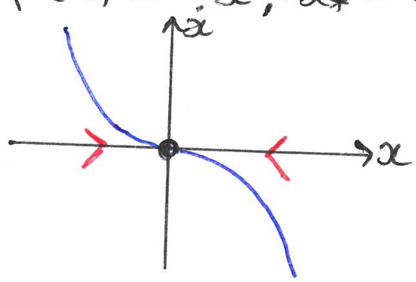
$\therefore f'(0) = r > 0 \to N_* = 0$ unstable

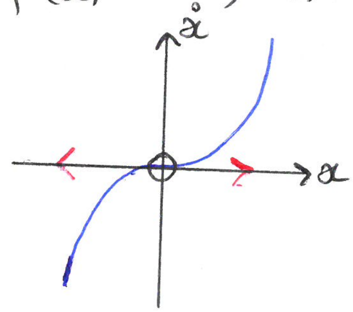
and $f'(K) = -r < 0 \to N_* = K$ stable.

In both cases, the characteristic timescale is $1/r$.

Example – when $f'(x_*) = 0$ [when linear stability analysis is invalid]

- $f(x) = -x^3$, $x_* = 0$ stable
- $f(x) = x^3$, $x_* = 0$ unstable





Potentials (gradient-driven motion $\leftrightarrow$ over-damped motion)

We define a potential $V(x)$ so that $f(x) = -V'(x)$ and $\dot{x} = -V'(x)$

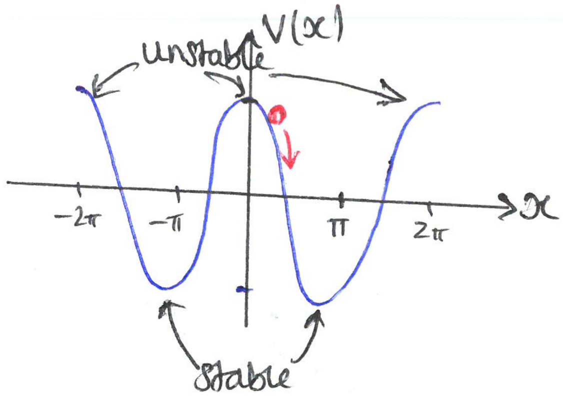
[note that shifting V up or down by a constant does not affect the dynamics!]

$\to$ picture: the particle always moves downhill in the potential

Observe: $V(x(t))$, $\frac{dV}{dt} = \frac{dV}{dx}\frac{dx}{dt} = V'(x)\left[-V'(x)\right] = -\left[V'(x)\right]^2 \leq 0$

So the potential height under the particle never increases, and remains a constant only at a fixed point.

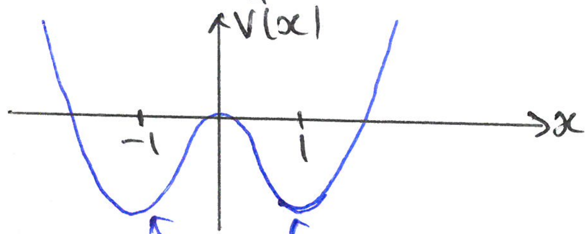
Example $f(x) = \sin(x)$, $V(x) = \cos(x)$



So the particle "falls down" the potential to stable fixed points $x_* \in \{\pm\pi, \pm 3\pi, \dots\}$, while the points $x_* \in \{0, \pm 2\pi, \pm 4\pi, \dots\}$ at the top of the potential are unstable

Note: Due to our over-damped assumption (from which inertia was neglected), the particle cannot oscillate whilst approaching a fixed point. ⊛ First-order systems preclude oscillations about a fixed point ⊛.

Example. $V(x) = -\frac{1}{2}x^2 + \frac{1}{4}x^4 + \text{constant.} = \frac{1}{2}x^2\left(1 - \frac{x^2}{2}\right)$



stable equilibria at $x_* = \pm 1$.

Existence and Uniqueness.

- unlike linear systems, the existence and uniqueness of non-linear systems is not guaranteed!

Example: $\dot{x} = x^{1/3}$, $x(0) = 0$.

① $x = 0$ is a solution for all $t > 0$

② Solving via seperation of variables, we obtain that

$$x(t) = \left(\frac{2}{3}t\right)^{3/2}$$ is also a solution.

- in fact, there are infinitely many solutions ⚠.

↳ Note: $f'(x)$ diverges as $x \to 0$ in this case.

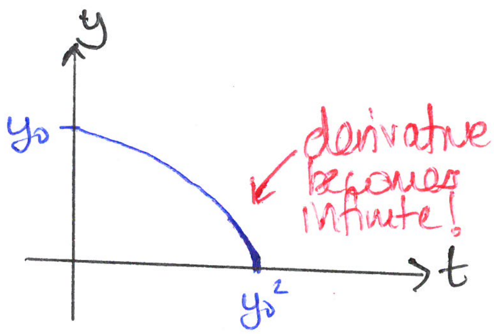
Theorem: Consider the initial value problem $\dot{x} = f(x)$, $x(0) = x_0$. Suppose that $f(x)$ and $f'(x)$ are continuous on the open interval R, where $x_0 \in R$. Then the initial value problem has a solution on some interval $t \in (-\tau, \tau)$ and the solution is unique.

- in our above example, $f'(x)$ has an asymptote (so discontinuity) at $x = 0$ $(= x_0)$, so the above conditions are violated!

Note, existence is only guaranteed up to a finite time!

Example: $\dot{y}(t) = -\frac{1}{2y}$, $y(0) = y_0 \neq 0$

$$\Rightarrow y\dot{y} + \frac{1}{2} = 0 \qquad \therefore \frac{d}{dt}(y^2) + 1 = 0$$



$$\Rightarrow \quad y^2 = \text{constant} - t$$

$$\Rightarrow y^2 = y_0^2 - t$$

So $\underline{y(t) = \text{sign}(y_0)\sqrt{y_0^2 - t}}$.

↑ solution becomes invalid at $t = y_0^2$.

Numerical methods.

To get a more quantitative answer, we may solve the equation $\dot{x} = f(x)$, $x(0) = x_0$ numerically.

By introducing a timestep $h > 0$ (or Δt) and a uniform time mesh $t_n = nh$ for $n = 0, 1, 2, \ldots$, we let the numerical approximation by $x_n \approx x(t_n)$.

Euler's method. $\dot{x} = f(x)$

Integrate $\int_{t_n}^{t_{n+1}} \cdots \, dt \Rightarrow x_{n+1} - x_n = \int_{t_n}^{t_{n+1}} f(x)\,dt$

approximate f by a rectangle

$$\approx h f(x_n)$$

$$\Rightarrow \underline{x_{n+1} = x_n + h f(x_n)}.$$

Explicit Euler.

- As $h \to 0$, the accuracy of the solution is $O(h)$.

Trapezium rule: Approximate $\int_{t_n}^{t_{n+1}} f(x(t))\,dt \approx \frac{h}{2}\left[f(x_n) + f(x_{n+1})\right]$

$$\Rightarrow \quad x_{n+1} = x_n + \frac{h}{2}\left[f(x_n) + \underbrace{f(x_{n+1})}_{\text{implicit}}\right]$$

$O(h^2)$ convergence, but method is implicit (Use Newton Raphson to find x_{n+1})

Improved Euler / RK2.

In the Trapezium rule, remove the implicit component with a suitable approximation, ~~mean~~ namely

Two-Stage Runge-Kutta method

$$\begin{cases} \bar{x}_{n+1} = x_n + h f(x_n) \\ x_{n+1} = x_n + \frac{h}{2}\left[f(x_n) + f(\bar{x}_{n+1})\right] \end{cases}$$

$O(h^2)$ convergence and explicit ☺.

Runge-Kutta 4th-order (with 4 stages).

Define
$$\begin{cases} k_1 = h f(x_n) \\ k_2 = h f(x_n + \frac{1}{2}k_1) \\ k_3 = h f(x_n + \frac{1}{2}k_2) \\ k_4 = h f(x_n + k_3) \end{cases}$$

Then $x_{n+1} = x_n + \frac{1}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right]$

- $O(h^4)$ convergence but more cost per timestep.