Proof that van der Pol has limit [redacted] cycle, using trapping regions.

$\epsilon > 0$

$$\dot{x} = y$$

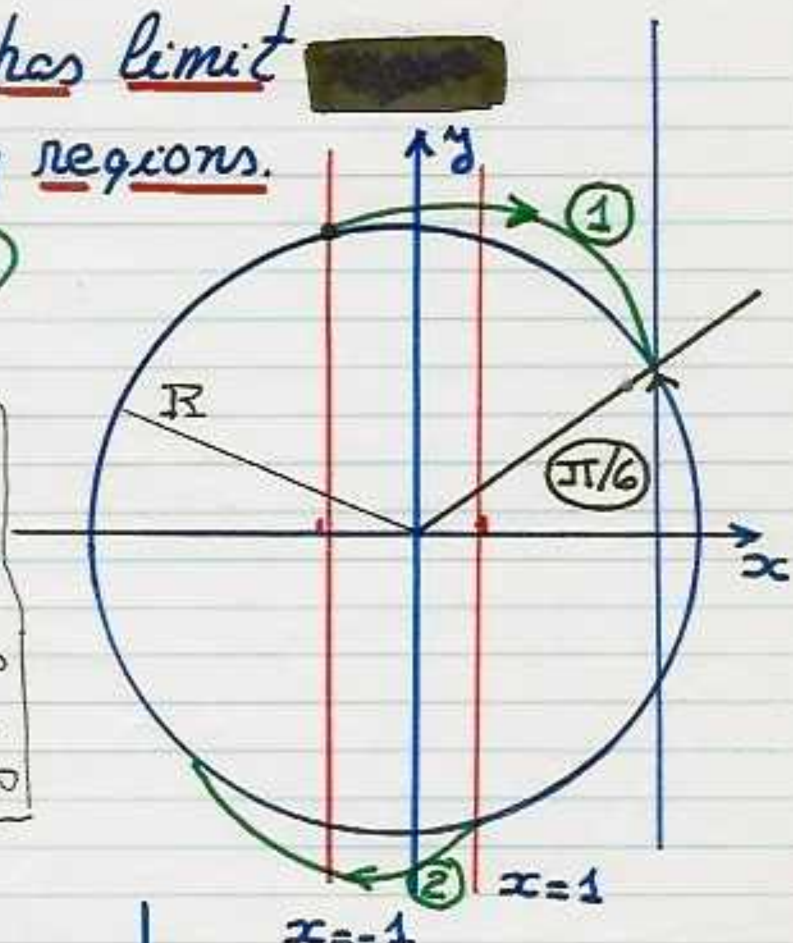
$$\dot{y} = -x + \epsilon(1 - x^2)y$$

$$r = \sqrt{x^2 + y^2}$$

$$r\frac{dr}{dt} = \epsilon(1 - x^2)y^2$$

$$\therefore \boxed{r\frac{dr}{dx} = \epsilon(1 - x^2)y} \quad (1)$$

Note symmetry $y \to -y$ & $x \to -x$



Note that for $\frac{1}{6}\pi < \theta < \frac{5}{6}\pi$

$$\boxed{\tfrac{1}{2}r < y = r\sin\theta \le r}$$

For $r > R$ this happens on $-\frac{\sqrt{3}}{2}R < x < \frac{\sqrt{3}}{2}R$ — $\sin\frac{\pi}{6} = \frac{1}{2}$, $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$,

For $|x| > 1$, flow is inward across $r = R$

Want to show orbit (1) & (2) do as shown

$\Rightarrow$ trapping region.

Show now (1) returns to $r \le R$ [redacted] before $x = x_0 = \frac{\sqrt{3}}{2}R$, as shown, for R large enough.

vdP-trapping

For $-1 < x < 1$: $\quad 0 < \cancel{r}\dfrac{dr}{dx} \leq \epsilon(1-x^2)\cancel{r}$

$$\Longrightarrow \quad \Delta r_1 = \int_{-1}^{1} \frac{dr}{dx}\, dx \leq (4/3)\epsilon$$

For $1 < x < x_0$: [redacted] $\cancel{r}\dfrac{dr}{dx} < \dfrac{\epsilon}{2}(1-x^2)\cancel{r}$

(as long as $\underbrace{r \geq R}$ and $\underbrace{y > \tfrac{1}{2} r}$ valid)

$$\therefore \quad \Delta r_2 = \int_{1}^{x_0} \frac{dr}{dx}\, dx < \frac{\epsilon}{2}\left\{x_0 - \frac{2}{3} - \frac{1}{3}x_0^3\right\}$$

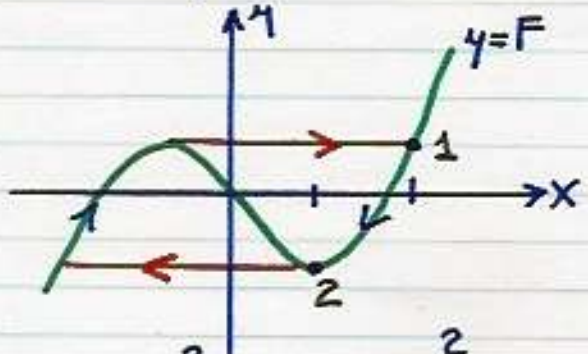
$$\therefore \quad \underbrace{\Delta r = \Delta r_1 + \Delta r_2} < \frac{\epsilon}{2}\left\{x_0 - \frac{1}{3}x_0^3 + 2\right\}$$

Now, as $R \longrightarrow \infty$, $x_0 \longrightarrow \infty$

$\therefore$ for R large enough, $\Delta r \leq 0$

and orbit must return to $r \leq R$!

Period of van der Pol Relaxation regime



$$\dot{x} = \mu(y - F(x))$$

$$\dot{y} = -\frac{1}{\mu} x$$

$$\mu \gg 1$$

$$T \cong 2\int_1^2 dt = 2\int_1^2 \underbrace{\frac{dt}{dx}}_{\frac{dt}{dy}\frac{dy}{dx} = -\frac{\mu}{x}\frac{dy}{dx}} dx = -2\mu \int_{x_1}^{x_2} \frac{F'(x)\,dx}{x}$$

For van der Pol $F = \frac{1}{3}x^3 - x$, $x_2 = 1$, $x_1 = 2$

$$\Rightarrow \quad T \cong 2\mu \int_1^2 \frac{x^2-1}{x}\,dx = \mu(3 - 2\ln 2)$$