Lecture # 21

Q = rational # (positive)

N = natural #

$\boxed{N \longleftrightarrow Q \text{ bijection}}$

What about R, cannot be ordered R > N

$[0,1]$ $x = 0.d_1 d_2 d_3 \ldots$

$x_1 = 0.a_1 a_2 \ldots a_n \ldots$

$x_2 = 0.b_1 b_2 \ldots b_n \ldots$

$x_3 = 0.c_1 c_2 \ldots c_n \ldots$

$x = 0.\alpha_1 \alpha_2 \alpha_3 \alpha_4 \ldots$

$\alpha_1 \neq a_1$

$\alpha_2 \neq b_2$

$\alpha_3 \neq c_3$

Parts of $(N) = \underline{2^N \approx R}$

$[\circ \; \circ \; \circ \; \circ]$ $\quad \emptyset \; 1$

4

$\binom{N}{1} \binom{N}{2} \quad C'$ $\quad$ 6

4

1

$\aleph_0 =$ cardinal of nat. numbers

$\aleph_{1?} =$ cardinal of the real #

$$\boxed{\aleph_1 < \aleph_{1?}}$$

Is there something in between i.e. set

cardinal $(N) <$ cardinal $(?)$ $<$ card. (R)

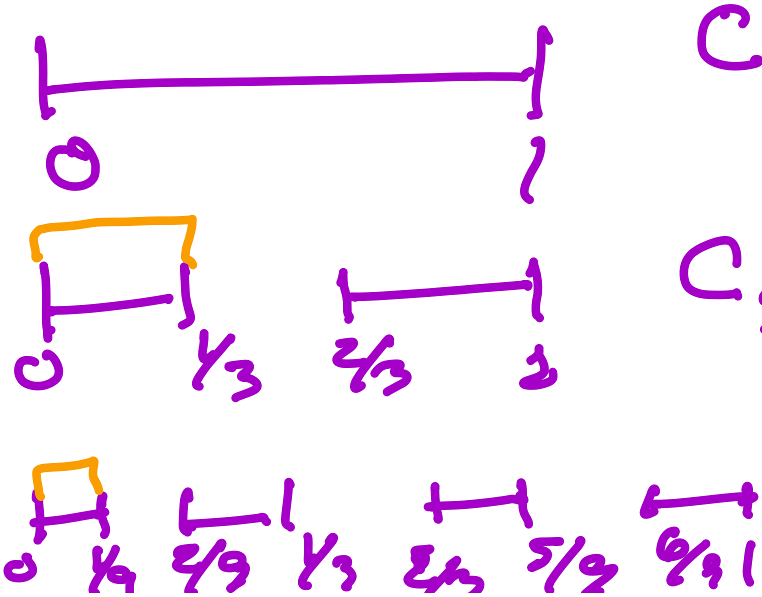
$\boxed{\text{Question of the continuum}}$

Gödel → can add
No there is none
consistent

Cohen → can add
Yes there is one
consistent

Cantor explores continuum:

Cantor Set (First Fractal)



C_0 2^0

C_1 2

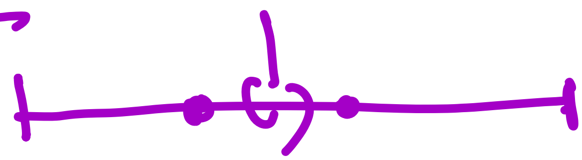
C_2 2^2

C_n has 2^n intervals of length $(1/3)^n$

$C_\infty = \bigcap C_n$ Cantor set

$\text{cardinal}(C_\infty) = \text{cardinal}([0,1])$

$\text{length (measure) } C_\infty = 0$



point in C are separate

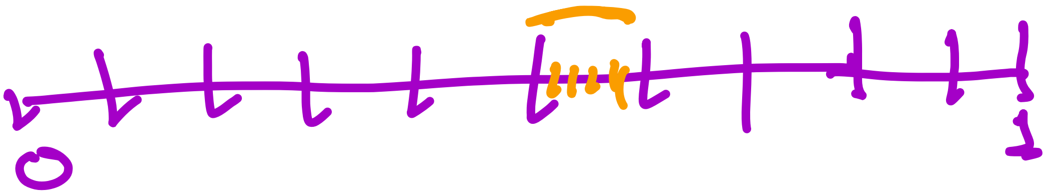
all points have infinity close neighbours

"dense" set

Self-similar $C_\infty \sim 2 \times \left[\frac{1}{3} C_\infty\right]$

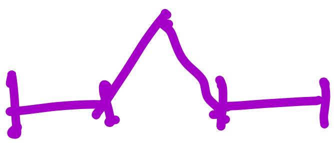
Proof Cardinal (C_∞) = continuum

digits $\boxed{x = 0.d_1 d_2 d_3 \ldots}$

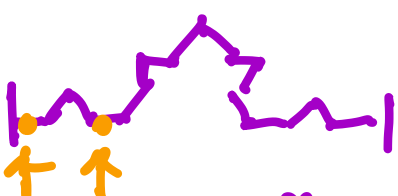


Koch curve

$K_1 = 1 \times 1$



$K_2 = 4 \times \frac{1}{3}$

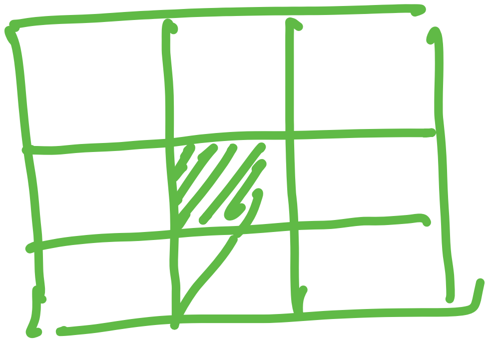


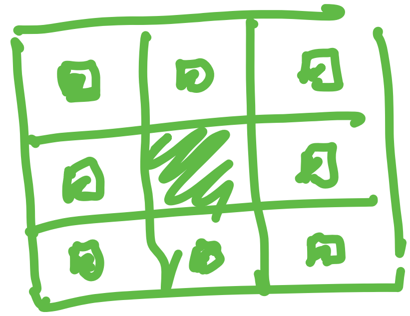
$K_3 = 4^2 \times \frac{1}{3^2} = \left(\frac{4}{3}\right)^2$

$K_n = \left(\frac{4}{3}\right)^n$ length $n \to \infty$

K_∞ Koch curve

"arclength" $ds = \infty$

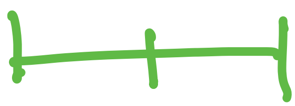




8 copies of size $1/3$

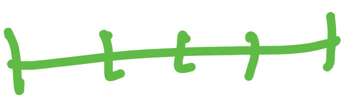
8^2 copies size $1/3^2$

→ Zero area ←



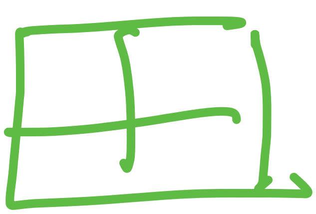
$N = 1/\ell$

$\ell = 2^{-N}$



$\boxed{N \sim 1/\ell}$

$\boxed{N \sim 1/\ell^{2}}$



Cube $N \sim 1/\ell^{3}$

Self similar dimension

N = number of parts
l = length of parts

$N \sim (1/l)^d$ then d = dimension

Cantor $l = \left(\frac{1}{3}\right)^n$ $N = 2^n$

$N \sim (1/l)^d$

$$\boxed{d = \frac{-\log N}{\log l}}$$

Take limit $l \to 0$ dimension

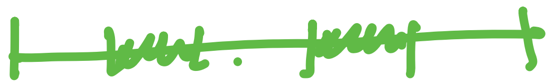
Cantor $d = \frac{\log 2}{\log 3} \sim 0.63$

Koch Curve $d = \frac{\log 4}{\log 3} \sim 1.26$

$\log 3$

Sierpinski $\quad d = \dfrac{\log 8}{\log 3}$

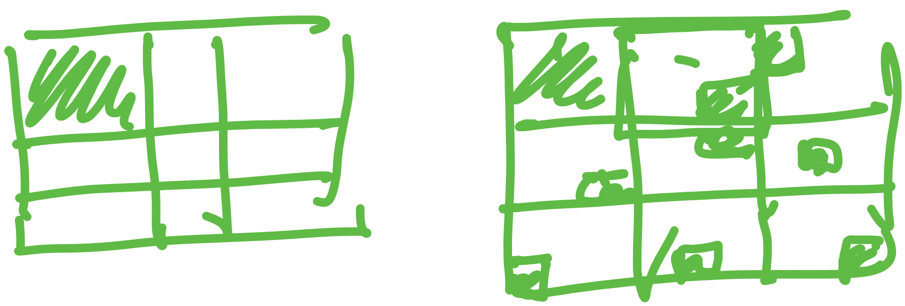
$$1 < d < 2$$



$l \sim (1/5)^n$

$N \sim 3^n$

$$\boxed{d = \log 3 / \log 5}$$



Box Dimension

Cover set with squares of side l $\quad l = 1/3 \quad N = 8$

$l = 1/3^2 \quad N = 8^2$

$l = 1/3^m \quad N = 8^N$

Box dimension of Q in [0,1]

$l = \frac{1}{2} \quad N = 2$

$l = \frac{1}{4} \quad N = 4$

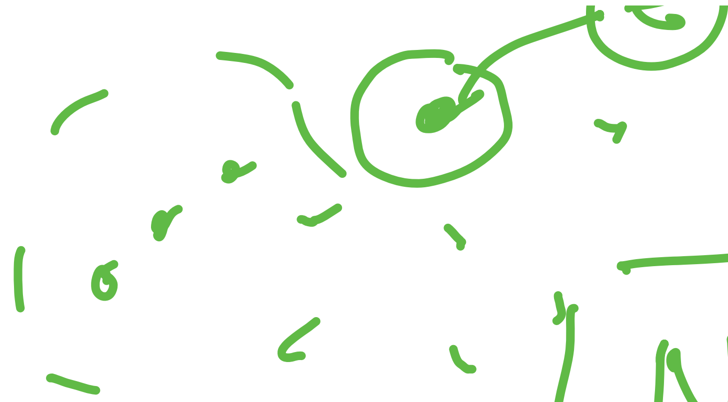
$\boxed{N \sim 1/l}$

Box dim = 1

Hausdorff dim

Pointwise d (For dynamical systems)

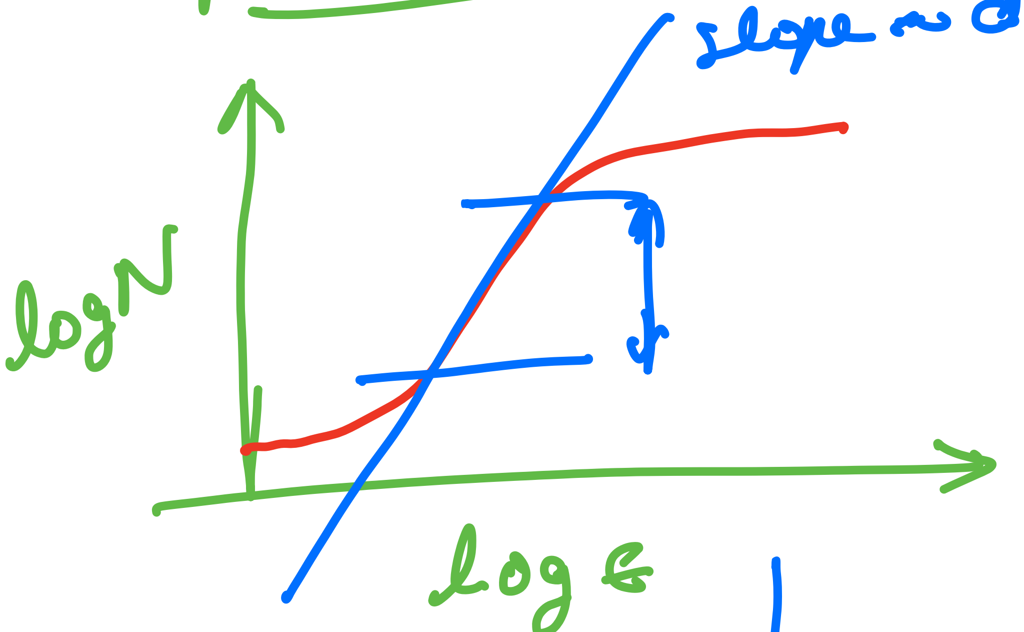
$x_1 \; --- \; x_{2a} \; —$



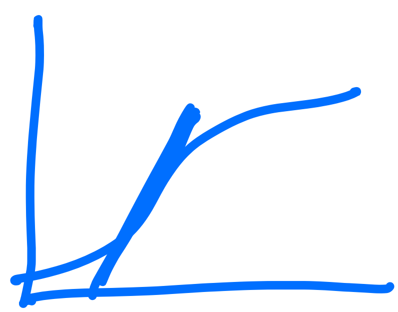
$N(\epsilon)$

$$\boxed{N \sim \epsilon^{d}}$$

d = point wise dim



$d = 1.7 \pm 0.05$



Logistic map

$d_{box} \approx 0.538$

$d_{corr} \approx 0.500$