Lorenz equations.

$$\begin{cases}\dot{x}=\sigma(y-x)\\ \dot{y}=rx-y-\underline{xz}\\ \dot{z}=\underline{xy}-bz.\end{cases}$$

nonlinearity.

Parameters $\sigma, r, b > 0$.

[σ = Prandtl number
r = Rayleigh number]

Symmetry: The system is invariant under the mapping $x \mapsto -x,\ y \mapsto -y$.

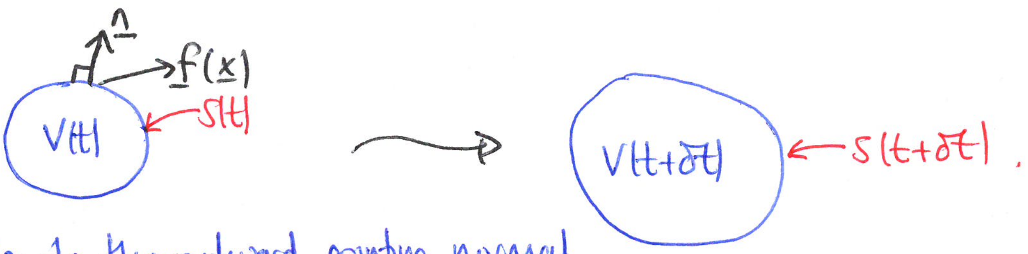
⇒ either solutions are symmetric themselves or a symmetric partner (depending on the initialisation).

Volume contraction: The Lorenz system is dissipative: volume in phase space contracts under the flow.

How does volume evolve in general 3D systems $\dot{\underline{x}} = \underline{f}(\underline{x})$?

- Consider a volume $V(t)$ in phase space whose surface $S(t)$ is the position of the trajectories [so $S(0)$ is the initial conditions of the trajectories].

How do $S(t)$ and $V(t)$ evolve?



- Let $\underline{n}$ denote the outward pointing normal vector on S.
- As $\underline{f} = \dot{\underline{x}}$ is the instaneous velocity, $\underline{f}\cdot\underline{n}$ is the outward normal component of velocity.

⇒ a patch of area δA on the surface swaps out a volume $(\underline{f}\cdot\underline{n}\,\delta t)\,\delta A$ in time δt

↳ summing over all patches gives

$$V(t+\delta t) - V(t) = \int_S (\underline{f}\cdot\underline{n}\,\delta t)\,dA \quad \Rightarrow \quad \frac{V(t+\delta t)-V(t)}{\delta t} = \int_S \underline{f}\cdot\underline{n}\,dA$$

$\lim_{\delta t \to 0}$ ⇒ $$\dot{V} = \int_S \underline{f}\cdot\underline{n}\,dA.$$

Divergence Theorem ⇒ $$\dot{V} = \int_V \underline{\nabla}\cdot\underline{f}\,dV.$$

For the Lorenz system,

$$\underline{\nabla}\cdot\underline{f} = \frac{\partial}{\partial x}\left[\sigma(y-x)\right] + \frac{\partial}{\partial y}\left[rx-y-xz\right] + \frac{\partial}{\partial z}\left[xy-bz\right]$$

$$= -\sigma \quad -1 \quad -b$$

$$= -(1+b+\sigma) < 0 \quad \text{everywhere}$$

$\Rightarrow \underline{\underline{\dot{V} = -(1+b+\sigma)V}}$ $\Longrightarrow$ the volume decreases exponentially fast!

i.e. contraction of trajectories to a set of <u>zero volume</u>.

$\left\{\begin{array}{l}\text{fixed points, limit}\\ \text{cycles, strange attractor}\end{array}\right.$ $\hookleftarrow$

↳ As a consequence, there are no <u>quasiperiodic</u> solutions to Lorenz equations

↳ i.e. trajectories that are nearly periodic but with a weak precession (i.e. rotation)

↳ Why? Such trajectories live on the torus, but the volume of the flow decreases ⨳.

↳ Another consequence is that the Lorenz system cannot have either repelling fixed points or repelling orbits

↳ Why? Repellers are sources of volume, yet all volumes contract! (But saddle-like structures are okay)

⊛ Conclusion: All fixed points must be attractors/saddles and any closed orbits must be stable or "saddle-like".

<u>Fixed points</u>:

① The origin $\forall \sigma, r, b > 0$

② For $r > 1$, <u>symmetric</u> pair of points $(+\sqrt{b(r-1)}, +\sqrt{b(r-1)}, r-1)$ ← C^+

and $(-\sqrt{b(r-1)}, -\sqrt{b(r-1)}, r-1)$ ← C^-.

↓ recall $x \mapsto -x, y \mapsto -y$ symmetry!

As $r \to 1^+$, $C^\pm$ coalesce with the origin via a pitchfork bifurcation.

Stability of the origin

① Linear stability → linearized system $\left\{\begin{array}{l} \dot{x} = \sigma(y-x) \\ \dot{y} = rx - y \\ \dot{z} = -bz \end{array}\right.$

$\left.\begin{array}{l}\dot{x} = \sigma(y-x) \\ \dot{y} = rx - y\end{array}\right\}$ effective 2D system

$\dot{z} = -bz$ ← decouples and decays exponentially

Trace $= -(\sigma+1) < 0$
Determinant $= (-\sigma)(-1) - \sigma r = \sigma(1-r)$

compute eigenvectors of 3D system.

↳ $r > 1 \Rightarrow$ saddle ← 3D, so one outgoing and two incoming directions.

$r < 1 \Rightarrow$ stable node as $T^2 - 4D = (\sigma-1)^2 + 4\sigma r > 0$

② Global stability (r<1) → construct a Lyapunov function $V(x,y,z)$. (not a volume!)

↳ Choose $V(x,y,z) = \frac{1}{\sigma}x^2 + y^2 + z^2 > 0$ except at origin.

We need to check that $\frac{dV}{dt} < 0$ along trajectories

A short calculation gives

$$\frac{1}{2}\dot{V} = -\left[x - \frac{r+1}{2}y\right]^2 - \left[1 - \left(\frac{r+1}{2}\right)^2\right]y^2 - bz^2. \le 0$$

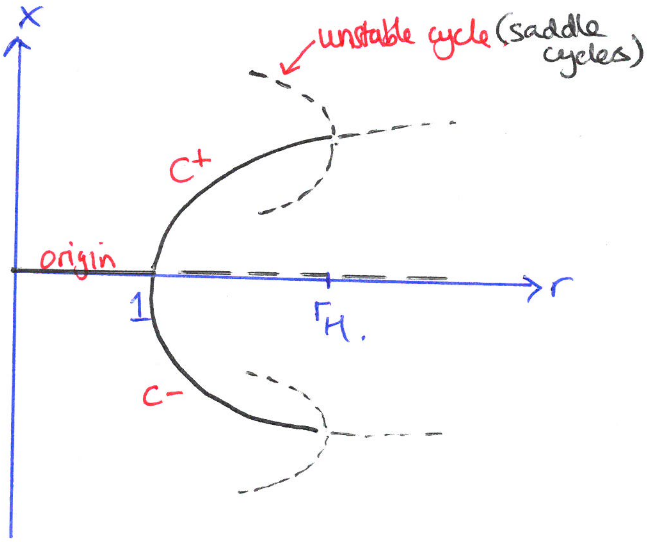
with equality iff $x=y=z=0$.

Stability of C^+ and C^-.

These fixed points are linearly stable for $1 < r < r_H = \dfrac{\sigma(\sigma+b+3)}{\sigma-b-1}$ (assuming $\sigma-b-1>0$)

At $r = r_H$, the fixed points undergo a subcritical Hopf bifurcation

(so a small-amplitude stable limit cycle does not form for $r \gtrsim r_H$)



What happens for $r \gtrsim r_H$?

- The trajectory settles onto a very thin set in phase space – zero volume, in accordance with the volume contraction of the system.
  ↳ fractal – zero volume but with infinite surface area.

Sensitivity to initial conditions. $\sigma = 10, \; b = \frac{8}{3}, \; r = 28$

- small changes are amplified over time – two initially adjacent trajectories will end up on opposite sides of the attractor at some later time.

Idea: start with a trajectory that already lies "on" the attractor.

↳ let the trajectory pass through point $\underline{x}(t)$ at time t.

↳ let $\underline{\delta}(t)$ be the vector to a nearby trajectory, $\underline{x}(t) + \underline{\delta}(t)$, where $\|\underline{\delta}_0\| << 1$. $[\underline{\delta}_0 = \underline{\delta}(0)]$

- From numerical simulation, one finds that $\|\underline{\delta}(t)\| \sim \|\underline{\delta}_0\| e^{\lambda t}$, where $\lambda > 0$ is the maximal Lyapunov exponent ($\lambda \approx 0.9$ in this case)

$\Rightarrow$ neighboring trajectories separate exponentially fast.

↳ distance saturates when $\|\underline{\delta}\|$ becomes comparable to the size of the attractor.

What are the other Lyapunov exponents?

↳ for an n-dimensional system there are n Lyapunov exponents.

- consider the evolution of an initially infinitesimal n-dimensional sphere of initial conditions
- during the evolution, the sphere becomes distorted to an n-dimensional ellipsoid, where the length of the kth principal axis is $\delta_k(t) > 0$
- Then $\delta_k(t) \sim \delta_k(0) e^{\lambda_k t}$, where λ_k are the Lyapunov exponents $(k = 1, .., N)$

Then the maximal $\lambda = \max_k \lambda_k$ dominates the growth!

NOTE: λ varies slightly between trajectories – need to get the true value of λ by averaging over many trajectories.

NOTE: Time horizon is the time at which two initially close trajectories become sufficiently far apart for the two trajectories to no longer approximate each other.

Chaos.

"Chaos is $\overbrace{\text{aperiodic long-term behaviour}}^{\circledast}$ in a $\overbrace{\text{deterministic}}^{\text{no noise}}$ system that exhibits $\underbrace{\text{sensitive dependence}}_{\text{positive maximal Lyapunov exponent}}$ on initial conditions"

$\circledast$ = ∃ trajectories that do not approach fixed points/periodic orbits/quasiperiodic orbits as $t \to \infty$ [Infinity can be thought of as a fixed point!]

Attractor - "a set to which all neighboring trajectories converge"

The set A is an attractor if it satisfies

① A is an invariant set - any trajectory that starts in A stays in A $\forall t$

② A attracts an open set of initial conditions

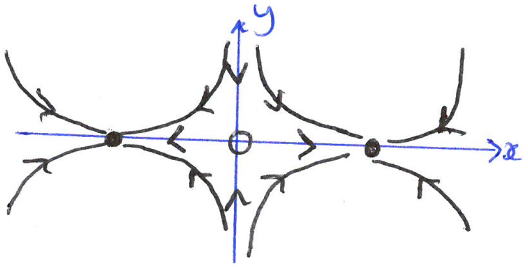
- If $\underline{x}(0) \in U \supset A$ then $\underline{x}(t)$ reaches A as $t \to \infty$

↳ The largest such U is the basin of attraction of A.

③ A is minimal - there is no proper subset of A that satisfies ① and ②.

Example.

$$\begin{cases} \dot{x} = x - x^3 \\ \dot{y} = -y \end{cases}$$



Consider $I = \{(x,y) : -1 \le x \le 1, y = 0\}$.

Phase portrait ⇒ ① I is an invariant set

② The basin of attraction $U \supset I$ is the entire (x,y) plane

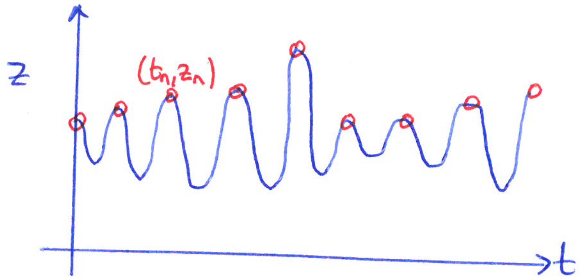
③ NOT minimal as stable fixed points $(\pm 1, 0)$ are subsets of I and satisfy ① and ②.

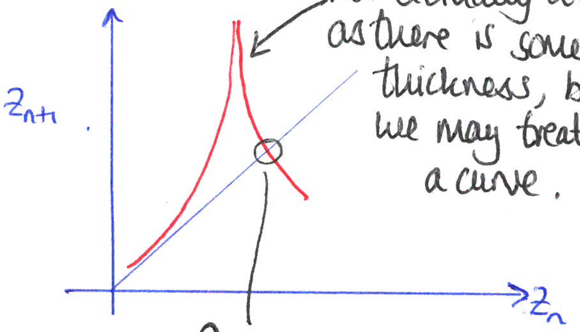
Strange attractor - "an attractor that exhibits sensitive dependence on initial conditions"

Lorenz map.

Lorenz observed that when a trajectory spirals away from C^+ (or C^-), the loops become larger until eventually the trajectory crosses to the other "wing of the butterfly". Crucially, the "amplitude" of one loop appears to determine the amplitude of the next loop.

This method allows us to extract order from chaos.





not actually a curve as there is some thickness, but we may treat it as a curve.

fixed point.

Consider: $z_{n+1} = f(z_n)$. ← Lorenz map.

Note: For Lorenz, $|f'(z)| > 1 \quad \forall z$.

Hence, there are no stable fixed points (via linear stability of iterative maps)
By considering the compound map, there are also no stable periodic trajectories.

- In different parameter regimes (e.g. $r=21$, $\sigma=10$, $b=8/3$), the Lorenz system can exhibit transient chaos.
  - chaotic-like behaviour initially, but eventually the system approaches one of the stable fixed points $C^\pm$
  - However, which of C^+ or C^- is attained depends sensitively on the initial conditions, and so is unpredictable.

- For large r, the system can have globally attracting limit cycles.

Synchrony of Chaotic systems.

- Remarkably, two chaotic systems may synchronize!
    - ↳ have a receiver that is forced by a chaotic transmitter.

Such a scenario can be constructed using circuits (see Strogatz)

↳ voltages u, v, w play the role of x, y, z in Lorenz equations (under a certain rescaling)

$$\Rightarrow \begin{cases} \dot{u} = \sigma(v-u) \\ \dot{v} = ru - v - 20uw \\ \dot{w} = 5uv - bw \end{cases}$$

(use Kirchoff's law and nondimensionalise)

TRANSMITTER.

Now we consider a receiver with voltages u_r, v_r, w_r that is driven by the transmitter

$$\Rightarrow \begin{cases} \dot{u}_r = \sigma(v_r - u_r) \\ \dot{v}_r = ru - v_r - 20uw_r \\ \dot{w}_r = 5uv_r - bw_r \end{cases}$$

RECEIVER

← note, this is not the Lorenz system itself, but the system is driven by just <u>one</u> of the Lorenz variables (u)

Aim: Show that the receiver asymptotically approaches perfect synchrony with the transmitter for any initial conditions.

i.e. show that the errors

$$\begin{cases} e_1 = u - u_r \to 0 \\ e_2 = v - v_r \to 0 \\ e_3 = w - w_r \to 0 \end{cases} \quad \text{as } t \to \infty$$

Note:

$$\begin{cases} ① & \dot{e}_1 = \sigma(e_2 - e_1) \\ ② & \dot{e}_2 = -e_2 - 20ue_3 \\ ③ & \dot{e}_3 = 5ue_2 - be_3 \end{cases}$$

← a linear system but with chaotic forcing [Note: fixed point $e_1 = e_2 = e_3 = 0$]

Method: construct a Lyapunov function so that the chaos cancels out.

Note: $e_2 \times ② + 4e_3 \times ③ \Rightarrow e_2\dot{e}_2 + 4e_3\dot{e}_3 = -e_2^2 - 4be_3^2$

$$\Rightarrow \frac{d}{dt}\frac{(e_2^2 + 4e_3^2)}{2} = -(e_2^2 + 4be_3^2)$$

Define the Lyapunov function $E = \frac{1}{2}\left(\frac{1}{\sigma}e_1^2 + e_2^2 + 4e_3^2\right) \geq 0$ except at $e_1 = e_2 = e_3$ (fixed point)

Need to check that $\dot{E} < 0$ except at fixed point along trajectories,

So ∧ $E = \frac{1}{\sigma}e_1\dot{e}_1 + (e_2\dot{e}_2 + 4e_3\dot{e}_3)$

$= e_1(e_2 - e_1) - e_2^2 - 4be_3^2$ substitute in from ODEs

$= \underbrace{-e_1^2 + e_1e_2}_{\downarrow} - e_2^2 - 4be_3^2$

$= -\left[e_1 - \frac{1}{2}e_2\right]^2 + \frac{1}{4}e_2^2 - e_2^2 - 4be_3^2$ complete the square

$$\underline{\underline{\dot{E} = -\left[e_1 - \frac{1}{2}e_2\right]^2 - \frac{3}{4}e_2^2 - 4be_3^2 \leq 0}}$$

and $\dot{E} = 0$ only at the fixed point.

Hence, E is a Lyapunov function, so the origin $e_1 = e_2 = e_3 = 0$ is globally attracting

Hence, we have asymptotic synchrony of the transmitter and the receiver.

Application: using chaos to send secret messages (Strogatz §9.6).