#1 Describe attractor
period doubling and so on { 2^n cycles appear! $1-2-4-8-16 \ldots$

#2 Point out period 3 window
and "darker lines"
Intermittency → explain behavior
→ $\boxed{\text{why next lecture}}$ ⊛

#3 Universality / U sequence

note get period doubling → orbits multiples of 2
then a period 5 window → multiple of 5
then " 3 window →
This "sequence" depends only on shape

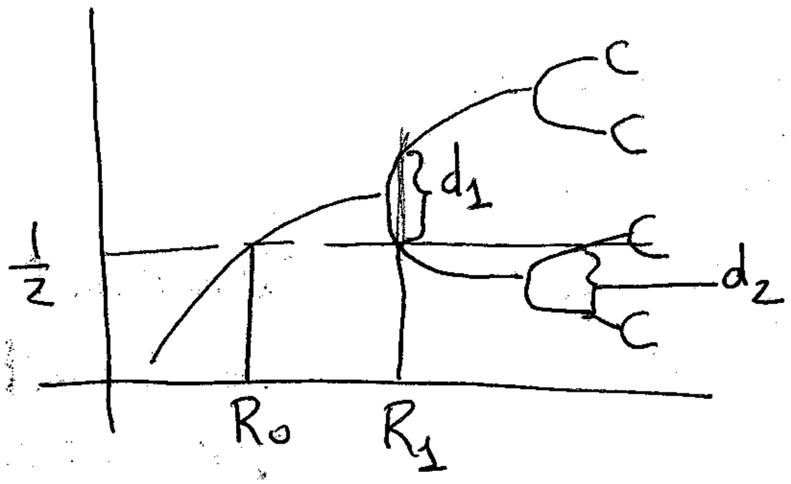
unimodal ($y = f(x)$ has single maximum with
$f'' < 0$ everywhere.

#4 Universality and Feigenbaum's self similarity
for period doubling

$$\delta = \lim_{n \to \infty} \left[\frac{r_n - r_{n-1}}{r_{n+1} - r_n} \right] = 4.669\ldots$$

again for all uni modal
r_n = value of r at which 2^n cycle born.

Let also R_n = value of r ~~at~~ which 2^n cycle is superstable [includes $x = \frac{1}{2}$!]



d_n = distance from $\frac{1}{2}$ to nearest other point in 2^n cycle

Then $\lim \frac{-d_n}{d_{n+1}} = 2.5024...$ also universal

[For generic unimodal $\frac{1}{2} \to$ value where max. is]

Note: d_n "alternate" sign

recall 2^n cycle = fixed point for ~~$f_n(x)$~~ $f_{2^n}(x)$

and $f_n'(x_j) = f'(x_1) f'(x_2) \cdots f'(x_{2^n})$

Lyapunov exponent

Consider two "infinitesimally" close trajectories

$$x_{n+1} = f(x_n) \qquad x_{n+1} + \delta_{n+1} = f(x_n + \delta_n)$$

$$\therefore \quad \delta_n = f^n(x_0 + \delta_0) - f^n(x_0) = (f^n)'(x_0)\,\delta_0$$

Let $\lambda_n = \frac{1}{n} \log\left|\frac{\delta_n}{\delta_0}\right| = \frac{1}{n} \log\left[(f^n)'(x_0)\right]$

$$= \frac{1}{n} \log\left|\prod_0^{n-1} f'(x_j)\right| = \frac{1}{n} \sum_0^{n-1} \log|f'(x_j)|$$

If the limit exists $\underbrace{\text{define} \quad \lambda = \lim_{n \to \infty} \lambda_n}_{\text{Liapunov exponent}}$

λ depends on X_0, but it is the same for any x_0 within a given attractor

$\boxed{\text{Note then} \quad |\delta_n| \sim e^{n\lambda} |\delta_0|}$

Note $\lambda < 0$ for stable fixed points and cycles [where limit exists]

— Show it —

$\lambda > 0$ for an attractor

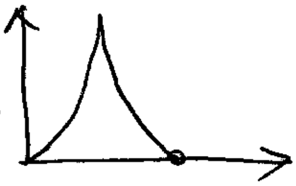
Chaotic

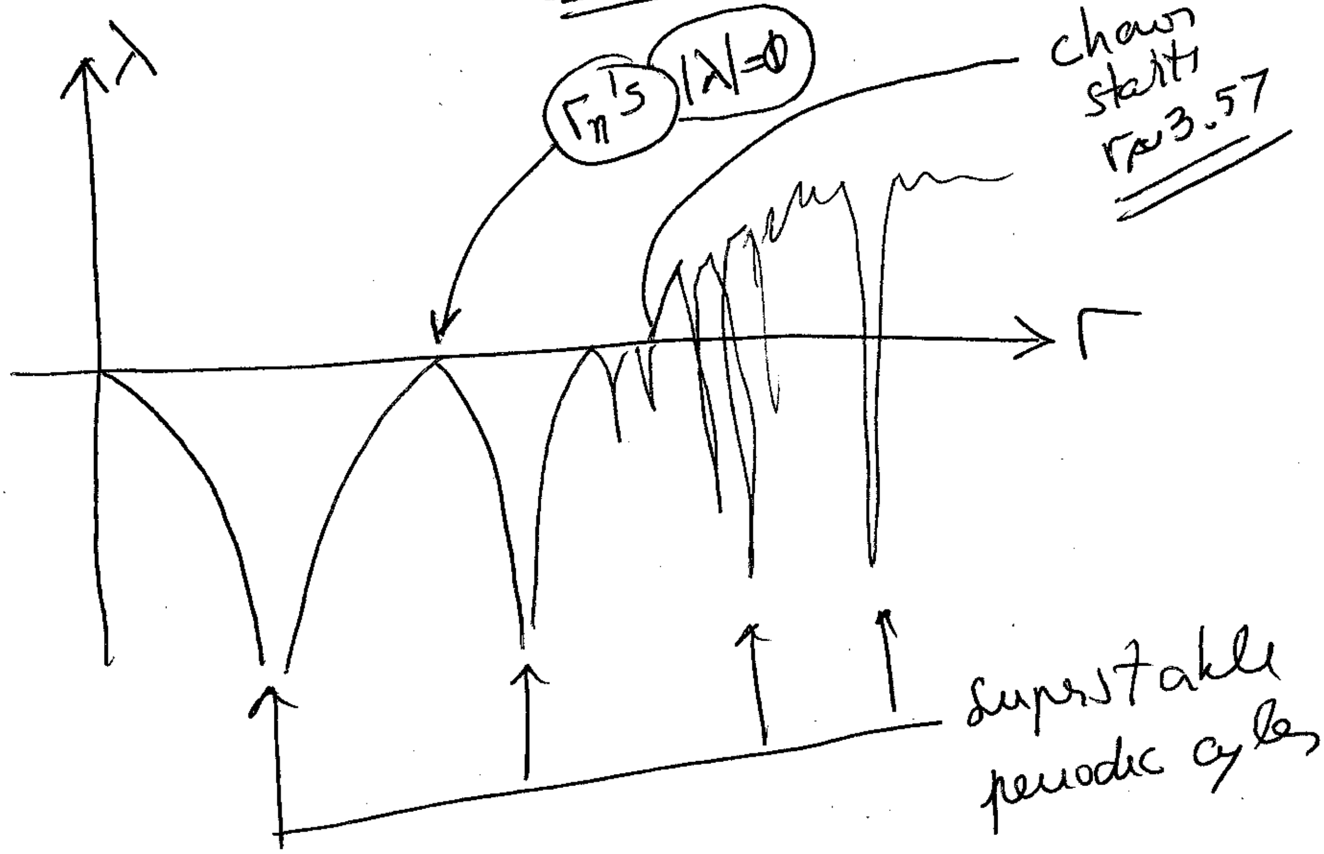
Example tent map $f = \begin{cases} rx & 0 \le x \le 1/2 \\ r(1-x) & 1/2 \le x \le 1 \end{cases}$

$|f'| \equiv r \quad \therefore \quad \lambda = \ln r \gtrless 0$

Chaotic for $r > 1$

Same for Lorenz





Prep to fractals | Cantor & ∞

Cardinality, Countable sets ~ N

Uncountable

can order them!

Rationals are countable

Even # are countable

Integers are countable

Real #'s are not countable!

"Cont. hypothesis"

Cantor set example of fractal

[geometric shape with structure at all scale

generally roughly self similar

Note: cube is self similar but has no "structure"

[what the heck is "structure"

vague definition]

Example Cantor set | Self similar

$\therefore$ structure at all scale

C has measure zero (zero length)

C is uncountable | Note $0.1 = 0.02222\ldots$

end points of intervals

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"First arrow" (on self similar, etc)

Topological Cantor sets:

"Totally disconnected" Contain no intervals

"Has no isolated points"

"Intuitively" Points are spread apart but also packed together! Actually not intuitive at all.

Cross-sections of the strange attractor for Lorenz and Rossler are "Topological Cantor sets"

"Second Arrow"

In the proof of Cantor $\sim [0,1]$ need to make trinary representation unique.

Either

① allow infinite sequence of 2's
$$X = 0.d_1 \cdots d_n 2222\ldots$$
and do not allow "last" digit to be 1
$$X = 0.d_1 \cdots d_n 0000$$
with $d_n = 1$

OR THE OPPOSITE

Self similar dimension intervals

Let us first look at squares, cubes, hypercubes

$$1 \text{ interval} \quad 2 \times \tfrac{1}{2} \to 2^n \times \tfrac{1}{2^n}$$

$$\text{square} \quad \underset{\#}{4} \times \underset{\text{size} = r}{\tfrac{1}{2}} \longrightarrow 4^n \times \tfrac{1}{2^n}$$

$$\text{cube} \quad 8 \times \tfrac{1}{2} \longrightarrow 8^n \times \tfrac{1}{2^n}$$

$\therefore$ log note $\# \sim \left(\frac{1}{r}\right)^d$

$$\therefore \ln \# \sim d \ln(1/r) \qquad \boxed{d = \frac{\ln \#}{-\ln r}}$$

Example Cantor $\log 2/\log 3 \sim 0.63$

Koch Curve $\frac{\log 4}{\log 3} \sim 1.26$

Note $\boxed{\text{Self similar dimension } \mathbb{Q} = 1}$ does not distinguish between $\mathbb{Q}$ and $\mathbb{R}$

What if we take out even-5th instead of middle 1/3 $\quad \frac{\log 3}{\log 5} \sim 0.68$