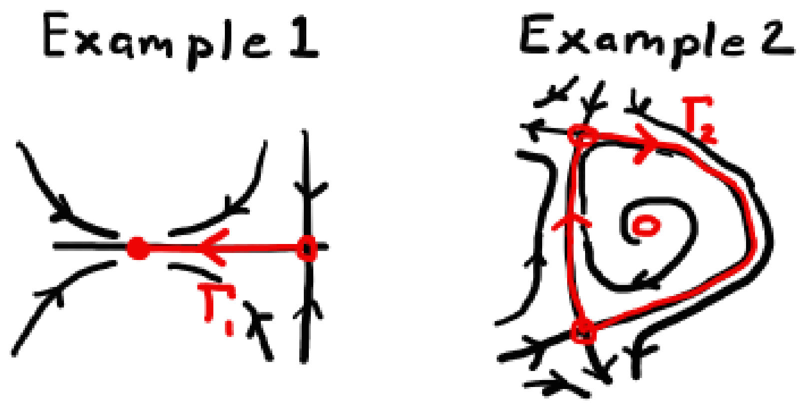
11 Strange attractors and Lyapunov dimension

11.1 Attractors

In previous lectures we have discussed a number of attractors as sets towards which closeby trajectories converge, for example stable fixed points and stable limit cycles. An attractor is defined as

1. an invariant set: trajectories starting on the attractor can not leave it.
2. attracting: there exists a set of initial conditions (basin of attraction) whose trajectories reach the attractor as $t \to \infty$.
3. minimal: No subset on the attractor satisfies conditions 1 and 2.

Example 1 Heteroclinic trajectory Γ_1 between (and including) a saddle and a stable node:



Γ_1 satisfies condition 1 (trajectories starting on Γ_1 remains on Γ_1) and Γ_1 is attractive. But it is not minimal, there is a subset (the node) that satisfies 1 and 2 $\Rightarrow$ the node is the attractor.

Example 2 Cycle of heteroclinic trajectories between two saddle points, Γ_2, surrounding an unstable spiral is an attractor. Which saddle a trajectory starting inside the cycle ends up at as $t \to \infty$ can not be determined (for any large t we can make t even larger in order to closely follow a heteroclinic trajectory to the opposite saddle). This implies, to satisfy condition 2, that both saddles and the interconnecting heteroclinic trajectories constitute the attractor.

11.2 Lyapunov exponents of attractors

For the attractors discussed previously in the course (stable fixed points, limit cycles, homoclinic orbits, cycles of heteroclinic trajectories), we expect that all Lyapunov exponents are non-positive.

11.2.1 Attracting fixed point

For trajectories in the basin of attraction of an attracting fixed point, $\mathrm{Re}\,\sigma_i < 0$, separations must in the long run shrink in all directions because, as shown in Lecture 10, for this case the stability exponents of separations $\tilde{\sigma}_i = \sigma_i$, and hence all Lyapunov exponents are negative, $\lambda_i < 0$, in a system with globally attracting fixed points.

11.2.2 Attracting limit cycle

For any bounded, autonomous (time-independent) flow $\boldsymbol{f}$ without attracting fixed points, one Lyapunov exponent is zero. This follows from (f_i are components of $\boldsymbol{f}$)

$$\dot{f}_i = \underbrace{\partial_t f_i}_{=0} + \sum_j \underbrace{\dot{x}_j}_{f_j} \underbrace{\partial_j f_i}_{J_{ij}} = \sum_j J_{ij} f_j \,,$$

i.e. the phase-space velocity $\dot{\boldsymbol{x}} = \boldsymbol{f}$ satisfies the same time evolution as $\mathbb{M}$ and $\boldsymbol{\delta}$ (tangent equations). For an initial separation $\boldsymbol{\delta}(0) = \boldsymbol{f}(0)$ the separation grows to $\boldsymbol{\delta}(t) = \boldsymbol{f}(t)$ at a later time t and the corresponding stretching rate is

$$\lambda = \lim_{t\to\infty} \frac{1}{t} \ln |\dot{\boldsymbol{x}}| \,.$$

This is zero unless $\dot{\boldsymbol{x}}$ depends exponentially on t for large t, which would happen close to a stable fixed point or if infinity is approached exponentially fast. But, in the bounded system considered here, regular trajectories do not diverge exponentially with time and λ must vanish. The vanishing $\lambda = 0$ must be equal to one of the Lyapunov exponents (by the decomposition in terms of eigenvectors $\boldsymbol{v}_i$ of $\mathbb{M}^{\mathrm{T}}\mathbb{M}$, $\boldsymbol{f} = \sum_j a_j \boldsymbol{v}_j$, λ approaches the largest Lyapunov exponent λ_i with non-zero a_i).

As a consequence, two trajectories starting close-by on a closed orbit (i.e. their separation points along the direction of velocity) does not on average separate or contract. $\Rightarrow$ attracting limit cycles have $\lambda_1 = 0$ and the remaining Lyapunov exponents are non-positive.
Similarly: Attracting m-frequency quasiperiodic orbit (motion on m-torus) has $\lambda_1 = \lambda_2 = \cdots = \lambda_m = 0$ and the remaining Lyapunov exponents are negative.

11.2.3 Volume conserving systems (no attractors)

The evolution of a small volume element in a flow is given by (lecture 5 or Strogatz Sec. 9.2):

$$\dot{\mathcal{V}} = \int_{\mathcal{V}} \mathrm{d}\mathcal{V} \boldsymbol{\nabla} \cdot \boldsymbol{f} = \int_{\mathcal{V}} \mathrm{d}\mathcal{V}\, \mathrm{tr}\mathbb{J}\,.$$

If the dissipation rate $\mathrm{tr}\mathbb{J}$ is constant, the equation simplifies to $\dot{\mathcal{V}} = \mathcal{V}\mathrm{tr}\mathbb{J}$, giving

$$\mathcal{V} = \mathcal{V}_0 e^{\mathrm{tr}\mathbb{J}\mathrm{t}}\,,$$

i.e. volumes shrink exponentially fast in dissipative systems ($\mathrm{tr}\mathbb{J} < 0$) and remain constant in volume-conserving systems ($\mathrm{tr}\mathbb{J} = 0$).

But in lecture 10 we found that an initially spherical shell around a test particle transforms into an ellipsoid with axes $\sim e^{\lambda_i t}$, i.e.

$$\mathcal{V} \sim \mathcal{V}_0 e^{\lambda_1 t} e^{\lambda_2 t} \cdots e^{\lambda_n t} = \mathcal{V}_0 e^{(\lambda_1 + \lambda_2 + \ldots \lambda_n)t}$$

Thus the sum $\lambda_1 + \ldots \lambda_n$ is zero in volume conserving systems.

11.3 Strange attractors

Summarizing the previous section for dimensionality $n = 3$:

Attractor	λ_1	λ_2	λ_3	Dimension
Fixed point	<0	<0	<0	0
Limit cycle	0	<0	<0	1
Limit torus	0	0	<0	2
Volume conserving (no attractor)	0	0	0	3

Now consider the case of a chaotic system ($\lambda_1 > 0$) with trajectories bounded in a finite region:

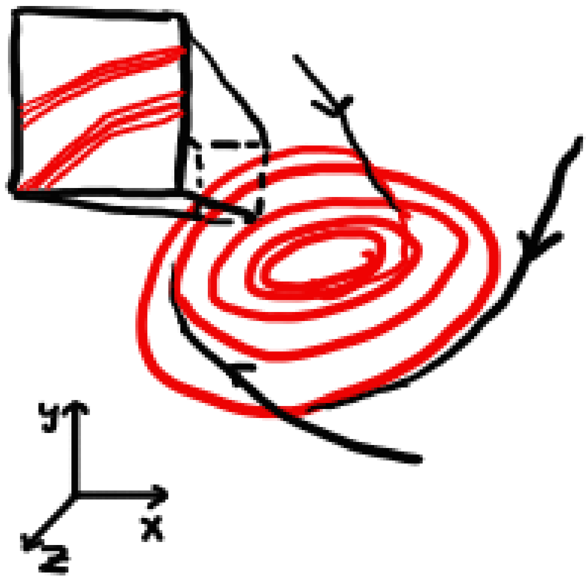
- If $\lambda_1 + \lambda_2 + \lambda_3 = 0$ (volume conserving chaotic system) then aperiodic chaotic trajectories fill out space uniformly. This is similar to the chaotic systems encountered for billiards, double pendulum, chaotic advection, ...

- If instead $\lambda_1 + \lambda_2 + \lambda_3 < 0$ (dissipative chaotic system) then phase-space volumes shrink.

For the case of dissipative chaotic dynamics, one Lyapunov exponent must be zero, $\lambda_2 = 0$ (see Section 11.2.2), and consequently $\lambda_3 < 0$. Now, since $\lambda_1 > 0$ and therefore $\lambda_1 + \lambda_2 > 0$, we expect small areas to grow and small volumes to shrink. What kind of attractor can we have? It cannot be a limit cycle|torus (because these have $\lambda_1 = 0$). The resolution is a new kind of attractor, strange attractor (fractal attractor), that takes on a fractional dimensionality, somewhere between 2 and 3.

11.3.1 Properties of strange attractors

Illustration of a strange attractor:



Some properties:

- Strange attractors show sensitive dependence of initial conditions. Close-by trajectories end up at different places on the attractor.

- The strange attractor is bounded but aperiodic (if it were periodic it would be a limit cycle)

- Strange attractors require a phase-space dimensionality of at least 3. In lower dimensions trajectories cannot pass and aperiodic motion is ruled out by the Poincaré-Bendixson theorem (Lecture 5).

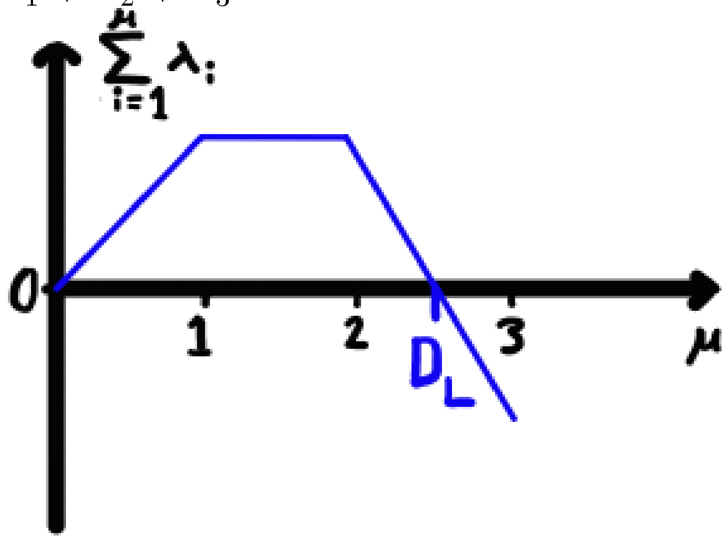
- The strange attractor has structure at all scales (since it outlines an infinitely long aperiodic trajectory in a confined region).

- Strange attractors cannot be plotted (there is always more structure if you zoom). Curves lying arbitrarily close to the attractor are obtained by choosing an initial point in the basin of attraction, solving the flow equations for some time to get close to the attractor, and then plot the aperiodic dynamics for a long time.

- A system may have one or several regular/strange attractors with different basins of attraction (the basin of attraction can itself be a fractal) .

11.3.2 Lyapunov dimension

There are several ways to define the fractal dimension of a strange attractor (next lecture). One estimate of the dimensionality of the strange attractor is the Lyapunov dimension $D_{\rm L}$. It is defined as the number of ordered Lyapunov exponents that sum to zero. For the attractors listed in the table above, $D_{\rm L}$ becomes 0 for the fixed point, 1 for the limit cycle, 2 for the limit torus and 3 for the volume conserving system. For the strange attractor above $\lambda_1+\lambda_2>0$ and $\lambda_1+\lambda_2+\lambda_3<0$ sum to non-zero numbers:



$D_{\rm L}$ is then determined by a linear interpolation $D_{\rm L}(\lambda_1+\lambda_2)=A+B(\lambda_1+\lambda_2)$ where A and B are determined by $D_{\rm L}(0)=2$ and $D_{\rm L}(-\lambda_3)=3$, i.e.

$$D_{\rm L}=2-\frac{\lambda_1+\lambda_2}{\lambda_3}\,.$$

This is a number between 2 and 3 as desired (seen from the constrains $\lambda_1+\lambda_2>0$ and $\lambda_1+\lambda_2+\lambda_3<0\Rightarrow\lambda_3<-(\lambda_1+\lambda_2)$).

Similarly, the Lyapunov dimension can be generalized to other dimensionalities.

11.3.3 Example of strange attractor: Lorenz attractor

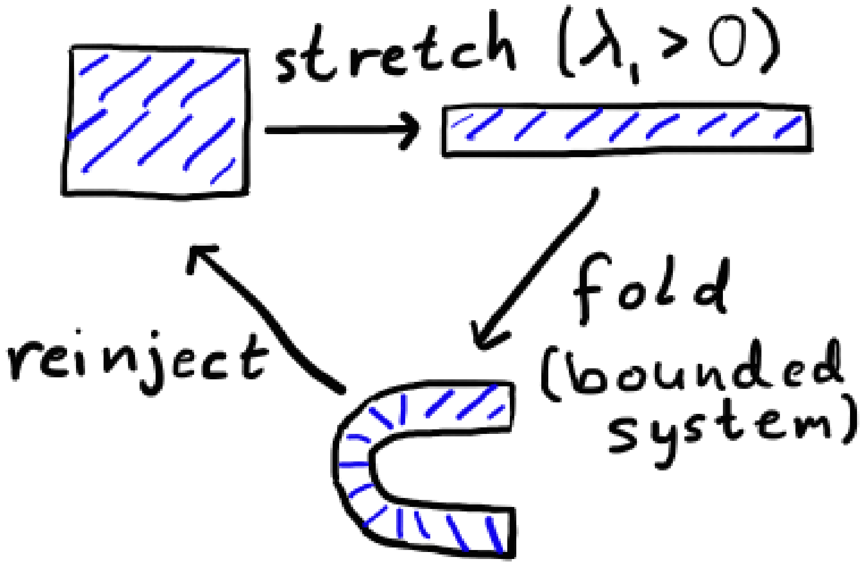
The standard example of system with a fractal attractor

$$\dot{x} = \sigma(y - x)$$
$$\dot{y} = rx - y - xz$$
$$\dot{z} = xy - bz$$

This is a toy model for convection rolls in the atmosphere It also describes the motion of a particular water wheel (Strogatz 9.1). Attracting fixed points exist for small r, but system jumps to a strange attractor after a subcritical Hopf Bifurcation $r_{\mathrm{H}} \approx 24.74$.

11.4 Formation of strange attractors

Typical behaviour in chaotic systems:



We typically have a strange attractor if the system is also dissipative.

Geometric illustration: Simple horseshoe map Repeated mapping of a rectangle into itself.

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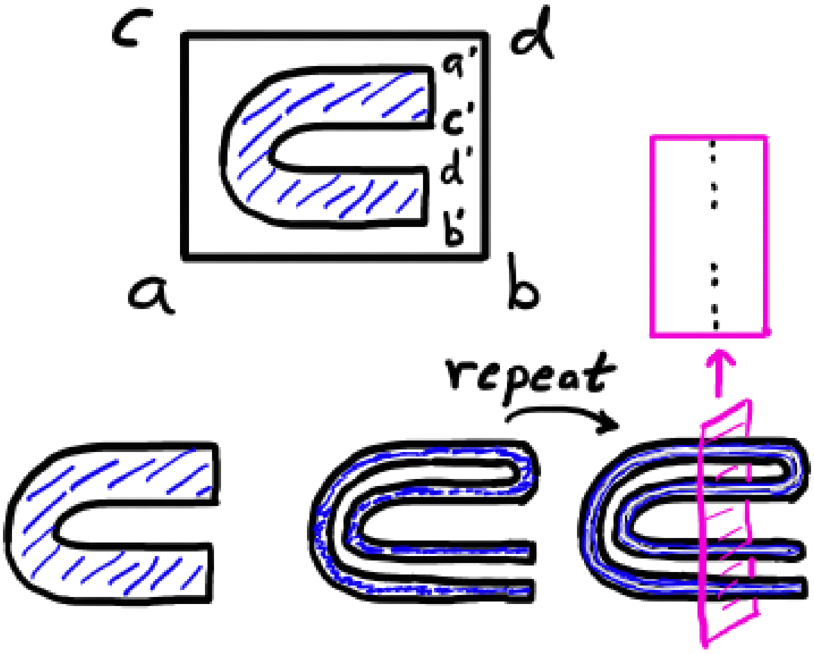
- Map rectangle *abcd* into a horseshoe $a'b'c'd'$ by stretching and folding as above. Area of horseshoe smaller than original image $\Rightarrow$ dissipation.
- Iterate the map. Stretch gives a rotated 'u'-shape which is folded.
- Iterate to get thinner and thinner filaments



After infinite iterations, a vertical cut through the middle resembles a fractal (a topological deformation of the Cantor set, next lecture).

This is a typical situation: locally the strange attractor consists of a bundle of a large (infinite) number of close-to parallel filaments. By putting a $n-1$-dimensional cross-section through the bundle some (fractal) pattern emerges.