Bifurcations in 2-dimensional systems.

- Topological changes in the phase plane as a parameter is varied } e.g., number of fixed points / closed orbits / saddle connections

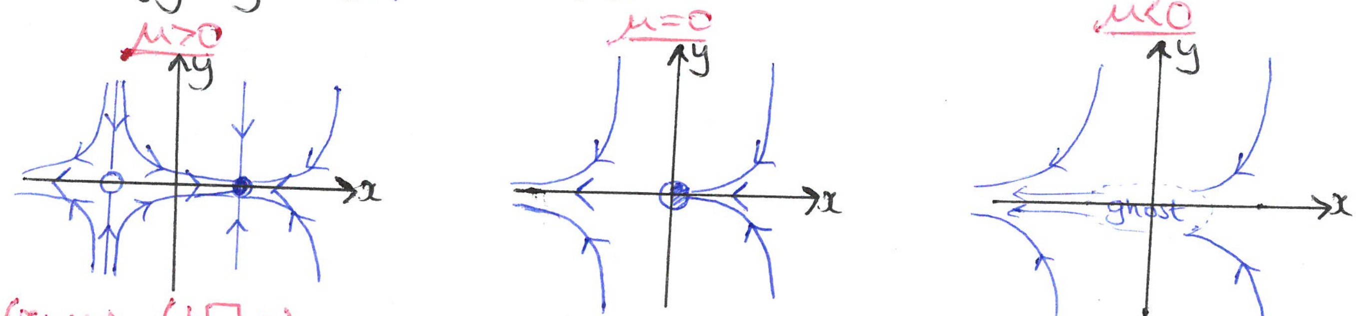
Extending 1D bifurcations to the phase plane.

- Typically the action occurs a one-dimensional subspace, so examples are easy to construct.

Saddle-Node Bifurcation (now the name becomes clear!)

- creation/destruction of fixed points [when nullclines have a tangential intersection]

Example: $\begin{cases} \dot{x} = \mu - x^2 & \text{[see 1D]} \\ \dot{y} = -y & \text{[exponential damping]} \end{cases}$

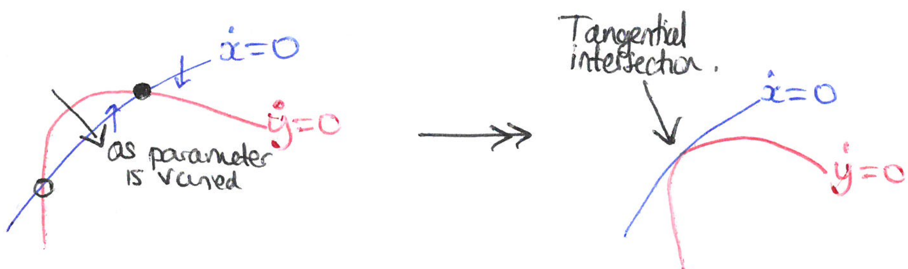


$(x_\pm, y_\pm) = (\pm\sqrt{\mu}, 0)$ - saddle and a node!

collision of fixed points [semi-stable]

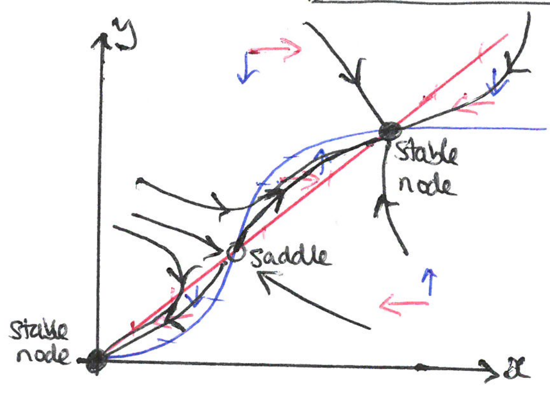
annihilation of fixed points. for $x = O(\sqrt{\mu})$ ← ghost point and flow is slow [bottleneck]

Generic case: consider the nullclines!



Example $\begin{cases} \dot{x} = -ax + y \\ \dot{y} = \dfrac{x^2}{1+x^2} - by \end{cases}$ $\boxed{\begin{array}{l} x, y > 0 \\ a, b > 0 \quad \text{parameters} \end{array}}$

Nullclines $\begin{cases} y = ax & \text{— } \dot{x} = 0 \\ y = \dfrac{1}{b}\dfrac{x^2}{1+x^2} & \text{— } \dot{y} = 0 \end{cases}$



[can check linear stability by computing the Jacobian]

- Note that a controls the slope of the line, so consider b fixed for simplicity.
  - We want to find when two roots coalese (for b fixed)
    - method 1: compute a tangential intersection, like earlier in the course
    - method 2: compute roots exactly!

↳ fixed points satisfy $ax = \frac{1}{b}\frac{x^2}{1+x^2} \Rightarrow x=0$ or $a = \frac{x}{b(1+x^2)}$

$x = 0 \Downarrow y=0$

$\Rightarrow ab(1+x^2) - x = 0$

$\Rightarrow x_{\pm} = \frac{1}{2ab}(1 \pm \sqrt{1-4a^2b^2})$

roots intersect when $\underline{2ab=1}$ ←

↳ so $a_c(b) = \frac{1}{2b}$.

From the phase portrait, we see that the fixed points move together along the unstable manifold of the saddle (which plays the role of the x-axis in our prototypical example).

⊛ In general, see center manifold theory ⊛

Transcritical and Pitchfork bifurcations: [bifurcation at $\mu = 0$] $\quad \mu < 0 \rightarrow \mu > 0$

- Transcritical: $\dot{x} = \mu x - x^2$, $\dot{y} = -y$ [Saddle/stable node exchange]
- Supercritical Pitchfork: $\dot{x} = \mu x - x^3$, $\dot{y} = -y$ [Stable node → saddle + 2 stable nodes]
- Subcritical Pitchfork: $\dot{x} = \mu x + x^3$, $\dot{y} = -y$. [stable node + 2 saddles → saddle].

Example [supercritical pitchfork]

$$\begin{cases} \dot{x} = \mu x + y + \sin x \\ \dot{y} = x - y \end{cases}$$

→ Show that a supercritical pitchfork bifurcation happens at the origin and find the critical parameter μ_c.

Note: invariance $x \mapsto -x$, $y \mapsto -y$: like in 1D, we have symmetry!

Fixed point at the origin: Jacobian $= \begin{pmatrix} \mu+1 & 1 \\ 1 & -1 \end{pmatrix}$ ∴ Trace $= \mu$, det $= -(\mu+2)$ $\Rightarrow \begin{cases} \mu < -2: \text{ stable node} \\ \mu > -2: \text{ saddle.} \end{cases}$

$\Rightarrow$ Pitchfork bifurcation at $\mu_c = -2$: Is it sub- or super-critical??

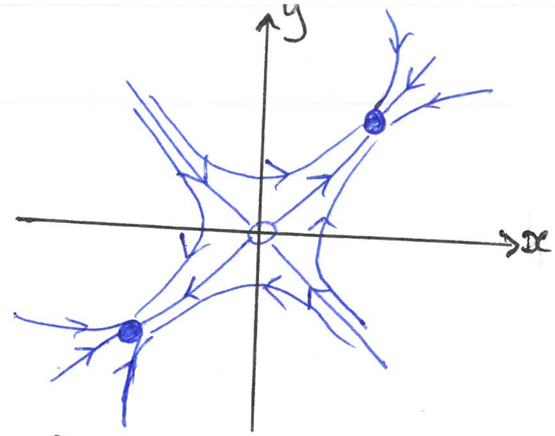
↳ need to find the other fixed points just after they emerge!

Note: $y_* = x_* \Rightarrow 0 = x_*\left[\mu + 1 + \frac{\sin x_*}{x_*}\right] \Rightarrow 0 = \underline{x_*\left[(\mu+2) - \frac{x_*^2}{6} + O(x_*^4)\right]}$

approximate normal form of a supercritical pitchfork.

↳ new fixed points are stable nodes for $\mu \gtrsim \mu_c = -2$.

Phase portrait for $\mu \gtrsim \mu_c = -2$.



NOTE THE SYMMETRY!

[For $\mu - \mu_c = O(1)$, the phase portrait could be very different!]

[Linear stability analysis $\Rightarrow$
stable manifold $\approx$ $y = -x$
unstable manifold $\approx$ $y = +x$]

NOTE: saddle-node, transcritical & pitchfork bifurcations all occur when a real eigenvalue passes through zero.

What happens when an instability instead occurs when a pair of complex conjugate eigenvalues cross the imaginary axis, i.e. & go from negative to positive real part?

Hopf bifurcation.

- supercritical; when a small-amplitude, sinusoidal, stable limit-cycle emerges for $\mu > \mu_c$, where $\begin{cases} \text{Re}[\lambda] < 0 & \text{for } \mu < \mu_c \\ \text{Re}[\lambda] > 0 & \text{for } \mu > \mu_c \end{cases}$.
- subcritical: for $\mu < \mu_c$, there exists an unstable limit cycle whose amplitude shrinks to zero at $\mu = \mu_c$.

Supercritical Hopf:

- a stable spiral $\longrightarrow$ unstable spiral, and a stable limit cycle emerges. [nearly elliptical!]

e.g. $\begin{cases} \dot{r} = \mu r - r^3 \\ \dot{\theta} = \omega + b r^2 \end{cases}$

- μ controls stability of the fixed point at the origin
- ω gives frequency of infinitesimal oscillations
- b relates frequency and amplitude for larger oscillations.

Let $x = r\cos\theta$, $y = r\sin\theta$ and linearise about the origin

$$\Rightarrow \begin{cases} \dot{x} \approx \mu x - \omega y \\ \dot{y} \approx \omega x + \mu y \end{cases} \Rightarrow \text{evalues of Jacobian are } \lambda_\pm = \mu \pm i\omega.$$

For $\begin{cases} \mu < 0: & \text{origin is a stable spiral} \\ \mu > 0: & \text{origin is an unstable spiral} - \end{cases}$ stable limit cycle forms with $r_* = \sqrt{\mu}$.

small-amplitude

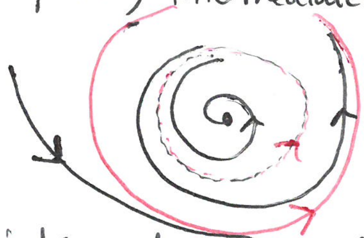
Generic rules of thumb. [bifurcation at $\mu = \mu_c$, eigenvalues of fixed point $\lambda(\mu)$]

1. For $\mu \gtrsim \mu_c$, the limit cycle amplitude grows like $\sim \sqrt{\mu - \mu_c}$.
2. Frequency $\omega = \underbrace{\text{Im}[\lambda(\mu_c)]}_{\text{from linear stability analysis.}} + O(\mu - \mu_c)$ $\therefore$ period $T = 2\pi/\omega$.
3. elliptical limit cycle

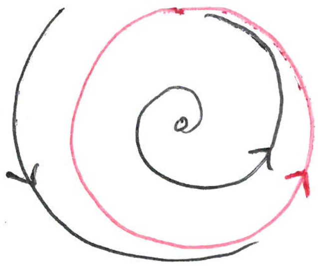
Subcritical Hopf: the system settles onto a distant attractor.

e.g. $\begin{cases} \dot{r} = \mu r + r^3 - r^5 \\ \dot{\theta} = \omega + br^2 \end{cases}$ ← cubic destabilizes.

$\mu<0$: origin stable spiral; intermediate unstable limit cycle; large stable limit cycle



$\mu>0$: origin unstable spiral; large stable limit cycle



NOTE: • For $0<\mu\ll 1$, the limit cycle amplitude is $O(1)$

- Like subcritical pitchfork bifurcations, this system exhibits hysteresis.
- At $\mu = -\frac{1}{4}$, the stable and unstable limit cycles collide and anihilate.

Degenerate Hopf: When a nonconservative system becomes conservative only at the bifurcation point.

e.g. $\ddot{x} + \mu\dot{x} + \sin x = 0$. $\begin{cases} \bullet\ \mu<0: \text{ origin unstable spiral} \\ \bullet\ \mu=0: \text{ origin nonlinear center [NOT limit cycles!!]} \\ \bullet\ \mu>0: \text{ origin stable spiral} \end{cases}$

Example. $\begin{cases} \dot{x} = \mu x - y + xy^2 \\ \dot{y} = x + \mu y + y^3 \end{cases}$ → use that $r\dot{r} = x\dot{x} + y\dot{y} \Rightarrow \dot{r} = \mu r + ry^2 \geq \mu r$ [repels for $\mu>0$]

- Consider the origin: Jacobian $= \begin{pmatrix} \mu & -1 \\ 1 & \mu \end{pmatrix} \Rightarrow \lambda = \mu \pm i$ ∴ stable → unstable spiral at $\mu=0$.
- As r grows without bound for $\mu>0$, the bifurcation is not supercritical!
- At $\mu=0$, $\dot{r}>0$ ∴ origin not a nonlinear center ⇒ not degenerate!

⇒ must be subcritical!

Example. $a,b>0$ $\begin{cases} \dot{x} = a - x - 4xy(1+x^2)^{-1} \\ \dot{y} = bx(1 - y(1+x^2)^{-1}) \end{cases}$

Aim: show that there is a closed orbit in the positive quadrant, and specify (a,b)

- Want to create a trapping region: $\dot{x}=0 \Leftrightarrow y = \dfrac{(a-x)(1+x^2)}{4x}$

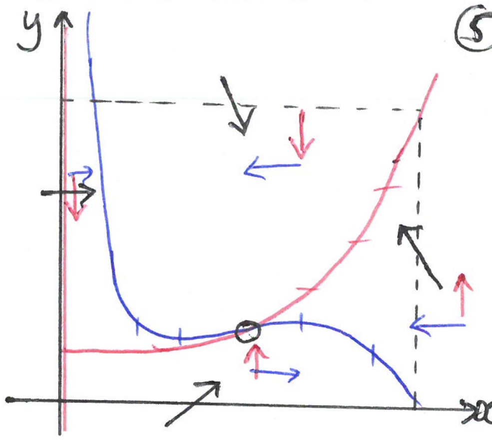
$\dot{y}=0 \Leftrightarrow x=0$ or $y = 1+x^2$.

Note: The ~~b~~ square gives a trapping region.

-Is the fixed point an attractor of repeller?

Linear stability analysis gives repeller if $\boxed{b < b_c = \frac{3}{5}a - \frac{25}{a}}$.

In this case, we consider a punctured box and apply the Poincaré-Bendixson Theorem to deduce that a closed orbit exists in the punctured box!



Numerical simulations ⇒ limit cycle is stable for $b < b_c(a)$ ⇒ supercritical Hopf bifurcation.

Global bifurcation of cycles.

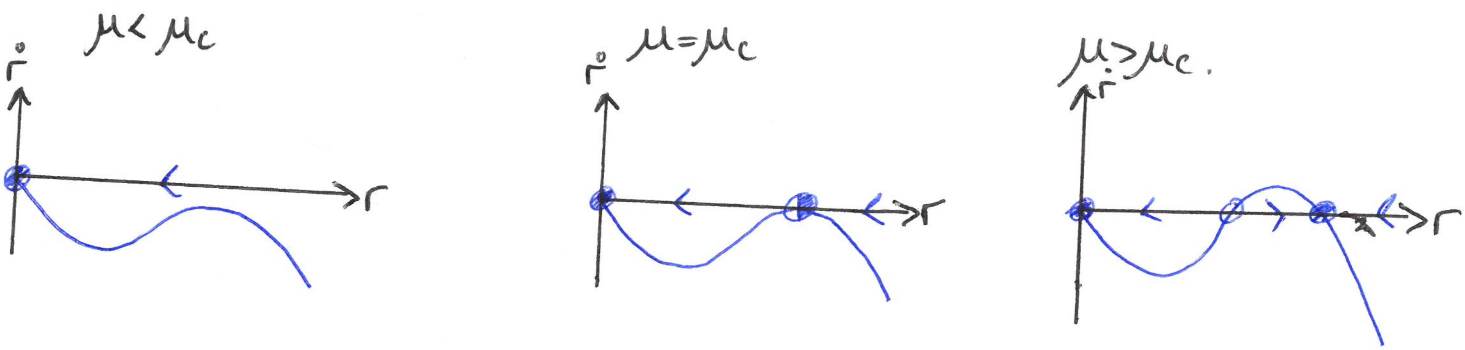
↳ creation/destruction of limit cycles via global changes in the phase plane.

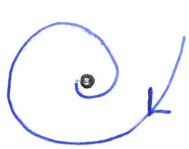
① Saddle-node bifurcation of cycles.

- when two limit cycles collide and annihilate each other

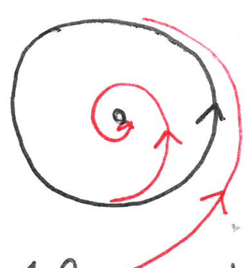
e.g. $\begin{cases} \dot{r} = \mu r + r^3 - r^5 \quad ⊛ \\ \dot{\theta} = \omega + b r^2 \end{cases}$ (previously showed that this system exhibits a subcritical Hopf bifurcation at $\mu = 0$)

⊛ ⇒ saddle-node bifurcation at $\mu = \mu_c = -\frac{1}{4}$ (when considered as a 1D system).

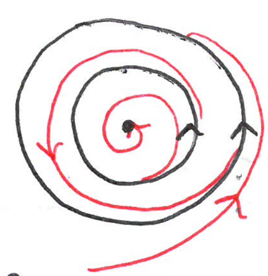




• 1 fixed point



• 1 fixed point, 1 half-stable limit cycle



• 1 fixed point, one stable limit cycle one unstable limit cycle.

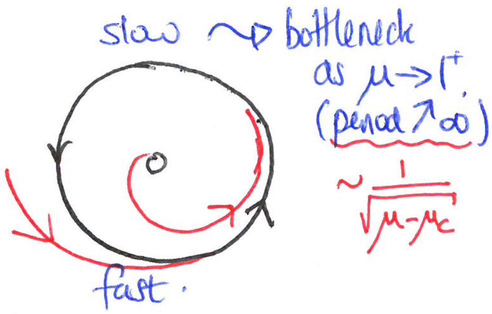
⊛ At the birth of the limit cycle, the amplitude is $O(1)$ [Hopf, amplitude as $O((\mu-\mu_c)^{1/2})$]

② Infinite Period bifurcation.

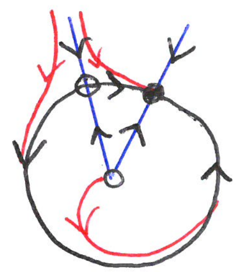
$$\begin{cases} \dot{r} = r(1-r^2) \\ \dot{\theta} = \mu - \sin\theta . \end{cases} \qquad \mu \geq 0$$

- $\mu > 1$: tangential motion counterclockwise everywhere $\begin{cases} \text{• rotation slowest at } \theta = \pi/2 \\ \text{• rotation fastest at } \theta = \frac{3}{2}\pi . \end{cases}$
- $\mu < 1$: emergence of two invariant rays.

$\mu > 1$



$\mu < 1$.



- saddle and a node appear along $r = 1$.

③ Homoclinic bifurcation.

See handout for figure!

- a homoclinic trajectory is created at the bifurcation point
- For $\mu > \mu_c$, the loop is destroyed.

Generic scaling laws:	Amplitude of stable limit cycle	Period of cycle.
Supercritical Hopf	$O(\mu^{1/2})$	$O(1)$
Saddle-node bifurcation of cycles	$O(1)$	$O(1)$
Infinite-period	$O(1)$	$O(\mu^{-1/2})$
Homoclinic.	$O(1)$	$O(\log \mu)$.

Coupled oscillators.

$$\begin{cases} \dot\theta_1 = f_1(\theta_1, \theta_2) \\ \dot\theta_2 = f_2(\theta_1, \theta_2) \end{cases}$$

← motion on the torus [f_1 and f_2 are periodic in both arguments]

Example

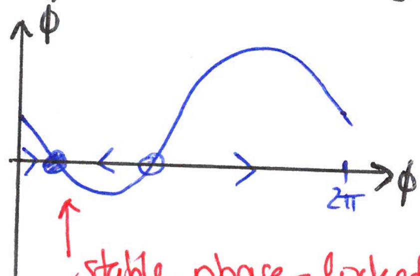
$$\begin{cases} \dot\theta_1 = \omega_1 + K_1 \sin(\theta_2 - \theta_1) \\ \dot\theta_2 = \omega_2 + K_2 \sin(\theta_1 - \theta_2) \end{cases}$$

- $\omega_1, \omega_2 > 0 \rightarrow$ natural frequencies
- $K_1, K_2 \geq 0 \rightarrow$ coupling constants

Phase difference $\phi = \theta_1 - \theta_2 \Rightarrow \dot\phi = (\omega_1 - \omega_2) - (K_1 + K_2)\sin\phi$ [non-uniform oscillator from earlier in course]

- Fixed points: $\begin{cases} \circ\ 2 \text{ if } |\omega_1 - \omega_2| < K_1 + K_2 \\ \circ\ 0 \text{ if } |\omega_1 - \omega_2| > K_1 + K_2. \end{cases}$ ← saddle-node bifurcation.

- When they exist, the fixed points satisfy $\sin\phi_* = \dfrac{(\omega_1 - \omega_2)}{(K_1 + K_2)}$



stable, phase-locked solution!

In this case, $\dot\theta_1 = \dot\theta_2 \ \forall t \Rightarrow \dot\theta_1 = \dot\theta_2 = \omega_2 + K_2 \sin\phi_*$

$$= \omega_2 + K_2 \frac{(\omega_1 - \omega_2)}{K_1 + K_2}$$

$$\Rightarrow \omega_* = \frac{K_1\omega_2 + K_2\omega_1}{K_1 + K_2}.$$

weighted average of the two frequencies

Note: the compromise frequency is not halfway between ω_1 and ω_2

In fact $\left| \dfrac{\omega_1 - \omega_*}{\omega_2 - \omega_*} \right| = \dfrac{K_1}{K_2}$.