An introduction to pilot-wave dynamics

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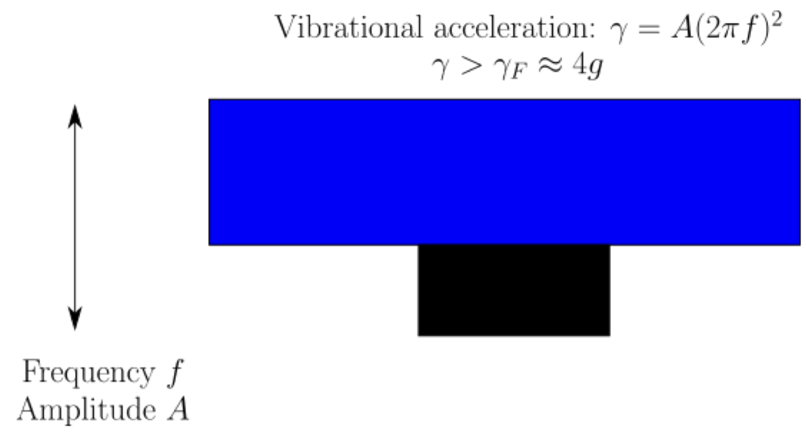
Faraday waves and bouncing droplets

- Oscillate a fluid bath vertically at a frequency f, amplitude A
- Above critical threshold, Faraday instability arises
- Below threshold, can bounce/walk a droplet on the decaying waves
- Wave field memory: how many bouncing periods a generated wave persists for
- Pilot wave: droplet is 'piloted' by the fluid

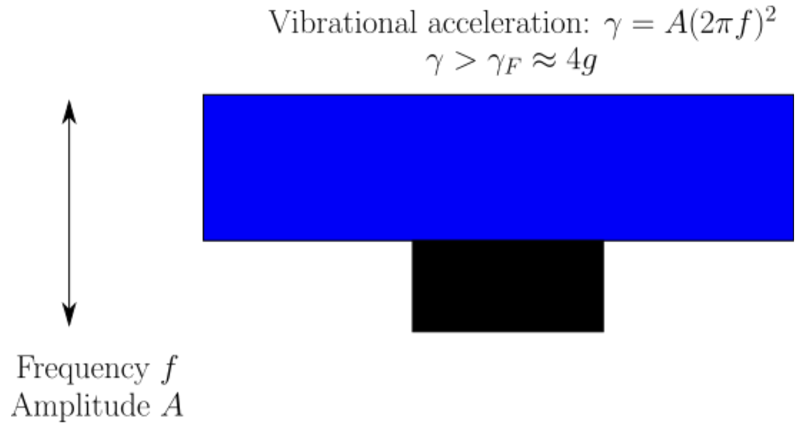


Origin of the Faraday instability

- Model fluid as inviscid, irrotational
- Fluid velocity governed by Laplace's equation
- Let fluid surface height be $\zeta(x, y)$
- Expand

$$\zeta(x, y) = \sum_{m=0}^{\infty} a_m S_m(x, y)$$

$$\nabla^2 S_m = -k_m^2 S_m$$



- Can show that, after non-dimensionalising,

$$\frac{d^2 a_m}{dt^2} + (p_m - 2q_m \cos(2t))a_m = 0$$

Benjamin, Ursell, 1954

Fluid inertia

Restoring forces due to gravity and surface tension

External driving of the fluid

Origin of the Faraday instability

$$\frac{d^2 a_m}{dt^2} + (p_m - 2q_m \cos(2t))a_m = 0$$

- Analysis of this equation requires Floquet theory, as there is a time-dependent forcing
- Results from Floquet theory:

$$\begin{bmatrix} a_m \\ \frac{da_m}{dt} \end{bmatrix} = Q(t)e^{tR}$$

for some periodic vector-valued function Q and real matrix R: eigenvalues of R determine growth rates of oscillations of a_m

- Instability when eigenvalues have positive real part

Trajectory equation for bouncing droplets

- Waves emitted by droplet exhibit radial symmetry
- Wavelength emitted by the droplet matches wavelength of Faraday instability, $\lambda_F = \frac{2\pi}{k_F}$
- Radially symmetric superposition of plane waves is a Bessel function:

$$J_0(r) = \frac{1}{2\pi} \int_0^{2\pi} e^{ir\cos(\theta)} d\theta$$

- Below Faraday threshold, expect waves to decay exponentially
- Inertia, drag affect particle
- Force due to the wave is proportional to the slope of the local wave field
- Superposition of waves generated by each impact
- Assume horizontal motion slower than vertical motion: integrate over particle trajectory, average over vertical particle motion
- Current state depends on the droplet's entire past history!

$$\kappa_0 \ddot{\boldsymbol{x}}_{\boldsymbol{p}} + \dot{\boldsymbol{x}}_{\boldsymbol{p}} = -2\nabla h\left(\boldsymbol{x}_{\boldsymbol{p}}(t), t\right)$$

$$h(\boldsymbol{x}, t) = \int_{-\infty}^{t} J_0\left(\left|\boldsymbol{x} - \boldsymbol{x}_{\boldsymbol{p}}(s)\right|\right) e^{-\epsilon(t-s)}\, ds$$

ϵ: memory, reciprocal of how many previous droplet impacts contribute to wave field

$\epsilon = 0$ is Faraday threshold

$\epsilon = 1$ is walking threshold

κ_0: non-dimensionalised mass

Instability of bouncing state

- A bouncing state is given by $\boldsymbol{x_p} = \boldsymbol{c}$
- Linearise: $\boldsymbol{x_p} = \boldsymbol{c} + \boldsymbol{X}$
- Linearised equations become (using that $J_1'(0) = \frac{1}{2}$)

$$\kappa_0 \boldsymbol{X}'' + \boldsymbol{X}' = \frac{1}{2}\int_{-\infty}^{t} \big(\boldsymbol{X}(t) - \boldsymbol{X}(s)\big) e^{-\epsilon(t-s)} ds$$

- Define now

$$\boldsymbol{Y}(t) = \int_{-\infty}^{t} \boldsymbol{X}(s) e^{-\epsilon(t-s)}\, ds$$

$$\kappa_0 \boldsymbol{X}'' + \boldsymbol{X}' = \frac{1}{2}(\boldsymbol{X} - \boldsymbol{Y})$$

$$\boldsymbol{Y}' = \boldsymbol{X} - \epsilon \boldsymbol{Y}$$

- Letting now $\boldsymbol{Z} = \boldsymbol{X}$, recast as a 3D linear system, find eigenvalues
- Bouncing state destabilises for $\epsilon < 1$
- Physically: bouncing state is stabilised by drag and destabilised by the wave force

Oza, Rosales, Bush (JFM) 2013

Walking state

- Consider a 1D walker, where $\boldsymbol{x_p} = ut\boldsymbol{\hat{e}_x}$
- Trajectory equation simplifies to

$$u = 2\int_{-\infty}^{t} J_1\big(u(t-s)\big)e^{-\epsilon(t-s)}ds$$

- Integral can be evaluated exactly; find

$$u = \frac{1}{\sqrt{2}}\left(4 - \epsilon^2 - \epsilon\sqrt{\epsilon^2 + 8}\right)^{\frac{1}{2}}$$

- Require that at $\epsilon = 1, u = 0$; indeed, find that $u = O(\sqrt{1-\epsilon})$, indicative of a supercritical pitchfork bifurcation

Walking state stability

- Imagine a droplet walking at constant velocity, but is perturbed at time $t = 0$
- Write $x_p(t) = ut + \eta x_1(t) H(t)$; think of it as adding an impulsive force to the particle at $t = 0$ so that $x_1(0) = 0, x_1'(0) = \frac{1}{\kappa_0}$
- Linearised equation:

$$\kappa_0 x_1''(t) + x_1'(t) = \epsilon x_1(t) - 2\int_0^\infty x_1(t-s) J_1'(us) e^{-\epsilon s} ds$$

- Convolution: use Laplace transforms and convolution theorem.

$$X(s) = \int_0^\infty x_1(t) e^{(-st)} dt$$

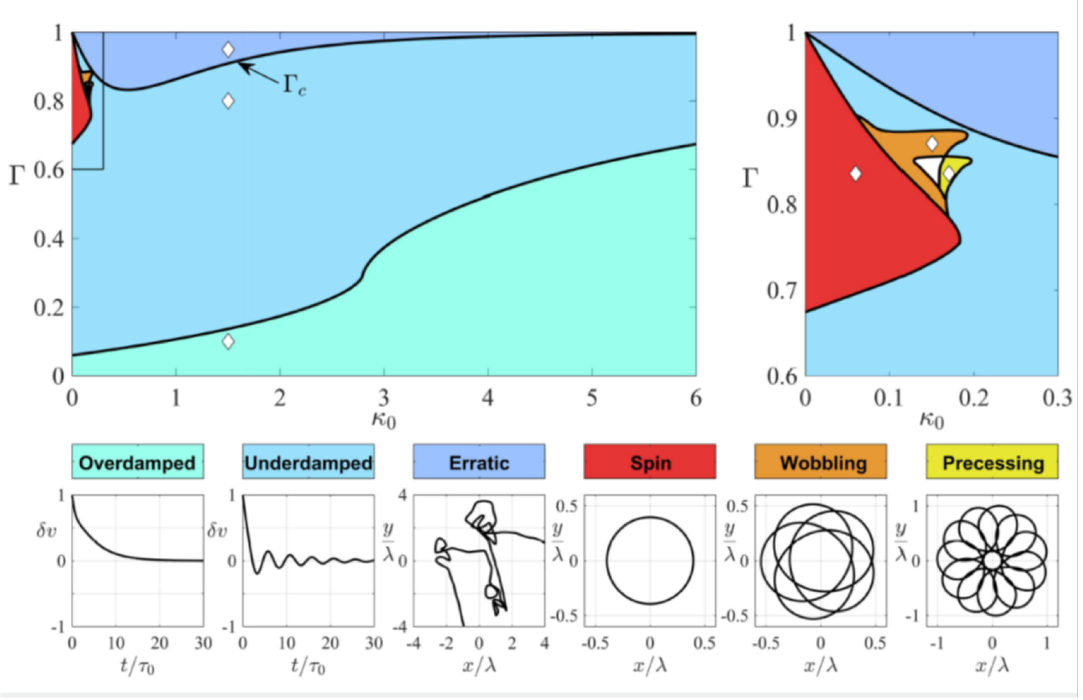
$$\kappa_0 \left(s^2 X(s) - \frac{s}{\kappa_0} \right) + sX(s) = \epsilon X(s) - 2X(s) \int_0^\infty J_1'(ut) e^{-t(s+\epsilon)}\, dt$$

$$X(s) = s\left(\kappa_0 + s - \epsilon + \frac{2(\epsilon + s)}{u^2} \left(1 - \frac{\epsilon + s}{\sqrt{(\epsilon + s)^2 + u^2}} \right) \right)^{-1}$$

- Observe that the Laplace transform of e^{at} is $\frac{1}{s-a}$; exponential growth rates correspond to singularities of Laplace transforms
- Want the singularities of $X(s)$ to determine exponential growth rates

Walking state stability diagram

- Eigenvalue with positive real part: unstable
- No eigenvalues with positive real parts: stable
  - Most unstable eigenvalue complex: underdamped oscillations
  - Most unstable eigenvalue real: overdamped oscillations
- Spin states possible! Circular orbits



Durey, Bush, Chaos 2021

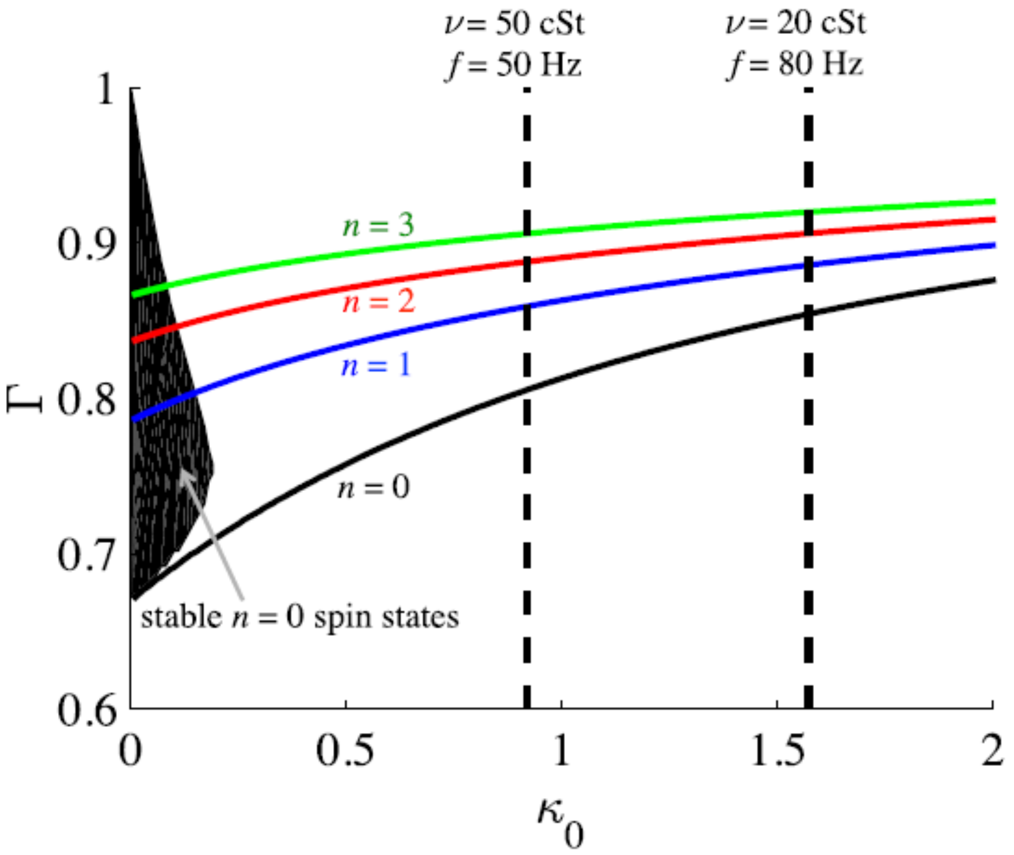
Spin states

- $\Gamma = 1 - \epsilon$; $\Gamma = 1$ is Faraday threshold
- Substituting $\boldsymbol{x_p} = (r_0 \cos(\omega t), r_0 \sin(\omega t))$ into trajectory equation,

$$-\kappa r_0 \omega^2 = 2\int_0^\infty J_1\left(2r_0 \sin\left(\frac{\omega s}{2}\right)\right) \sin\left(\frac{\omega s}{2}\right) e^{-\epsilon s} ds$$

$$r_0 \omega = 2\int_0^\infty J_1\left(2r_0 \sin\left(\frac{\omega s}{2}\right)\right) \cos\left(\frac{\omega s}{2}\right) e^{-\epsilon s} ds$$

- Small region of stability for smallest radius spin states
- What happens if you add rotation to the system?



Oza, Rosales and Bush (Chaos) 2018

Rotating frame

- Externally rotate the system with angular velocity $\boldsymbol{\Omega}$
- In the (non-inertial) frame of the fluid, Coriolis and centripetal forces arise
- Can show that the fluid will develop a parabolic surface to cancel the centripetal force; only consider Coriolis

$$\kappa_0 \ddot{\boldsymbol{x}}_{\boldsymbol{p}} + \dot{\boldsymbol{x}}_{\boldsymbol{p}} = -2\nabla h\left(\boldsymbol{x}_{\boldsymbol{p}}(t), t\right) - \vec{\boldsymbol{\Omega}} \times \dot{\boldsymbol{x}}_{\boldsymbol{p}}$$

$$h(\boldsymbol{x}, t) = \int_{-\infty}^{t} J_0\left(\left|\boldsymbol{x} - \boldsymbol{x}_{\boldsymbol{p}}(s)\right|\right) e^{-\epsilon(t-s)}\, ds$$

- Straight line walking states become circular orbits

Stability analysis

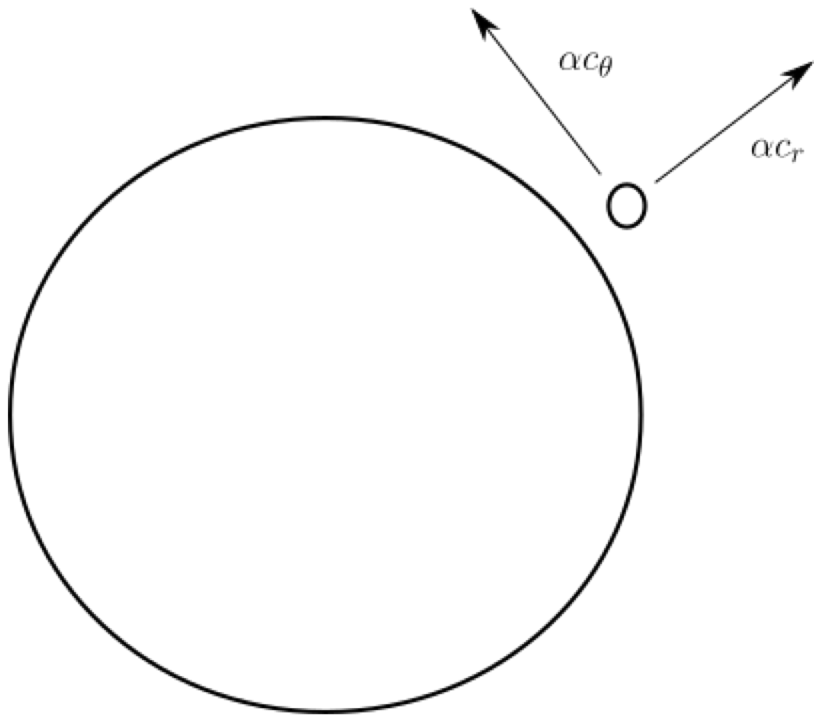
- Consider an impulse acting at $t = 0$, of the form $\eta c_r \hat{r} + \eta c_\theta \hat{\theta}, \eta \ll 1$
- Linearise

$$r(t) = r_0 + \eta r_1(t) H(t)$$
$$\theta(t) = \omega t + \eta \theta_1(t) H(t)$$

- Obtain a 2x2 linear system for the Laplace transforms $R(s), \Theta(s)$ of $r_1(t), \theta_1(t)$

$$\begin{bmatrix} A(s) & -B(s) \\ C(s) & D(s) \end{bmatrix} \begin{bmatrix} R(s) \\ r_0 \Theta(s) \end{bmatrix} = \begin{bmatrix} c_r \\ r_0 c_\theta \end{bmatrix}$$

- Eigenvalues occur for values of s when the Laplace transforms are singular: determinant must vanish
- Solve $\mathrm{F}(s) = A(s)D(s) + B(s)C(s) = 0$



Stability function

$$A(s) = \kappa_0(s^2 - 2\omega^2) + \epsilon + s - 2\Omega\omega + \int_0^\infty \left(J_0\left(2r_0 \sin\left(\frac{\omega t}{2}\right)\right)\cos(\omega t) + J_2\left(2r_0 \sin\left(\frac{\omega t}{2}\right)\right)\right) e^{-(\epsilon+s)t} dt - 2\int_0^\infty J_0\left(2r_0 \sin\left(\frac{\omega t}{2}\right)\right) e^{-\epsilon t} dt$$

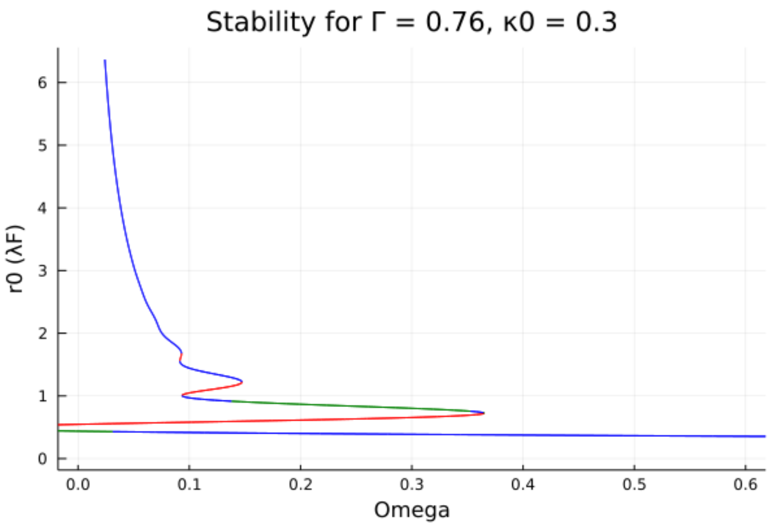
$$D(s) = \kappa_0 s^2 + s - \epsilon + \int_0^\infty \left(J_0\left(2r_0 \sin\left(\frac{\omega t}{2}\right)\right)\cos(\omega t) - J_2\left(2r_0 \sin\left(\frac{\omega t}{2}\right)\right)\right) e^{-(\epsilon+s)t} dt$$

$$B(s) = 2\kappa_0\omega s + \Omega s - \epsilon(\omega\kappa_0 + \Omega) - \int_0^\infty J_0\left(2r_0 \sin\left(\frac{\omega t}{2}\right)\right) \sin(\omega t)\, e^{-(\epsilon+s)t} dt$$

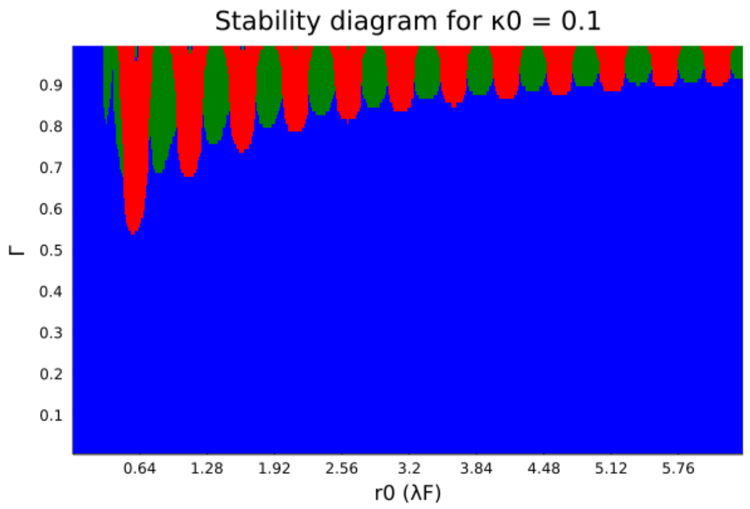
$$C(s) = 2\kappa_0\omega s + 2\omega + \Omega s + \epsilon(\omega\kappa_0 + \Omega) - \int_0^\infty J_0\left(2r_0 \sin\left(\frac{\omega t}{2}\right)\right) \sin(\omega t)\, e^{-(\epsilon+s)t} dt$$

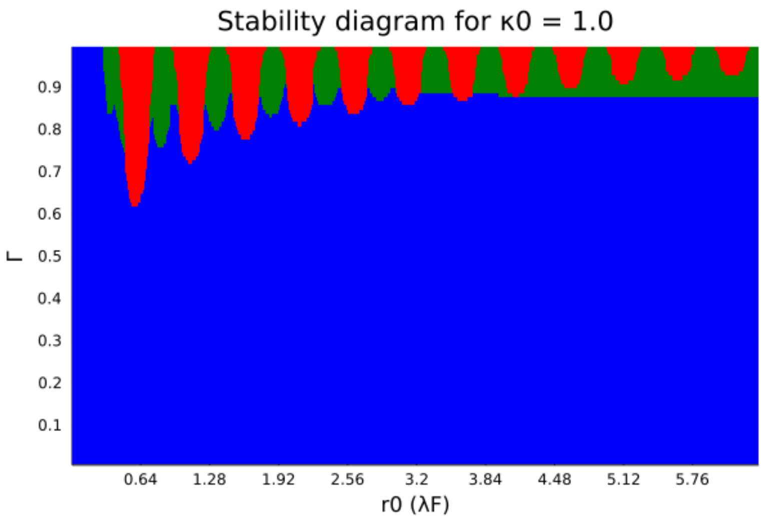
$$F(s) = A(s)D(s) + B(s)C(s)$$

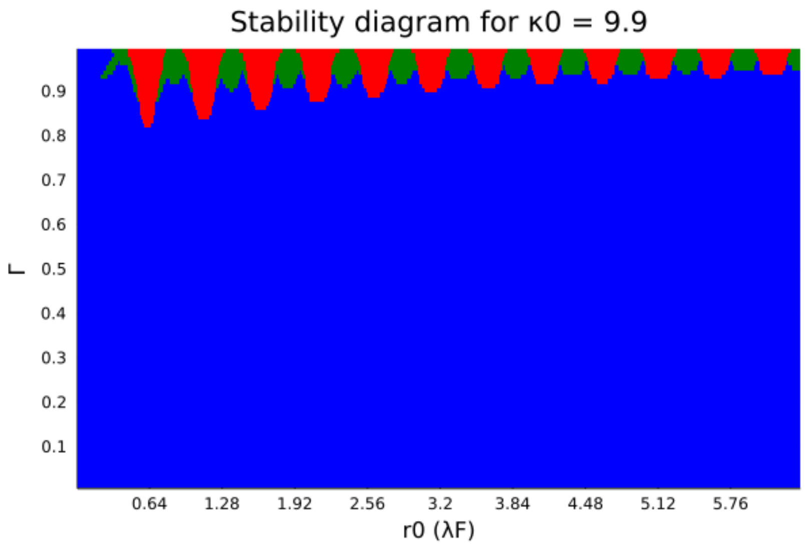
- In the absence of rotation, A, D represent the stability functions for inline and lateral perturbations to 2D straight line walking
- First derived by Oza et al (2014, JFM)
- $F(0) = F(\pm i\omega) = 0$; trivial eigenvalues
- These represent translational and rotational symmetry



Regime diagrams



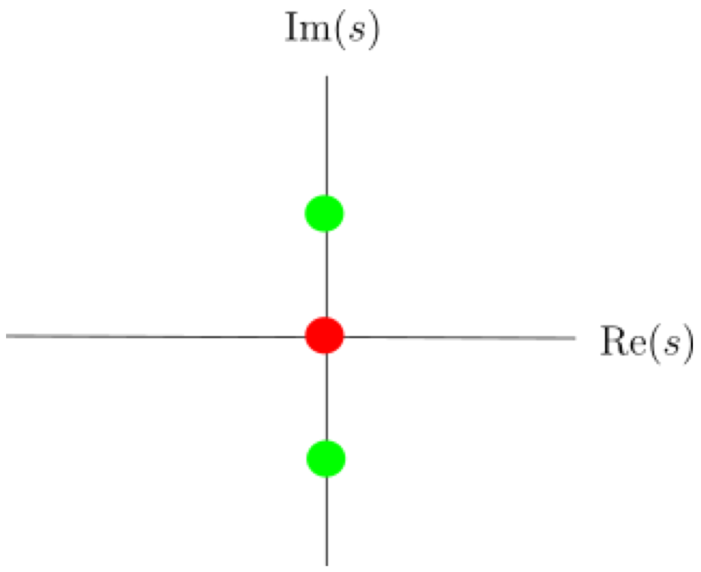


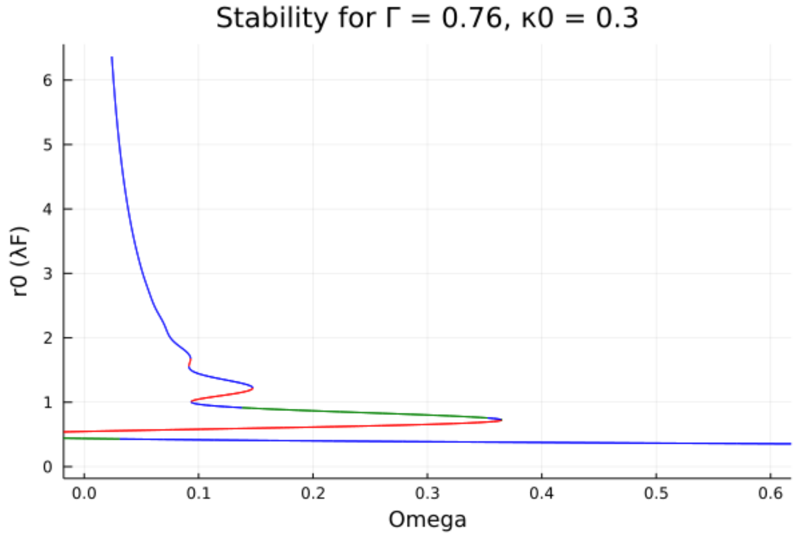


- Colour scheme: stable, oscillatory unstable, non-oscillatory unstable
- Horizontal slices result in the snake curves on the previous slides
- For larger mass, more stable orbital states can be observed

Stability boundaries

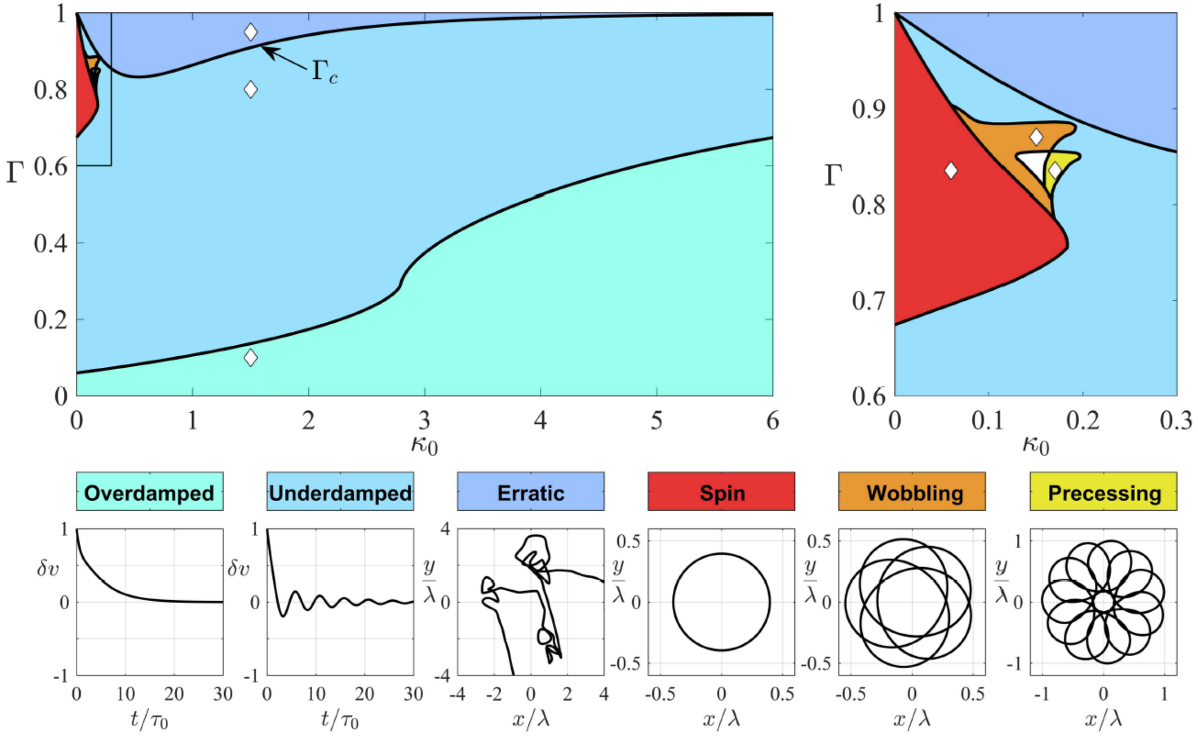
- Two types of linear instabilities
  - Non-oscillatory instabilities: dominant eigenvalue is real and positive
  - Oscillatory instabilities: dominant eigenvalue is complex with positive real part
- For non-oscillatory instabilities, stability boundary occurs when 0 is the dominant (non-trivial) eigenvalue
  - 0 is a non-trivial eigenvalue when $F'(0) = 0$
- For oscillatory instabilities, stability boundary occurs when the dominant non-trivial eigenvalue is imaginary
- On the snake curves, stability boundaries are where blue changes colour

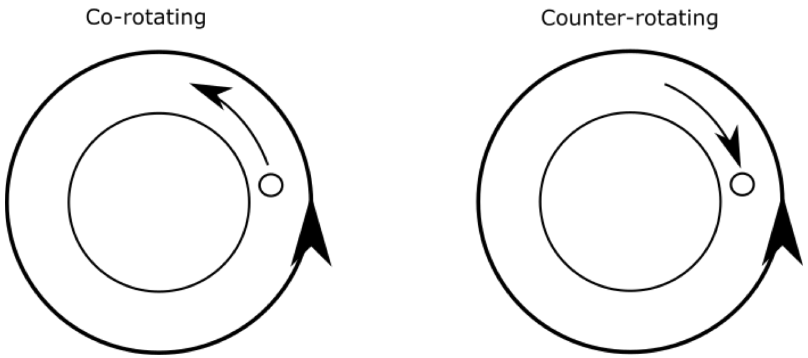




Effects of small rotation

- At zero rotation and high memory, always exists range of κ_0 for which spin state is stable
- Not true with small rotation
- Bath rotation destroys the symmetry of the two directions of orbital states

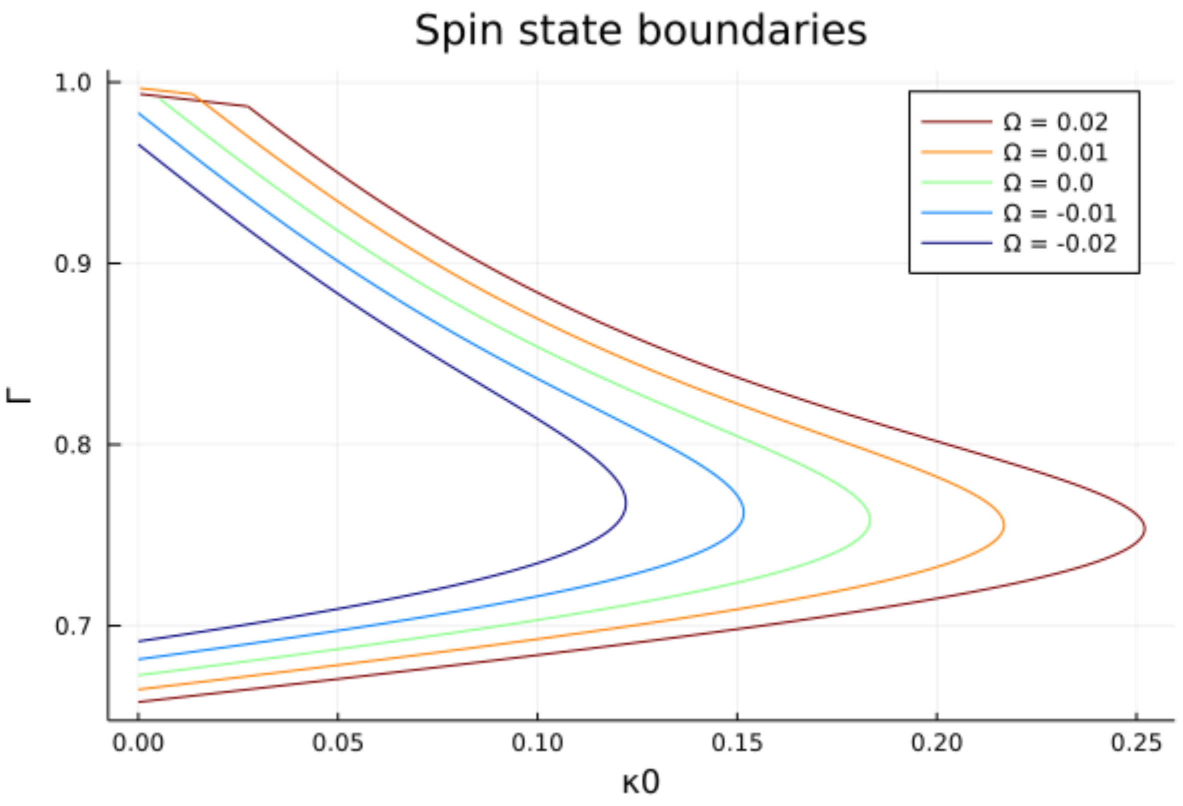




Consider bath rotation direction vs orbital direction

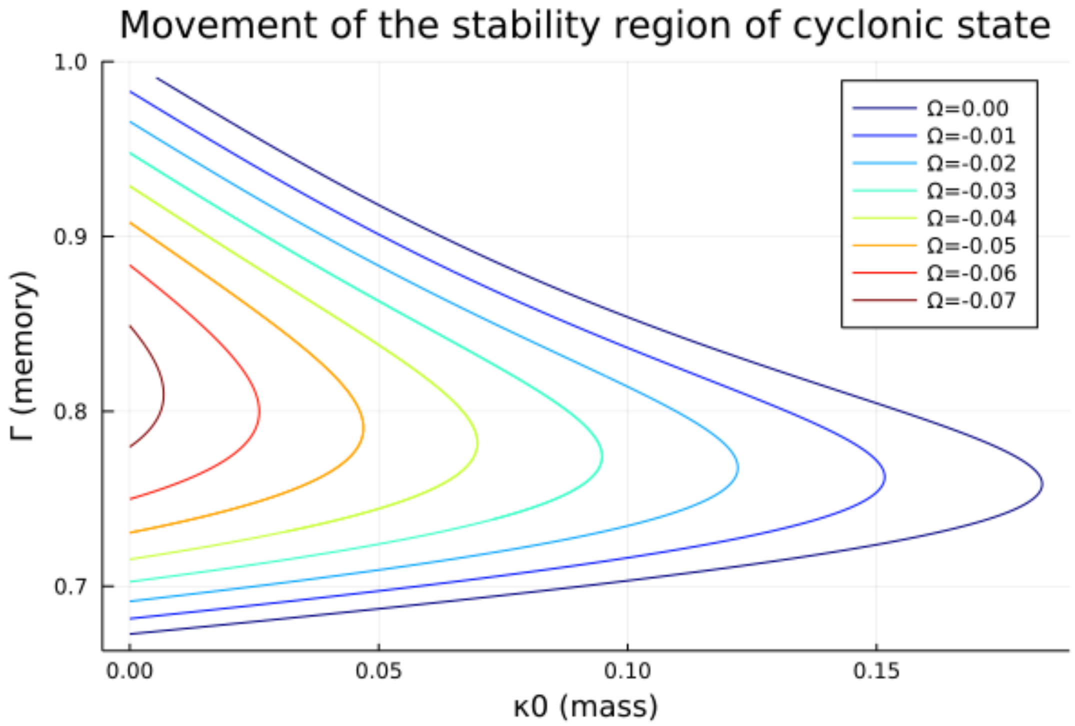
Effects of small rotation on stability boundaries

- Here, $\omega < 0$ for all states, so $\Omega > 0$ corresponds to counter-rotating etc
- Counter-rotating spin states ($\Omega = 0, \omega < 0$) are stabilised by weak rotation
- Co-rotating spin states are destabilised by weak rotation
- Can show asymptotically that $\epsilon \sim 0.322\Omega$ for the stability boundary at $\kappa_0 = 0$ for counter-rotating state, but $\epsilon \sim -1.697\Omega$ for the co-rotating state
  - Mathematical justification for preference of counter-rotating state
- Can solve for location of the cusp asymptotically for small Ω too



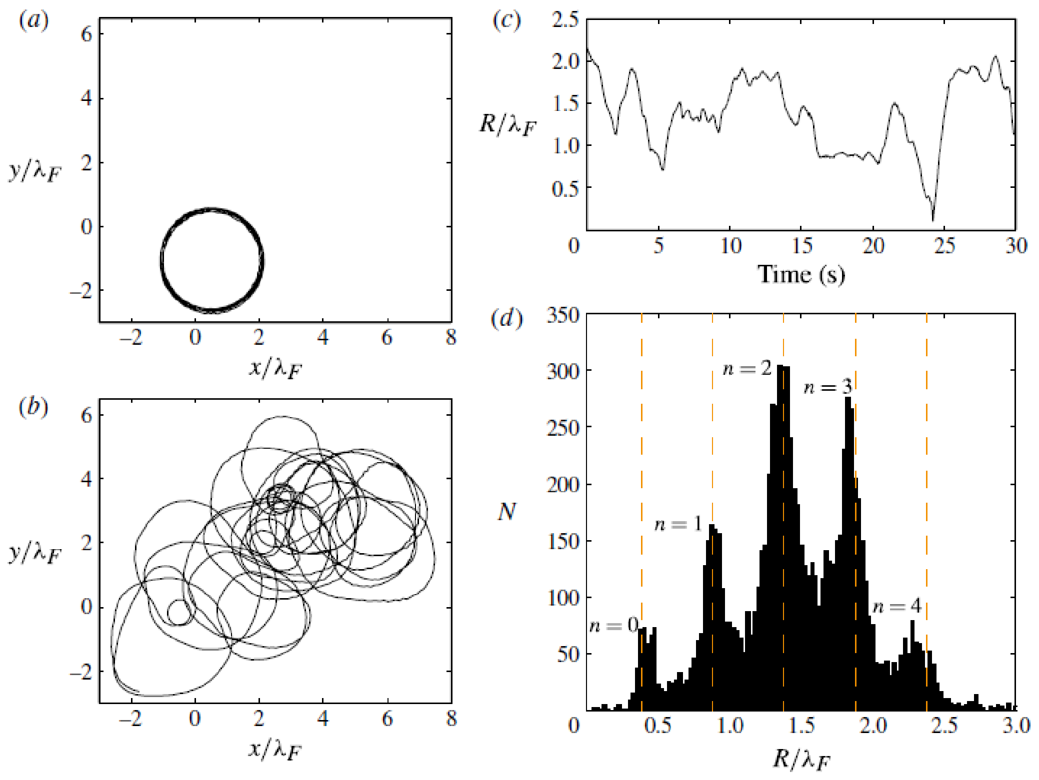
Co-rotating state stability

- Stability region of co-rotating state shrinks with bath rotation
- Critical Ω for which stability region vanishes is found by imposing $\kappa_0 = \frac{\mathrm{d}\kappa_0}{d\epsilon} = 0$ at the stability boundary
- Result: $\Omega = -0.0732$
- No stable larger radius cyclonic states were found



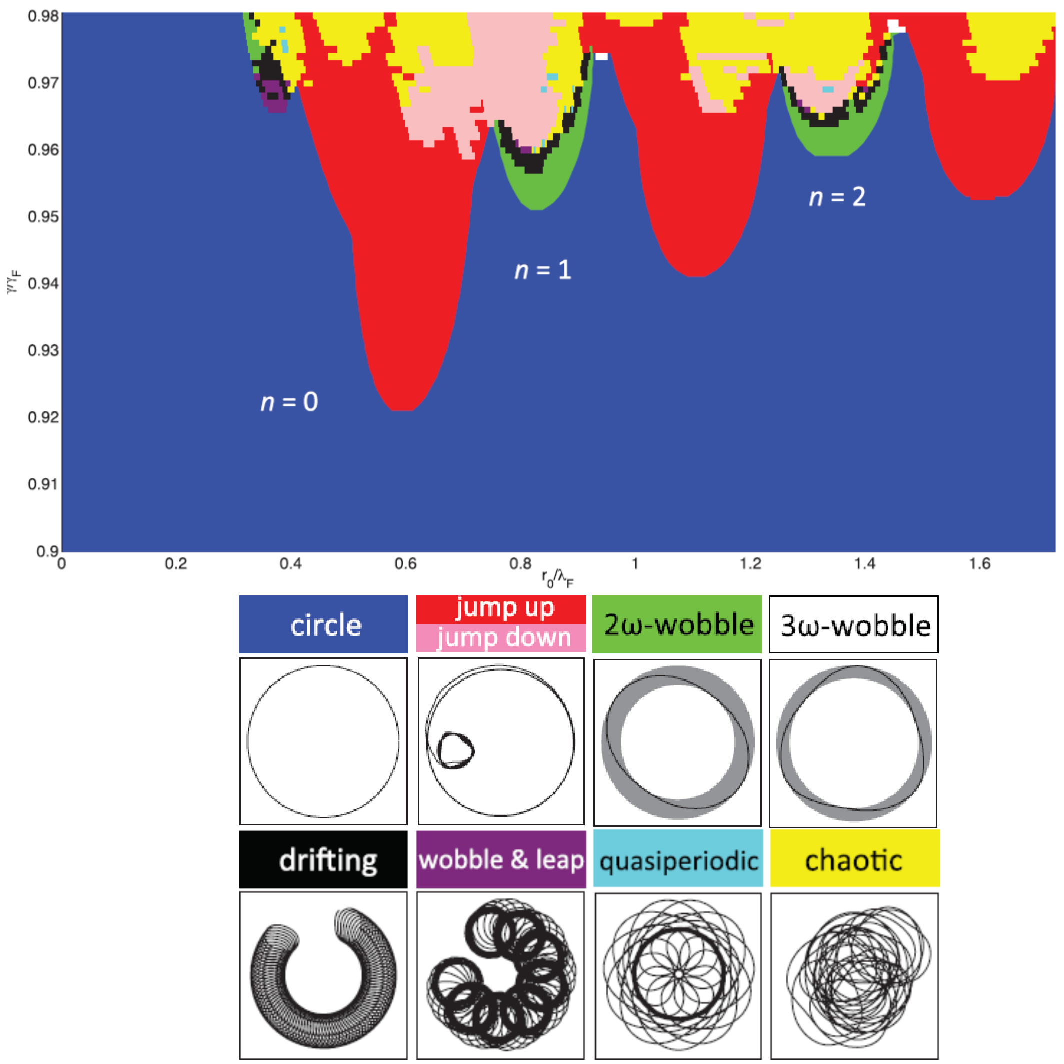
Nonlinear dynamics: experiments (Harris et al, JFM 2014)

- Performed experiments on walking droplets in a rotating frame in high memory
- Describes onset of orbital quantization at low memory
- Wobbling orbits, chaotic orbits
  - Think of these as supercritical Hopf bifurcations of circular orbits
- Multimodal statistics when tracing histograms of radius of curvature



Nonlinear dynamics: Oza et al. (2014) (PoF)

- Numerical solutions of trajectory equation
- Blue regions are stable, everything else linearly unstable
- Describes nonlinear behaviour in linearly unstable regions
- Simulated orbits were qualitatively in agreement with experiments of Harris



Other developments (past, present and future)

- Past
  - Hydrodynamic quantum analogs: use this system as an analogy of quantum mechanics, e.g. quantum tunnelling, quantum computers, double slit interference
  - Lattices of bouncing droplets
- Present
  - Me: investigating stability boundaries of rotating frame system in various asymptotic limits, like large rotation, large radius (first one is done, second one is partially done)
  - Others: extension to 3D pilot wave systems, Bell's inequality
- Future
  - Me: nonlinear dynamics of rotating frame, probability distributions of bouncing droplets and any relationships with quantum mechanics