One-dimensional maps.

- $x_{n+1} = f(x_n)$, f smooth, $f : \mathbb{R} \to \mathbb{R}$.

uses: – analysing differential equations (Poincaré map / Lorenz map)
– models for impulsively driven mechanical systems
– a simple example of chaos

⊕ The non-smoothness of map trajectories can allow for a greater range of dynamics than differential equations (where flow is continuous) ⊕

Fixed points.

A fixed point x_* satisfies $x_* = f(x_*)$ [equilibrium]

To study the linear stability, consider $x_n = x_* + \varepsilon_n$ for $|\varepsilon_n| \ll 1$.

Then $x_{n+1} = f(x_n) \implies x_* + \varepsilon_{n+1} = f(x_* + \varepsilon_n)$

$$\implies \cancel{x_*} + \varepsilon_{n+1} \approx \cancel{f(x_*)} + \varepsilon_n f'(x_*) + O(\varepsilon_n^2)$$

Linearise $\implies \varepsilon_{n+1} = \varepsilon_n f'(x_*)$.

MULTIPLIER

Define $\lambda = f'(x_*)$ $\implies \varepsilon_{n+1} = \lambda \varepsilon_n \implies \dots \implies \varepsilon_n = \lambda^n \varepsilon_0$.

- For linear stability, we need $|\lambda| < 1$ so that $|\varepsilon_n| \to 0$ as $n \to \infty$
- If $|\lambda| > 1$ then x_* is unstable
- For the marginal case $|\lambda| = 1$, need to keep quadratic terms in the Taylor expansion to determine (nonlinear) stability, i.e. $\varepsilon_{n+1} = \varepsilon_n f'(x_*) + \frac{1}{2}\varepsilon_n^2 f''(x_*)$

  (see $f(x) = \sin x$ example)

  ↳ or use graphical techniques.

Example. $f(x) = x^2$

Fixed points $x_* = 0$ or $x_* = 1$

$$f'(x) = 2x \quad \therefore \begin{cases} \text{for } x_* = 0, \ \lambda = 0 \implies \text{stable} \\ \text{for } x_* = 1, \ \lambda = 2 \implies \text{unstable.} \end{cases}$$

In fact, when the multiplier is $\lambda = 0$ then the system is superstable and exhibits rapid decay $\varepsilon_n \sim \varepsilon_0^{(2^n)}$

Why? Consider the system $x_{n+1} = x_n^2$

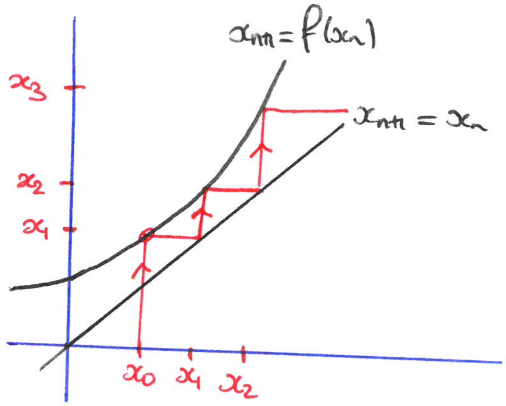
$$\Rightarrow \log x_{n+1} = 2\log x_n$$

Define $y_n = \log x_n \Rightarrow y_{n+1} = 2y_n \Rightarrow y_n = 2^n y_0$

$$\Rightarrow \log x_n = 2^n \log x_0 = \log\left(x_0^{2^n}\right)$$

$$\Rightarrow \underline{\underline{x_n = x_0^{(2^n)}}}$$

Cobwebs (a graphical method).

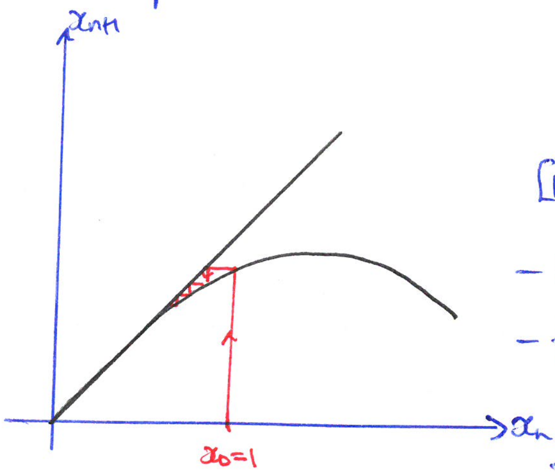


- useful when linear analysis fails

Example. $x_{n+1} = \sin(x_n)$

- aim: show that $x_* = 0$ is globally stable. [NOTE: $\lambda = 1$ in this case so linear analysis fails!]

First consider the map evolution with $x_0 = 1$.

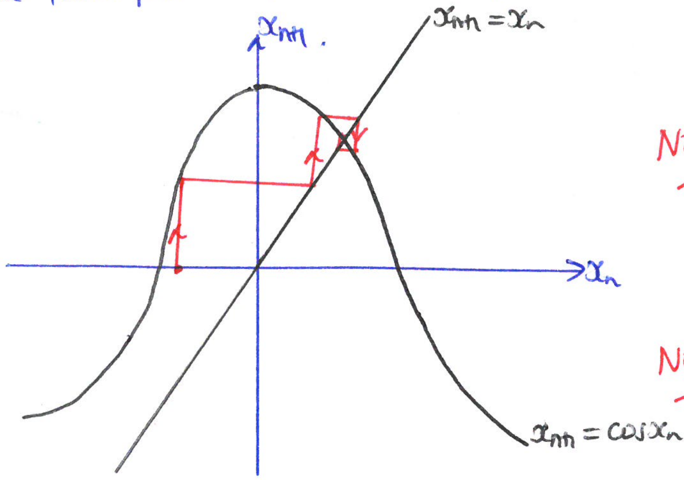


[we see a similar cobweb for $x_0 = -1$]

- Cobwebbing suggests local stability
- to see global stability, note that for any $x_0 \in \mathbb{R}$, $x_1 = \sin x_0$ satisfies $\underline{|x_1| \leq 1}$
- Then apply cobwebbing!

Example: $x_{n+1} = \cos x_n$

NOTE: the fixed point x_* is the solution of the transcendental equation $x = \cos x$.

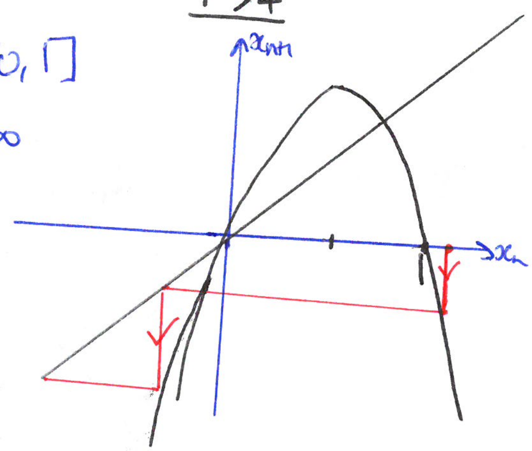


NOTE: Spiral motion as $-1 < \lambda < 0$ [f slopes downward]
↳ the perturbation alternates sign at each iteration.

NOTE: monotonic convergence for $0 < \lambda < 1$

Logistic map. $x_{n+1} = r x_n (1 - x_n)$ (analogous to continuous model)
- $x_n \geq 0$ is the population
- $r \geq 0$ is small population growth rate

• For $0 \leq r \leq 4 \rightarrow$ if $x_n \in [0,1]$ then $x_{n+1} \in [0,1]$
• For $r > 4 \rightarrow$ blow up towards $x_n \rightarrow -\infty$



Period doubling.

• $0 < r < 1 \rightarrow x_* = 0$ is globally stable

• $1 < r < 3 \rightarrow$ A non-zero stable fixed point appears
$$x_* = 1 - 1/r \geq 0$$

• For $r \gtrsim 3$ (e.g. $r = 3.3$), a 2-cycle exists, $x_{n+2} = x_n \ \forall n$
$r = 3.5$, a 4-cycle exists, $x_{n+4} = x_n \ \forall n$.

In fact, we have a sequence r_n at which the 2^n-cycle is born, where
$$r_1 = 3, \ r_2 = 3.449, \ \ldots\ldots, \ r_\infty = 3.5699\ldots$$

↳ geometric convergence with $\delta = \lim_{n \to \infty} \frac{r_n - r_{n-1}}{r_{n+1} - r_n} = 4.669\ldots$

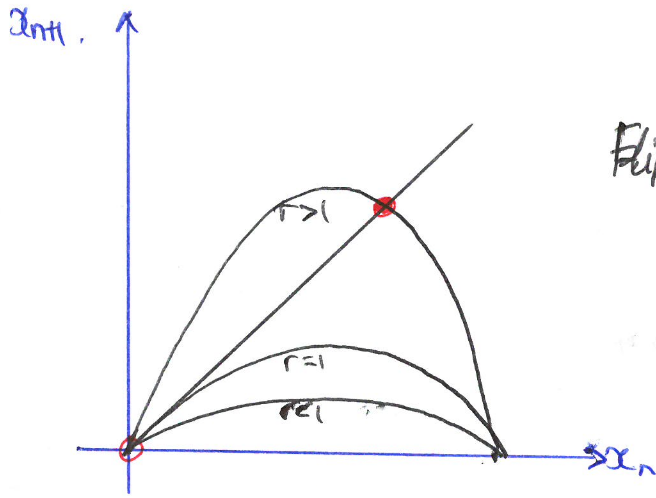
Analysis: $x_{n+1} = r x_n (1 - x_n)$ for $0 \le x_n \le 1$, $0 \le r \le 4$.

Fixed points:
$$\begin{cases} x_0 = 0 \quad \forall r \in [0,4] \\ x_* = 1 - \frac{1}{r} \text{ for } r \ge 1 \text{ (in desired range of } x) \end{cases}$$

$x_0 \rightarrow$ stable for $0 \le r < 1$, unstable for $r > 1$

Transcritical bifurcation at $r = 1$ (eigenvalue passes through 1)

$\hookrightarrow f'(x_*) = r - 2r\left(1 - \frac{1}{r}\right) = 2 - r \Rightarrow$ stable for $1 < r < 3$.



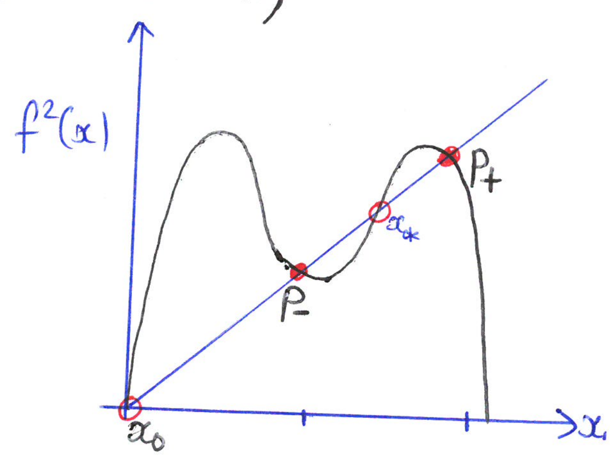
Flip bifurcation at $r = 3$ (associated with period-doubling).

$\hookrightarrow$ eigenvalue passes through -1

Two-cycle.

Want $x_{n+2} = x_n \ \forall n$, ie. $x_* = f(f(x_*)) = f^2(x_*)$ ← composite map.

NOTE:
$$\begin{aligned} f^2(x_*) &= f(r x_n (1 - x_n)) \\ &= r[r x_n (1 - x_n)](1 - [r x_n (1 - x_n)]) \quad \leftarrow \text{quartic!} \\ &= r^2 x_n \left(1 - (r+1) x_n + 2 r x_n^2 - r x_n^3\right) \end{aligned}$$



Note: it turns out that (trivially) x_0 and x_* are also fixed points of f^2.

So finding fixed points $x = f^2(x)$ can be reduced to finding roots of a quadratic polynomial.

$$x = f^2(x) \Leftrightarrow 0 = r^2 x \left(1 - \frac{1}{r^2} - (r+1) x + 2 r x^2 - r x^3\right)$$
$$\Leftrightarrow \boxed{0 = r^2 x \left(x - \left(1 - \frac{1}{r}\right)\right)\left(-r x^2 + (r+1) x - \left(1 + \frac{1}{r}\right)\right)}$$

Fixed points are $p_\pm = \dfrac{(r+1) \pm \sqrt{(r-3)(r+1)}}{2r}$ → roots exist only for $r>3$ (otherwise complex)

↳ as $r \to 3^+$, $p_\pm \to x_*$ ∴ 2-cycle forms continuously from x_*.

NOTE: If $x_n = p_+$ then $x_{n+1} = p_-$ and $x_{n+2} = p_+$, etc. [so $p_+ = f(p_-)$ and $p_- = f(p_+)$]

NOTE: like pitchfork bifurcations, flip bifurcations can also have a subcritical form (i.e. there exists an unstable 2-cycle below the bifurcation point)

2-cycle Stability.

- Want to show that the 2-cycle is stable for $r_1 = 3 < r < r_2 = 1+\sqrt{6} \simeq 3.449$

Method: determine stability of composite map f^2

Multiplier $\lambda = \dfrac{d}{dx}\left[f(f(x))\right]\Big|_{x=p_+} = f'(\underbrace{f(p_+)}_{=p_-})f'(p_+) = \underbrace{f'(p_-)f'(p_+)}_{\text{symmetric}}$

so λ the same for p_+ and p_- ∴ simultaneously destabilize

$$\Rightarrow \lambda = r(1-2p_+)r(1-2p_-)$$
$$= r^2\left(1-2(p_+ + p_-) + 4p_+p_-\right) = \cdots$$
$$= 4 + 2r - r^2$$

∴ 2-cycle stable for $|\lambda|<1$, i.e. $|4+2r-r^2|<1 \Rightarrow 3 < r < 1+\sqrt{6}$

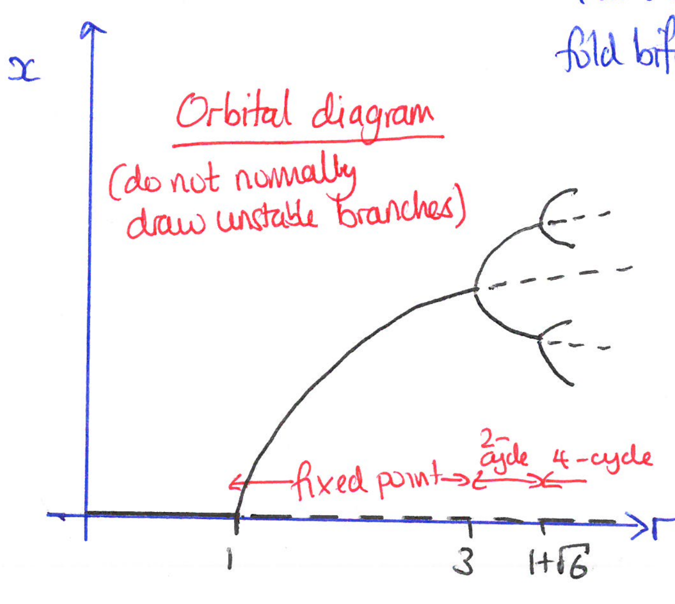
positive root of $4+2r-r^2 = 1$

positive root of $4+2r-r^2 = -1$

So the multiplier passes through -1 at the instability, giving another fold bifurcation and period-doubling.

NOTE: • $p_+ + p_- = \dfrac{r+1}{r}$

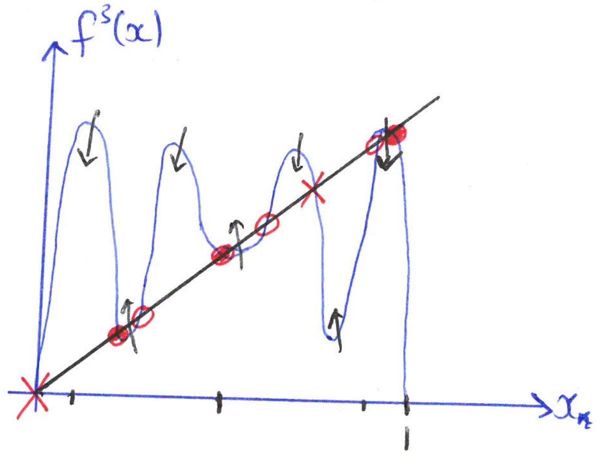
• $p_+p_- = \dfrac{1}{4r^2}\left[(r+1)^2 - (r+1)(r-3)\right]$



Periodic windows

- How can we explain the presence of the 3-cycle that exists in small interval of r for $r > r_\infty$ (i.e. within the chaotic regime) [other cycles appear by a similar mechanism]

- Consider $x_{n+3} = f^3(x_n)$ ← seek fixed points.



× corresponds to usual fixed point of f

○ unstable 3-cycle

● stable 3-cycle

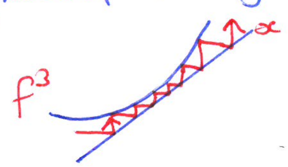
→ r decreasing

- At some critical value of r, the stable and unstable 3-cycles coalesce and annihilate in a tangent bifurcation (c.f. saddle-node bifurcation)
  ↳ as r is increased, this bifurcation marks the beginning of the period window.

Intermittency

- For r just below the threshold r_c at which a 3-cycle appears, we see a ghost of a 3-cycle (like with ghosts for saddle-node bifurcations)

- Why?



We have $f^3(x_n) \approx x_n$ during the passage and so the orbit looks like a 3-cycle.

- eventually the orbit escapes the channel and moves chaotically until returning to the channel at some unpredictable time later.

↳ Intermittency is a common feature of systems where the transition from periodic to chaotic takes place by a saddle-node bifurcation of cycles.

- intermittency route to chaos.

Lyapunov exponent.

- Consider the separation δ_n of two trajectories, where δ_0 is very small,

  ↳ We want to find the exponent λ s.t. $|\delta_n| \approx |\delta_0| e^{n\lambda}$

  ↳ $\lambda > 0$ is a signature of chaos

Note: • $x_n = f^n(x_0)$

• $x_n + \delta_n = f^n(x_0 + \delta_0)$

$$\left.\begin{array}{l} x_n = f^n(x_0) \\ x_n + \delta_n = f^n(x_0+\delta_0) \end{array}\right\} \Rightarrow \delta_n = f^n(x_0+\delta_0) - f^n(x_0).$$

But if $\dfrac{|\delta_n|}{|\delta_0|} \approx e^{n\lambda}$ then $\lambda \approx \dfrac{1}{n}\log\left|\dfrac{\delta_n}{\delta_0}\right| = \dfrac{1}{n}\log\left|\dfrac{f^{n}(x_0+\delta_0) - f^{n}(x_0)}{\delta_0}\right|$

Take the limit $\delta_0 \to 0 \Rightarrow \lambda \approx \dfrac{1}{n}\log\left|\dfrac{d}{dx}f^n\Big|_{x=x_0}\right|$

By the chain rule, $\dfrac{d}{dx}f^n(x_0) = \prod_{j=0}^{n-1} f'(x_j)$ where $x_j = f^j(x_0)$.

$$\Rightarrow \lambda \approx \frac{1}{n}\log\left|\prod_{j=0}^{n-1} f'(x_j)\right|$$
$$= \frac{1}{n}\sum_{j=0}^{n-1} \log|f'(x_j)|$$

If the limit as $n \to \infty$ exists then we define the Lyapunov exponent

$$\lambda = \lim_{n\to\infty}\left\{\frac{1}{n}\sum_{j=0}^{n-1}|f'(x_j)|\right\}$$

NOTE: λ depends on x_0, but λ is the same for all x_0 in the basin of attraction of a given attractor.

- $\lambda < 0$: stable fixed points and cycles
- $\lambda > 0$: chaotic attractors.

Example.

Suppose that f has a stable p-cycle containing the point x_0

Aim: Show that $\lambda < 0$

Note: $x_0 = f^p(x_0)$, and, by stability, $\left|\frac{d}{dx} f^p(x_0)\right| < 1$

Hence $\underline{\log\left|\frac{d}{dx} f^p(x_0)\right| < 0}$ ✱

Note: $\lambda = \lim_{n\to\infty}\left\{\frac{1}{n}\sum_{j=0}^{n-1}\log|f'(x_j)|\right\} = \frac{1}{p}\sum_{j=0}^{p-1}\log|f'(x_j)|$ ← for a p-cycle

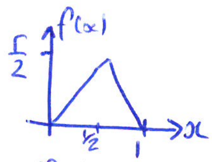
$= \frac{1}{p}\log\left|\prod_{j=0}^{p-1} f'(x_j)\right|$

reverse chain rule

$= \frac{1}{p}\log\left|\frac{d}{dx} f^p(x_0)\right| < 0 \quad \Rightarrow \lambda < 0$

Example.

Tent map $f(x) = \begin{cases} rx & 0 \le x \le 1/2 \\ r - rx & \frac{1}{2} \le x \le 1 \end{cases}$ for $0 \le r \le 2$, $0 \le x \le 1$ [simplified version of logistic map]



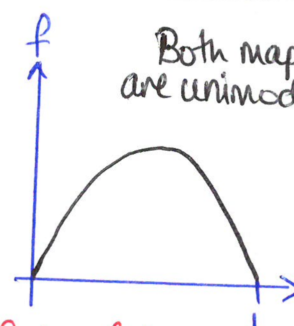
Note: $f'(x) = \pm r \quad \forall x$

$\Rightarrow \lambda = \lim_{n\to\infty}\left\{\frac{1}{n}\sum_{j=0}^{n-1}\log(r)\right\} = \log(r)$ $\Rightarrow$ Tent map has chaotic solutions for $r > 1$.

Universality and experiments.

- logistic map: $x_{n+1} = r x_n (1 - x_n)$ $\quad 0 \le r \le 4$, $0 \le x \le 1$
- sine map: $x_{n+1} = r \sin(\pi x_n)$ $\quad 0 \le r \le 1$, $0 \le x \le 1$

Note: logistic$\big|_{x_n = 1/2} = \frac{r}{4}$, sine$\big|_{x_n = 1/2} = r$ ← so r is scaled by a factor of 4.



Both maps are unimodal

- The orbital diagrams are qualitatively similar.

The U-sequence.

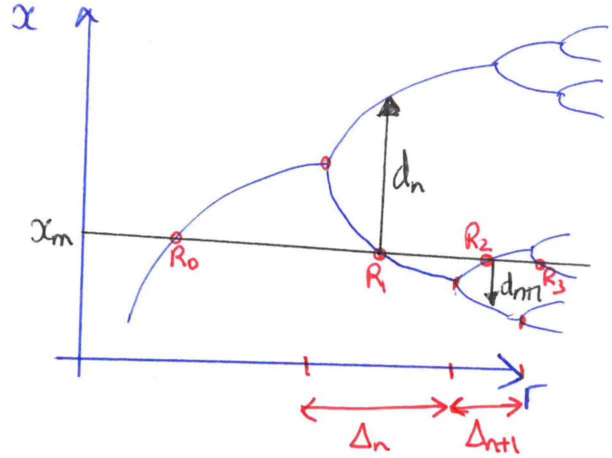
Consider $x_{n+1} = f(x_n)$ where $f(0) = f(1) = 0$ and is unimodal

Metropolis: As r is varied, the order in which stable periodic solutions appear is independent of the unimodal map f.

↳ gives qualitative information, but when do the cycles appear?

Feigenbaum's constant: $\delta = \lim_{n \to \infty} \left[\frac{r_n - r_{n-1}}{r_{n+1} - r_n} \right] = 4.669\ldots$ ← a mathematical constant.

The convergence rate is independent of the map, provided that the map is <u>unimodal</u>



- x_m maximises f
- d_n = distance from x_m to the <u>nearest</u> point in a 2^n-cycle

Then $\lim_{n \to \infty} \frac{d_n}{d_{n+1}} \to \alpha = -2.5029\ldots$

(another universal limit)

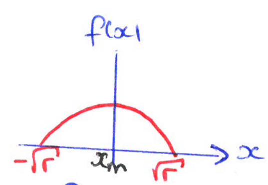
(*) Nearest 2^n-cycle alternatives between being above and below x_m (*)

<u>Renormalization</u> – let $f(x,r)$ be a unimodal map that undergoes a PDC as r increases, and x_m maximises r.

- r_n: value of r at each a 2^n-cycle is born
- R_n: value of r at which the 2^n-cycle is superstable [i.e. multiplier = 0]

<u>Example</u> (superstable cycles)

– Find R_0 and R_1 for $f(x,r) = r - x^2$



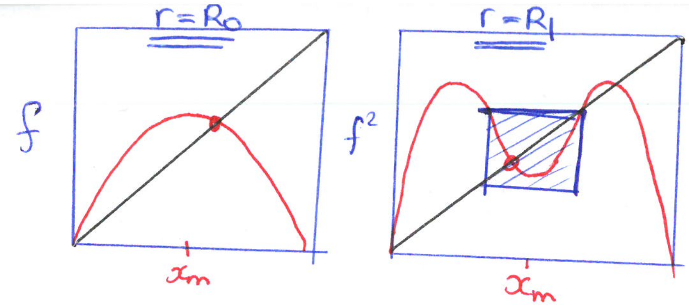
- R_0 ⇒ superstable fixed point ⇝ $f' = -2x = 0 \Leftrightarrow x_* = 0$.
  But fixed points satisfy $x_* = r^2 - x_*^2$ ⇒ $\underline{R_0 = 0}$

- R_1 ⇒ superstable 2-cycle, denoted by the points p and q
  Multiplier $\lambda = f'(p) f'(q) = 4pq = 0$ ∴ need $p = 0$ or $q = 0$.
  But $f^2 = r - (r - x^2)^2$, so $f^2\big|_{x=0} = r - r^2 = 0$ if $r = \underline{R_1 = 1}$ [other solution is a fixed point]

<u>General rule</u> – a superstable cycle of a unimodal map always contains x_m as one of its points [in the above example, $x_m = 0$]

– renormalization is based on self-similarity of the orbital diagram – each branch is like a scaled down version of previous branches.

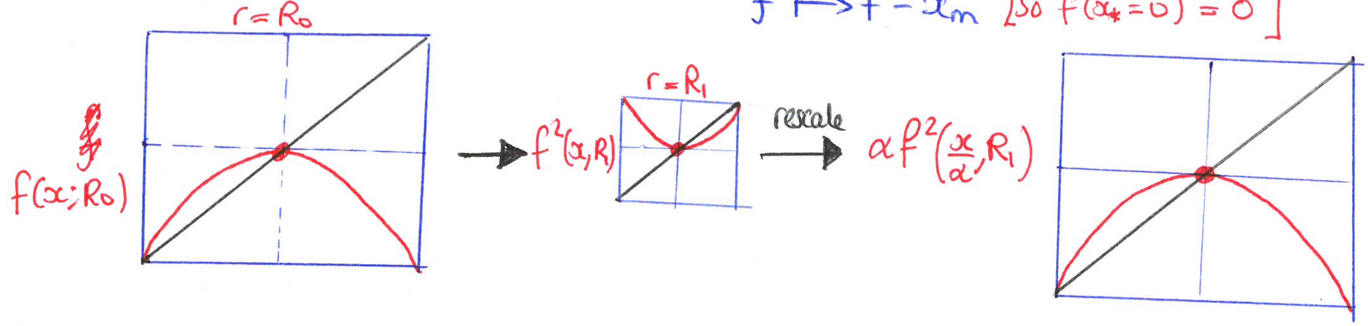
Consider



← inner box is similar to f, but scaled down and flipped – yield similar dynamics

Step 1: translate the origin of x to x_m, ie. $x \mapsto x - x_m$

$f \mapsto f - x_m$ [so $f(x_*=0)=0$]



Step 2: rescale $f(x,R_0) \approx \alpha f^2\left(\frac{x}{\alpha}, R_1\right)$, where $\alpha = -2.5\ldots$ ($\alpha<0$, so flips $x \mapsto -x, y \mapsto -y$)

stretching

Step 3: $f^2\left(\frac{x}{\alpha}, R_1\right) \approx \alpha \; f^4\left(\frac{1}{\alpha}\frac{x}{\alpha}, R_2\right)$ $\Rightarrow f(x,R_0) \approx \alpha^2 f^4\left(\frac{x}{\alpha^2}, R_2\right)$

$$\Rightarrow f(x,R_0) \approx \alpha^n f^{2^n}\left(\frac{x}{\alpha^n}, R_n\right)$$

Step 4: Define the universal function $g_0(x)$ with a superstable fixed point such that

$$\lim_{n\to\infty}\left[\alpha^n f^{(2^n)}\left(\frac{x}{\alpha^n}, R_n\right)\right] = g_0(x)$$

← limit exists only if $\alpha = -2.5029\ldots$

independent of f provided that f has a quadratic maximum

NOTE: g_0 depends on f only through its behaviour near $x=0$ since that is all that survives in the argument as $n\to\infty$ [as $|\alpha|>1$]

Step 5: Repeat process starting at $f(x,R_m)$ to get a universal function $g_m(x)$ with a superstable 2^m-cycle:

$$g_m(x) = \lim_{n\to\infty}\left[\alpha^n f^{(2^n)}\left(\frac{x}{\alpha^n}, R_{n+m}\right)\right]$$

In particular, at $R_m = R_\infty$, we no longer need to shift r when normalizing, giving

$$g(x) = \alpha g^2\left(\frac{x}{\alpha}\right)$$

← we can express g in terms of a power series expansion.

↳ also get $\alpha \approx -2.5029\ldots$

Example: renormalization for pedestrians.

Let $f(x,\mu)$ be a unimodal map that gives a PDC to chaos

↳ shift variables so that the 2-cycle is born at $x=0$ and $\mu=0$

$$\Rightarrow \quad x_{n+1} \approx -(1+\mu)x_n + a x_n^2 + \ldots$$ [gives a multiplier $\lambda=-1$ at bifurcation corresponding to a fold bifurcation]

↳ map $x \mapsto \frac{x}{a} \Rightarrow \underline{a=1 \text{ WLOG}}$

$\therefore$ normal form: $\underline{x_{n+1} = -(1+\mu)x_n + x_n^2 + \ldots}$ (*) $\quad \boxed{f(x) = -(1+\mu)x + x^2}$

Method: look at successive period doublings to get a map with the same algebraic form as (*)
[the renormalization step gives approximations of α and δ]

Step 1. Define period-2 points p and q s.t.
$$\begin{cases} p = -(1+\mu)q + q^2 \\ q = -(1+\mu)p + p^2 \end{cases} \Rightarrow \begin{cases} p = \frac{1}{2}\left[\mu + \sqrt{\mu^2+4\mu}\right] \\ q = \frac{1}{2}\left[\mu - \sqrt{\mu^2+4\mu}\right] \end{cases}$$

Step 2: Shift the origin to p and look at the local dynamics

Consider linearisation of fixed point p using map f^2

$$\Rightarrow \underline{\eta_{n+1} = (1-4\mu-\mu^2)\eta_n + C\eta_n^2 + (\text{hot})} \quad (\dagger) \leftarrow \text{same algebraic form as } (*)$$

where $\underline{C = 4\mu + \mu^2 - 3\sqrt{\mu^2+4\mu}}$.

Step 3: rescale η_n are shift μ

↳ let $\underline{\tilde{x}_n = C\eta_n} \Rightarrow \tilde{x}_{n+1} = \underbrace{(1-4\mu-\mu^2)}\tilde{x}_n + \tilde{x}_n^2 + \ldots$

Choose $\tilde{\mu}$ so this term is $-(1+\tilde{\mu})$

↳ let $-(1+\tilde{\mu}) = 1+4\mu-\mu^2 \Rightarrow \underline{\tilde{\mu} = \mu^2+4\mu-2}$.

Hence: $\underline{\tilde{x}_{n+1} = -(1+\tilde{\mu})\tilde{x}_n + \tilde{x}_n^2}$.

↳ the renormalized map undergoes a flip bifurcation at $\tilde{\mu}=0$.

Step 4: Compute μ at which $\tilde{\mu}=0$, i.e. $\mu^2+4\mu-2=0 \Rightarrow \underline{\mu = -2+\sqrt{6}}$ (positive root)

same as logistic map for $\mu = r-3$

Step 5: Let μ_k denote μ at which a 2^k-cycle is born. By iteration, it turns out that $\underline{\mu_k = -2+\sqrt{6+\mu_{k-1}}}$

- **Step 6**: Find that $\mu_k \to \mu_* = \frac{1}{2}(-3+\sqrt{17}) \approx 0.56$ as $k \to \infty$

↑ onset of chaos

(recall: $r_c = 3.57$ for logistic map)

Step 7: For $k \gg 1$, expect geometric convergence to μ_* at a rate δ

$$\Rightarrow \delta \approx \frac{(\mu_{k+1} - \mu_*)}{(\mu_k - \mu_*)} \approx \left.\frac{d\mu_{k-1}}{d\mu_k}\right|_{\mu=\mu_*} = \underline{2\mu_* + 4} \approx 1+\sqrt{17} \approx \underline{\underline{5.12}}$$

↑ L'hôpital's rule

True value 4.669

Step 8: Rescaling parameter was $C = 4\mu + \mu^2 - 3\sqrt{\mu^2 + 4\mu}$

Set $\mu = \mu_* \Rightarrow C = \frac{1+\sqrt{17}}{2} - 3\left[\frac{1+\sqrt{17}}{2}\right]^{1/2} \approx \underline{-2.24}$

↑ True value $\alpha \approx -2.50$