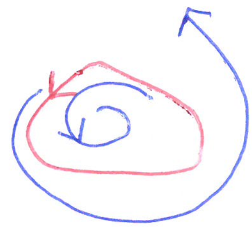
Limit cycles.

- an isolated closed trajectory.

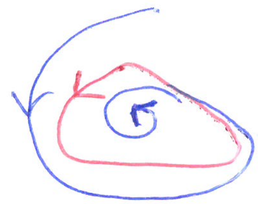
  ↑ means that neighboring trajectories cannot also be closed, so spiral towards/away from the limit cycle. Non-isolated closed trajectories are linear/nonlinear centers, for example

Stable limit cycle

Unstable limit cycle



half-stable limit cycle (rare!)



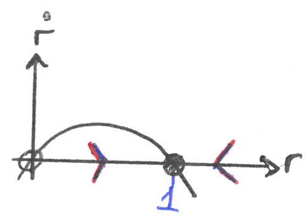
- appear in systems with self-sustained oscillations [nonlinear phenomena]

  e.g. - beating of a heart, self-excitations of bridges/airplane wings.

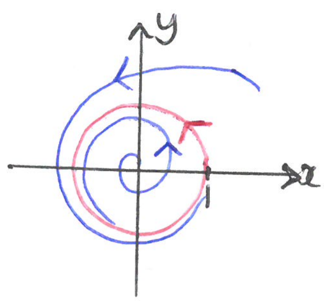
  ↳ oscillation has preferred period, waveform and amplitude. [a property of the system, not the initial conditions]

Example:

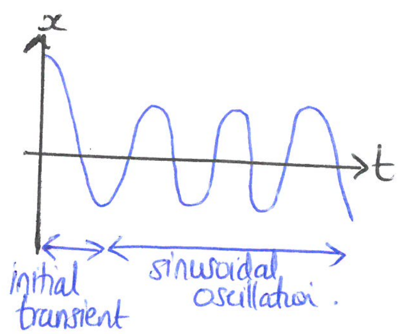
$\dot{r} = r(1-r^2)$, $\dot{\theta} = 1$, $r \geq 0$.



- stable limit cycle at $r_* = 1$,
- $r_* = 0$ unstable fixed point.



[as $\dot{\theta} = 1 > 0$]



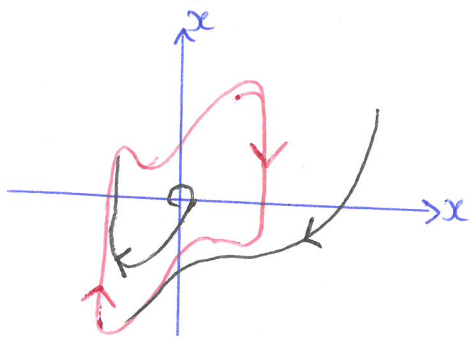
Example (van der Pol oscillator). [appears in circuits/seismology]

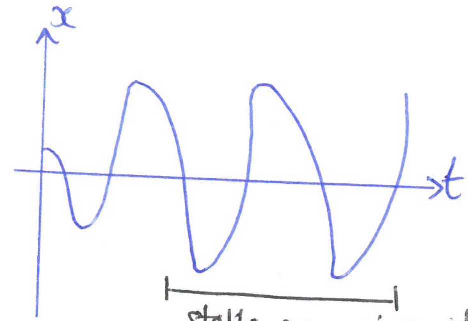
$$\ddot{x} + \underbrace{\mu(x^2-1)}_{\text{nonlinear damping coefficient}}\dot{x} + x = 0 \quad , \quad \text{parameter } \mu \geq 0.$$

nonlinear damping coefficient ⟶ large-amplitude oscillations decay (i.e. for $|x|>1$)

⟶ pumps up small-amplitude oscillations (i.e. for $|x|<1$).

Numerical simulation gives the limit cycle (stable)





Ruling out closed orbits.

- already used Index Theory to rule out the existence of closed orbits in given systems. There are three more common methods.

(1a) Gradient Systems.

Find a potential function $V(\underline{x})$ so that $\dot{\underline{x}} = -\nabla V(\underline{x})$.

need V continuously differentiable, single-valued, scalar.

- Closed orbits are impossible in gradient systems. Why?

Suppose there is a closed orbit with period $T>0$.

Then ΔV over one period is zero as V is single-valued. $[\Delta V = 0]$

Alternatively, $\Delta V = \int_0^T \frac{dV}{dt}\,dt = \int_0^T \dot{\underline{x}} \cdot \nabla V(\underline{x}(t))\,dt = -\int_0^T |\dot{\underline{x}}|^2\,dt < 0$.

contradiction !!

(*) Problem: need to construct $V(\underline{x})$, which may not always exist (*)

Example. $\begin{cases} \dot{x} = \sin y \\ \dot{y} = x\cos y \end{cases} \Leftarrow V(x,y) = -x\sin y \;(+\text{ constant}).$

(1b) Energy-like function

Like with the potential function, we aim to consider the change in energy to derive a contradiction.

Example.

Show that $\ddot{x} + \dot{x}^3 + x = 0$ has no periodic solutions.

- Choose the energy function $E(x,\dot{x}) = \frac{1}{2}(x^2 + \dot{x}^2)$ [not necessarily related to the ODE]
- Note that $\Delta E = 0$ on a closed orbit
- But $\frac{dE}{dt} = \dot{x}(x + \ddot{x}) = -\dot{x}^4 \leq 0$ [using ODE] So $\Delta E = \int_0^T \frac{dE}{dt}\,dt = -\int_0^T \dot{x}^4\,dt \leq 0$

one period

equality iff $\dot{x} \equiv 0$ [not an orbit]

Hence, $\Delta E < 0$, leading to a contradiction $\#$.

2 Lyapunov functions.

– Aim: construct an energy-like function that decreases along trajectories.

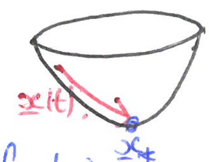
⊕ If a Lyapunov function exists then closed orbits are forbidden (like in previous example). ⊕

Consider $\dot{\underline{x}} = \underline{f}(\underline{x})$ with a fixed point $\underline{x}_*$.

A Lyapunov function $V(\underline{x})$ is continuously differentiable and real, satisfying

$$\begin{cases} 1. & V(\underline{x}) > \quad \forall \underline{x} \neq \underline{x}_* \quad \text{and } V(\underline{x}_*) = 0 \quad [\text{"positive definite"}] \\ 2. & \dfrac{dV}{dt} < 0 \quad \forall \underline{x} \neq \underline{x}_* \quad [\text{all trajectories flow downhill to } \underline{x}_*] \end{cases}$$

In such cases, $\underline{x}_*$ is globally asymptotically stable.



– sum of squares is a good way of constructing Lyapunov functions.

Example. $\begin{cases} \dot{x} = -x + 4y \\ \dot{y} = -x - y^3 \end{cases}$ ← show that this system has no closed orbits.
← only fixed point at $(x_*, y_*) = (0, 0)$.

Consider $V(x, y) = x^2 + ay^2$, where a is to be chosen.

$$\Rightarrow \frac{dV}{dt} = 2x\dot{x} + 2ay\dot{y} = 2x[-x + 4y] + 2ay[-x - y^3] = -2x^2 + \underbrace{(8 - 2a)xy}_{\text{want to set this term to zero } \Rightarrow \, a = 4} - 2ay^4.$$

So $\begin{cases} V = x^2 + 4y^2 & \Rightarrow V > 0 \quad \text{not at fixed point} \\ \dot{V} = -2(x^2 + 4y^4) & \Rightarrow \dot{V} < 0 \quad \text{not at fixed point.} \end{cases}$

Hence, V is a Lyapunov function, so no closed orbits exist.

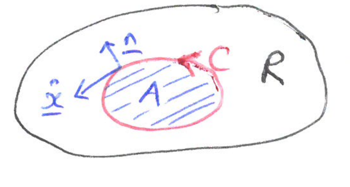
3 Dulac's Criterion.

Let $\dot{\underline{x}} = f(\underline{x})$ be a continuously differentiable vector field on a simply-connected subset R of $\mathbb{R}^2$.
If there exists a continuously differentiable, real-valued function $g(\underline{x})$ such that

$$\underline{\nabla} \cdot (g \dot{\underline{x}}) \text{ has one sign throughout } R,$$

then there are no closed orbits lying entirely in R.

Proof: Suppose a closed orbit C lies entirely within R.
Let A be the region bounded by C



Green's Theorem $\Rightarrow \underbrace{\iint_A \underline{\nabla} \cdot (g \dot{\underline{x}}) \, dA}_{\neq 0 \text{ by assumption}} = \underbrace{\oint_C g \dot{\underline{x}} \cdot \underline{n} \, ds}_{= 0 \text{ as } \dot{\underline{x}} \cdot \underline{n} = 0 \text{ along } C.}$ $\#$ $\square$

↳ candidate functions: $g = 1$, $g = x^{-a} y^{-b}$, e^{ax}, e^{by}.

Example.

Show that $\begin{cases} \dot{x} = x(2-x-y) \\ \dot{y} = y(4x-x^2-3) \end{cases}$ has no closed orbits in the quadrant $x, y > 0$.

↳ Pick $g(x,y) = \frac{1}{xy} \Rightarrow \nabla \cdot (g\dot{\underline{x}}) = \frac{\partial}{\partial x}(g\dot{x}) + \frac{\partial}{\partial y}(g\dot{y})$

$$= \frac{\partial}{\partial x}\left[\frac{2-x-y}{y}\right] + \frac{\partial}{\partial y}\left[\frac{4x-x^2-3}{x}\right]$$

$$= -\frac{1}{y} + 0 < 0 \quad \forall y > 0.$$

As the quadrant $x, y > 0$ is simply connected, and $\underline{f}$ and g satisfy the smoothness conditions, Dulac's condition implies that no closed orbits exist in the positive quadrant.

Poincaré – Bendixson Theorem – proving that closed orbits exist!

Suppose that

(i) R is a closed, bounded subset of $\mathbb{R}^2$

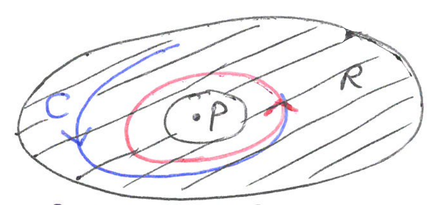
(ii) $\dot{\underline{x}} = \underline{f}(\underline{x})$ is a continuously differentiable vector field on an open set contained in R

(iii) R does not contain any fixed points

(iv) There is a trajectory C that is "confined" in R $\forall t > 0$

Then either C is a closed orbit or it approaches a closed orbit as $t \to \infty$.

Either case, R contains a closed orbit!



- P fixed point [a closed orbit contains one!]
- R is a subset of $\mathbb{R}^2$
- C is a trajectory
- — limit cycle.

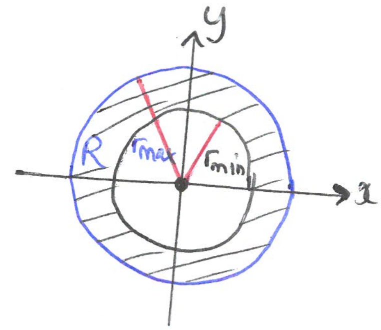
⊛ cannot have chaos in the phase plane ⊛.

– Application: need to construct a trapping region R to satisfy (iv)

↳ all vectors point into R along the boundary of R.

Example. $\begin{cases} \dot{r} = r(1-r^2) + \mu r\cos\theta \\ \dot{\theta} = 1. \end{cases}$ → $1 > \mu > 0$ is a parameter

(Note: $r=0$ is a fixed point!)



Technique: find $\begin{cases} r_{min} \text{ s.t. } \dot{r} > 0 \text{ for } r = r_{min} \ \forall\theta \\ r_{max} \text{ s.t. } \dot{r} < 0 \text{ for } r = r_{max}, \ \forall\theta \end{cases}$

Note: No fixed points for $r_{min} < r < r_{max}$ as $\dot{\theta} = 1 \neq 0$.

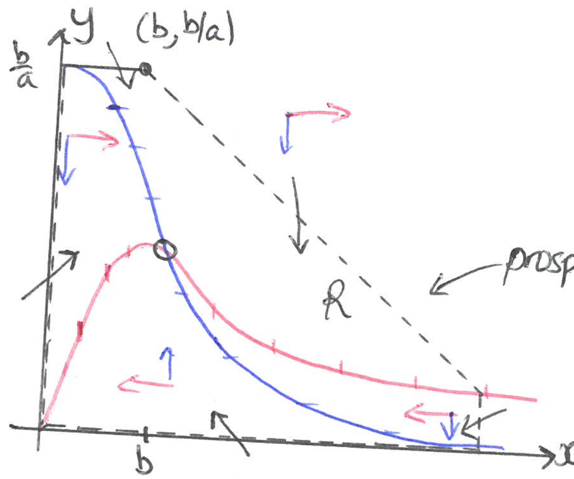
- r_{min}: Want $r(1-r^2) + \mu r\cos\theta > 0 \ \forall\theta \Leftarrow 1 - r^2 - \mu > 0$

  Hence, choose $0 < r_{min} < \sqrt{1-\mu}$

- r_{max}: Want $r(1-r^2) + \mu r\cos\theta < 0 \ \forall\theta \Leftarrow 1 - r^2 + \mu < 0$ ∴ want $r_{max} > \sqrt{1+\mu}$.

Hence, a closed orbit exists and lies in the annulus $r_{min} < r < r_{max}$.

Example. $\begin{cases} \dot{x} = -x + ay + x^2 y \\ \dot{y} = b - ay - x^2 y. \end{cases}$ parameters $a, b > 0$. Variables $x, y \geq 0$



Nullclines

$\dot{x} = 0 \Leftrightarrow y = \frac{x}{a + x^2}$

$\dot{y} = 0 \Leftrightarrow y = \frac{b}{a + x^2}$

We want to show that our trajectories crossing the trapping region are inward.

- The diagonal line is hardest to justify!

Note: If $x \gg 1$ and $y \gg 1$, then $\dot{x} \approx x^2 y$ and $\dot{y} \approx -x^2 y$ $\therefore \frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} \approx -1$ along trajectories

parallel to diagonal line

we want $\frac{dy}{dx} < -1$ on diagonal boundary, ie. $\dot{y} < -\dot{x} \Leftrightarrow \dot{x} < -\dot{y}$.
$\Leftrightarrow \dot{x} + \dot{y} < 0$

But $\dot{x} + \dot{y} = b - x < 0 \Leftrightarrow \underline{x > b}$. ← So trajectories are inward pointing on diagonal line
$\Rightarrow$ we have a trapping region!

→ We need to do more work to show that we have a limit cycle, as the region contains a fixed point! If the fixed point is a repeller then we can remove a small region of R about the fixed point and the Poincaré-Bendixson Theorem applies.

When is the fixed point a repeller?

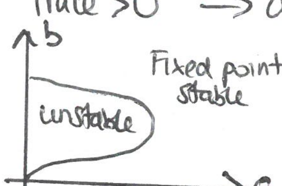
Jacobian $J(x,y) = \begin{pmatrix} -1 + 2xy & a + x^2 \\ -2xy & -(a + x^2) \end{pmatrix}$.

Fixed point $\begin{cases} 0 = -x + ay + x^2 y \\ 0 = b - ay - x^2 y \end{cases} \Rightarrow (x_*, y_*) = \left(b, \frac{b}{a + b^2}\right)$

So $J(x_*, y_*)$ satisfies .... $\det = a + b^2 > 0$

$\text{Trace} = -\frac{1}{a + b^2}\left[b^4 + (2a-1)b^2 + (a + a^2)\right]$.

For instability, we want Trace > 0 → dividing curve is $b^2 = \frac{1}{2}\left\{1 - 2a \pm \sqrt{1 - 8a}\right\}$.



→ When the fixed point is unstable, a (stable) limit cycle exists in R.
← from numerical solutions.

Liénard Systems.

nonlinear damping — restoring force

Many oscillating circuits may be modelled as $\ddot{x} + \underbrace{f(x)\dot{x}} + \underbrace{g(x)} = 0$ [Liénard's equation]

e.g. van der Pol: $f(x) = \mu(x^2-1)$, $g(x) = x$

System $\begin{cases} \dot{x} = y \\ \dot{y} = -g(x) - f(x)y \end{cases}$

Liénard's Theorem.

Suppose that $f(x)$ and $g(x)$ satisfy the following properties:

(1) $f(x)$ and $g(x)$ are continuously differentiable for all x

(2) $g(-x) = -g(x)$ $\forall x$ [ie. g is an odd function]

(3) $g(x) > 0$ $\forall x > 0$

(4) $f(-x) = f(x)$ $\forall x$ [ie f is an even function]

(5) The odd function $F(x) = \int_0^x f(u)\,du$ has (i) exactly one positive zero at $x = a$

(ii) is negative for $0 < x < a$

(iii) is positive and non-decreasing for $x > a$

(iv) $F(x) \to \infty$ as $x \to \infty$.

Then the system has a unique, stable limit cycle surrounding the origin in the phase plane.

Interpretation: – $g(x)$ acts like a spring, reducing any displacement

– $f(x)$ means that small oscillations are pumped up and large oscillations are damped down.

$\rightsquigarrow$ self-sustained oscillations!

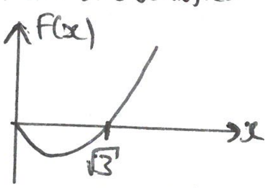
Example: van der Pol equation

$f(x) = \mu(x^2-1)$, $g(x) = x$ $(\mu > 0)$

→ conditions (1)-(4) of Liénard's Theorem are satisfied

→ $F(x) = \int_0^x f(u)\,du = \frac{1}{3}\mu x(x^2-3)$

↳ satisfies 5(i)-(iv) with $a = \sqrt{3}$.



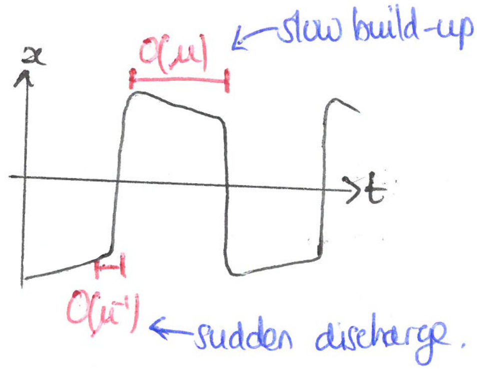
∴ Theorem conditions are satisfied and the van der Pol equation has a unique, stable limit cycle.

Relaxation Oscillators.

- given a closed orbit exists, what can we say about its shape and period?

Example: van der Pol in the strongly nonlinear limit.

$$\ddot{x} + \mu(x^2-1)\dot{x} + x = 0 \quad , \quad \mu \gg 1.$$



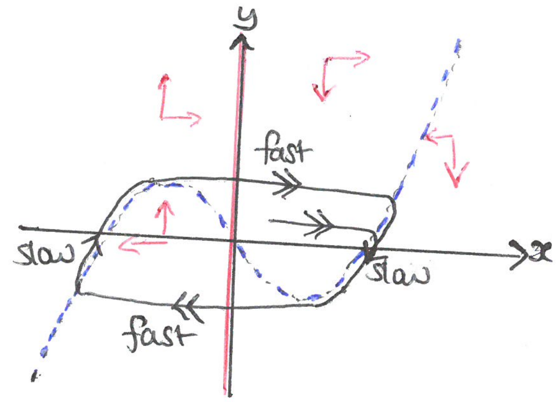
It's convenient to consider the phase portrait in a new pair of variables

- note: $\frac{d}{dt}\Big[\underbrace{\dot{x} + \mu\big[\tfrac{1}{3}x^3 - x\big]}_{w}\Big] = \ddot{x} + \mu\dot{x}(x^2-1)$
$= -x$

So $\begin{cases} \dot{x} = w - \mu F(x) \\ \dot{w} = -x \end{cases}$, where $F(x) = \frac{1}{3}x^3 - x$.

Then we define $y = \frac{w}{\mu}$

$\Rightarrow \begin{cases} \dot{x} = \mu\big[y - F(x)\big] \\ \dot{y} = -\frac{1}{\mu}x \end{cases}$ — nullcline $y = F(x)$ — nullcline $\dot{x} = 0$



- Suppose that the initial condition is ~~not~~ s.t. $y - F(x) = O(1) \Rightarrow \begin{cases} \dot{x} = O(\mu) \gg 1 \\ \dot{y} = O(1/\mu) \ll 1 \end{cases}$

$\therefore$ trajectory moves nearly horizontally in the phase plane

- If $y - F(x) \sim O(\mu^{-2})$ then $\dot{x}$ and $\dot{y}$ become comparable [size $O(\mu^{-1})$]
$\therefore$ trajectory moves slowly near the nullcline until it reaches the knee and can jump sideways again.

$\Rightarrow$ Two timescales: $\begin{cases} \text{Fast} & O(\mu^{-1}) \\ \text{Slow} & O(\mu) \end{cases}$

What is the period of the oscillations for $\mu \gg 1$?

- As the slow branches dominate the period, the period $T \approx 2 \times$ time spent on slow branches.

- on the slow branches, $y \approx F(x) \Rightarrow \frac{dy}{dt} \approx F'(x)\frac{dx}{dt} = (x^2-1)\frac{dx}{dt}$.

But $\dot{y} = -\frac{x}{\mu} \Rightarrow \dot{x} \approx \dfrac{-x}{\mu(x^2-1)}$ ⊕

At what values of x does the slow stage start/end?

Note: $\frac{dF}{dx} = x^2-1$ $\therefore$ $F(x)$ has minima at $x = \pm x_B$, where $x_B = 1$.

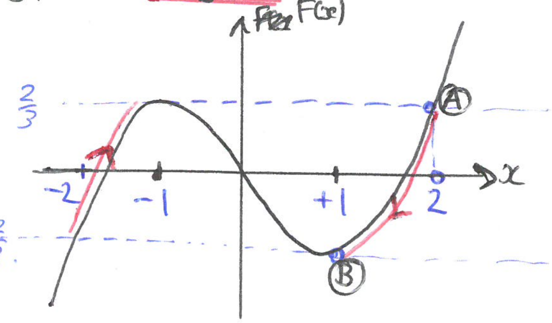
$F(-1) = -\frac{1}{3}+1 = \frac{2}{3}$

$F(x) = \frac{2}{3} \Rightarrow \frac{1}{3}x^3 - x - \frac{2}{3} = 0 \Rightarrow x^3-3x-2=0$

$\Rightarrow (x+1)(x^2-x-2) = 0$

$\Rightarrow (x+1)(x-2)(x+1) = 0$

$\therefore$ $\therefore x_A = +2$



So $T \approx 2\int_{t_A}^{t_B} dt \approx 2\int_{x_A}^{x_B} \underbrace{\frac{dt}{dx}}_{\text{use } (*)} dx \approx 2\int_2^1 \frac{-\mu(x^2-1)}{x}\,dx = 2\mu\int_1^2 x - \frac{1}{x}\,dx$

$= 2\mu\left[\frac{1}{2}x^2 - \log x\right]_1^2$

$\Rightarrow T \approx \mu[3-2\log 2]$ for $\mu \gg 1$.

Weakly nonlinear oscillators

We consider equations of the form: $\ddot{x} + x + \underbrace{\varepsilon h(x,\dot{x})}_{\text{small perturbation to the harmonic oscillator.}} = 0$, $0<\varepsilon\ll 1$, h smooth.

e.g. van der Pol: $\ddot{x} + x + \varepsilon(x^2-1)\dot{x} = 0$

Duffing : $\ddot{x} + x + \varepsilon x^3 = 0$

- from simulations, we find the emergence of a stable limit cycle – what is the shape/radius/period?

nearly circular.

Asymptotic analysis – the failure of regular perturbation theory.

Power series in ε: $x(t,\varepsilon) = \underbrace{x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t)}_{\text{to be determined.}} + \dots$ (*)

Aim: find an asymptotic expansion to the solution of $\ddot{x} + 2\varepsilon\dot{x} + x = 0$, $\begin{cases} x(0) = 0 \\ \dot{x}(0) = 1. \end{cases}$

Substitute in (*) $\Rightarrow [\ddot{x}_0 + x_0] + \varepsilon[\ddot{x}_1 + 2\dot{x}_0 + x_1] + O(\varepsilon^2) = 0$ $\forall \varepsilon > 0$.

means that we consider each power of ε separately.

ODE $\Rightarrow \begin{cases} O(1): & \ddot{x}_0 + x_0 = 0 \\ O(\varepsilon): & \ddot{x}_1 + x_1 = -2\dot{x}_0 \\ O(\varepsilon^2) & \leftarrow \text{small correction that we hope to ignore.} \end{cases}$

Initial conditions

$O(1)$: $x_0(0) = 0$, $\dot{x}_0(0) = 1$

$O(\varepsilon)$: $x_1(0) = 0$, $\dot{x}_1(0) = 0$.

$O(1) \Rightarrow x_0(t) = \sin t$

Substitute into $O(\varepsilon) \Rightarrow \begin{cases} \ddot{x}_1 + x_1 = \underline{-2\cos t} & \leftarrow \text{resonant forcing!} \\ x_1(0) = 0, \ \dot{x}_1(0) = 0 \end{cases}$

$\Rightarrow x_1(t) = \underline{-t\sin t}$.

↑ secular term ← grows without bound as $t \to \infty$

So $x(t,\varepsilon) = \sin t \underbrace{- \varepsilon t \sin t}_{} + O(\varepsilon^2)$

we need $\varepsilon t \ll 1$ for the correction term to be small compared to the $O(1)$ solution

∴ expansion quickly breaks down as soon as $t = O(\varepsilon^{-1})$. ‼

How do we get an approximate solution valid for all t?

Note: $\ddot{x} + 2\varepsilon\dot{x} + x = 0$ has two timescales (i) $O(1)$ oscillations (ii) $O(1/\varepsilon)$ decay.] Two Timescales!!

Method of Multiple Scales (Two-Timing)

- let $\tau = t$ be the fast timescale
  $T = \varepsilon t$ be the slow timescale. } we treat τ and T as independent variables!!!

Expand $x(t,\varepsilon) = x_0(\tau, T) + \varepsilon x_1(\tau, T) + O(\varepsilon^2)$. (*)

Note: $\frac{dx}{dt} = \frac{\partial x}{\partial \tau}\underbrace{\frac{\partial \tau}{dt}}_{=1} + \frac{\partial x}{\partial T}\underbrace{\frac{\partial T}{\partial t}}_{\varepsilon} = \partial_\tau x + \varepsilon \partial_T x$.

Substitute in (*) $\Rightarrow$ (†) $\begin{cases} \dot{x} = \partial_\tau x_0 + \varepsilon(\partial_T x_0 + \partial_\tau x_1) + O(\varepsilon^2) \\ \ddot{x} = \partial_{\tau\tau} x_0 + \varepsilon(\partial_{\tau\tau} x_1 + 2\partial_{T\tau} x_0) + O(\varepsilon^2). \end{cases}$

Consider $\ddot{x} + 2\varepsilon\dot{x} + x = 0, \quad x(0) = 0, \ \dot{x}(0) = 1$

Substitute in (†) $\Rightarrow \begin{cases} \partial_{\tau\tau} x_0 + \varepsilon(\partial_{\tau\tau} x_1 + 2\partial_{T\tau} x_0) + 2\varepsilon\partial_\tau x_0 + x_0 + \varepsilon x_1 = O(\varepsilon^2). \\ 0 = x_0(0,0) + \varepsilon x_1(0,0) + O(\varepsilon^2) \\ 1 = \partial_\tau x_0(0,0) + \varepsilon[\partial_T x_0(0,0) + \partial_\tau x_1(0,0)] + O(\varepsilon^2). \end{cases}$

Holds $\forall \varepsilon > 0$

Powers of $\varepsilon \Rightarrow O(1)$: $\begin{cases} \partial_{\tau\tau} x_0 + x_0 = 0 \\ x_0(0,0) = 0 \\ \partial_\tau x_0(0,0) = 1 \end{cases}$

$$O(\varepsilon): \begin{cases} \partial_{\tau\tau} x_1 + x_1 = -2\left[\partial_{\tau T} x_0 + \partial_\tau x_0\right] \\ x_1(0,0) = 0 \\ \partial_\tau x_1(0,0) = \partial_\tau x_0(0,0). \end{cases}$$

Solve for $O(1)$ first: Note, we have a PDE, not an ODE!!

$$\Rightarrow x_0(\tau, T) = A(T)\sin\tau + B(T)\cos\tau .$$

$$\text{ICs} \Rightarrow \begin{cases} 0 = B(0) \\ 1 = A(0). \end{cases}$$

⊛ Need to find functions A and B ⊛

Solve for $O(\varepsilon)$: $\partial_{\tau\tau} x_1 + x_1 = -2\left(A'(T) + A(T)\right)\cos\tau + 2\left(B'(T) + B(T)\right)\sin\tau .$

secular terms! :(

This is where the flexibility of two time-scales helps!
We can choose $A(T)$ and $B(T)$ so that the coefficients of the secular terms vanish, ie.

$$\begin{cases} A'(T) + A(T) = 0 \\ B'(T) + B(T) = 0 . \end{cases}$$

We now solve for A and B :)

$$\text{So} \begin{cases} A(T) = e^{-T} A(0) \\ B(T) = e^{-T} B(0) \end{cases} \longrightarrow \text{but} \begin{cases} A(0) = 1 \\ B(0) = 0 \end{cases} \Rightarrow \begin{cases} A(T) = e^{-T} \\ B(T) = 0 \quad \forall T. \end{cases}$$

Hence $x_0(\tau, T) = e^{-T}\sin\tau .$

The approximate solution to the full problem is: $x(t) \approx e^{-\varepsilon t}\sin(t) + O(\varepsilon).$

Note: • We could also solve for x_1, which would give an improved approximation.
• This equation remains valid for large t.

Example: radius and period of the van der Pol oscillator.

$$\ddot{x} + x + \varepsilon(x^2-1)\dot{x} = 0$$

Expansion: $x(t) = x_0(\tau, T) + \varepsilon x_1(\tau, T)$

Chain rule for derivatives $\Rightarrow \begin{cases} O(1): & \partial_{\tau\tau} x_0 + x_0 = 0 \\ O(\varepsilon): & \partial_{\tau\tau} x_1 + x_1 = -2\partial_{\tau T} x_0 - (x_0^2 - 1)\partial_\tau x_0 \end{cases}$

$O(1) \Rightarrow x_0(\tau, T) = A(T)e^{i\tau} + \text{c.c.}$

want x_0 real

$= Ae^{i\tau} + \bar{A}e^{-i\tau}$

Write $A(T) = r(T)e^{i\phi(T)}$

$\Rightarrow x_0 = r(T)\left[e^{i(\tau+\phi(T))} + \text{c.c.}\right]$

$= 2r(T)\cos(\tau + \phi(T))$

$O(\varepsilon) \Rightarrow \partial_{\tau\tau} x_1 + x_1 = -2\left[iA'(T)e^{i\tau} + \text{c.c.}\right] - \left[A^2 e^{2i\tau} + \bar{A}^2 e^{-2i\tau} + (2|A|^2 - 1)\right] \times \left[iAe^{i\tau} - i\bar{A}e^{-i\tau}\right]$

$= \underbrace{\text{secular terms}}$ $+$ non-secular terms.

$\left\{-2iA'(T) - iA(2|A|^2 - 1) + iA^2\bar{A}\right\}e^{i\tau}$ $+ \text{c.c.}$

We want the coefficient of $e^{i\tau}$ to be zero!!

Hence, we want $A(T)$ to satisfy: $-2\cancel{i}A'(T) - \cancel{i}A(2|A|^2 - 1) + \cancel{i}|A|^2 A = 0$

$\Rightarrow 2A'(T) = A - 2A|A|^2 + A|A|^2$

$\Rightarrow \boxed{2A'(T) = A - A|A|^2}.$

Write $A(T) = r(T)e^{i\phi(T)}$

$\Rightarrow 2\left[r'(T) + ir(T)\phi'(T)\right]\cancel{e^{i\phi(T)}} = r(T)\cancel{e^{i\phi(T)}} - r^3\cancel{e^{i\phi(T)}}$

Real part $\Rightarrow 2r'(T) = r(T) - r^3(T)$

Imag. part $\Rightarrow r(T)\phi'(T) = 0$

Hence, steady state $r = 1$, $\phi =$ constant

So $x_0 \to 2\cos(t + \phi_0)$ as $t \to \infty$

$\begin{cases} \text{Frequency} = 1 + O(\varepsilon^2). \\ \text{Amplitude} = 2 + O(\varepsilon) \end{cases}$

What about the initial value problem: $x(0)=1$, $\dot{x}(0)=0$?

$\Rightarrow x_0(0,0)=1$, $\partial_\tau x_0(0,0)=0$

But $x_0(\tau;T) = Ae^{i\tau} + \bar{A}e^{-i\tau}$

$$\Rightarrow \begin{cases} 1 = A(0) + \bar{A}(0) \\ 0 = iA(0) - i\bar{A}(0). \end{cases} \quad \Rightarrow 1 = 2A(0) \Rightarrow \quad A(0) = \tfrac{1}{2}$$

$\Rightarrow A(0)$ real

But $A(T) = r(T)\,e^{i\phi(T)}$ $\Rightarrow r(0) = \tfrac{1}{2}$, $\phi(0) = 0$.

But $r'(T) = \frac{1}{2} r(T)\left[1 - r^2(T)\right]$

$\phi'(T) = 0 \quad \Rightarrow \phi = 0 \quad \forall T$

Also $r(T) = \dfrac{1}{\sqrt{1+3e^{-T}}}$

So $x_0(\tau, T) = \dfrac{2}{\sqrt{1+3e^{-T}}} \cos\tau \quad \Rightarrow x(t) = \dfrac{2}{\sqrt{1+3e^{-\varepsilon t}}} \cos t + O(\varepsilon)$.