18.325: Vortex Dynamics

Problem Sheet 1

(Euler equation, streamfunction-vorticity relations, Biot-Savart integral)

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1. Show that the vorticity equation for the vorticity $\underline{\omega}$ of a barotropic fluid of density ρ (so that the fluid pressure $p = p(\rho)$) is given by

$$\frac{D}{Dt}\left(\frac{\underline{\omega}}{\rho}\right) = \frac{\underline{\omega}}{\rho}.\nabla\underline{u}.$$

2. By modifying the proof in the case of an ideal fluid, prove that Kelvin's circulation theorem also holds for a barotropic fluid where $p = p(\rho)$.

3. Derive the modified forms of the steady and unsteady Bernoulli theorems for a barotropic fluid where $p = p(\rho)$.

4. Let $u_i(\mathbf{x})$ $\rho(\mathbf{x})$ and $P(\mathbf{x})$ be the velocity, density and pressure fields of any three-dimensional steady solution of the incompressible Euler equation, i.e.,

$$u_j \frac{\partial u_i}{\partial x_j} + \frac{1}{\rho}\frac{\partial P}{\partial x_i} = 0, \tag{1}$$

and

$$\frac{\partial\left(\rho u_i\right)}{\partial x_i} = 0. \tag{2}$$

Consider now the modified velocity, density and pressure fields $\hat{u}_i$, $\hat{\rho}$ and $\hat{P}$ given by

$$\hat{u}_i = \lambda u_i \tag{3}$$

$$\hat{\rho} = \frac{\rho}{\lambda^2} \tag{4}$$

$$\hat{P} = P \tag{5}$$

where $\lambda(\mathbf{x})$ is some scalar function. Show that (3)–(5) also represent a steady solution of the Euler equation provided that λ satisfies

$$u_i \frac{\partial \lambda}{\partial x_i} = 0. \tag{6}$$

That is, provided $\lambda(\mathbf{x})$ is constant on streamlines.

Note: This (little-known) construction can be used to produce new equilibrium solutions of the Euler equations (in particular, with variable density fields) from known solutions.

5. Consider the special case of *axisymmetric flow without swirl* in which the velocity field has the special form

$$\underline{u}(r, z, t) = (u_r(r, z, t), 0, u_z(r, z, t))$$

in *cylindrical* polar coordinates. Show that the vorticity vector has the form

$$\underline{\omega}(r, z, t) = (0, \omega(r, z, t), 0)$$

and show that the equation for the scalar function $\omega(r, z, t)$ is

$$\frac{D}{Dt}\left(\frac{\omega}{r}\right) = 0$$

Why does this result indicate that vorticity intensifies as vortex lines are "stretched"?

6. In *spherical* polar coordinates, an incompressible axisymmetric flow without swirl has the form

$$\underline{u}(r, \theta, t) = (u_r(r, \theta, t), u_\theta(r, \theta, t), 0)$$

Define a streamfunction Ψ for this flow and establish that the vorticity $\underline{\omega}$ has the form

$$\underline{\omega}(r, \theta, t) = (0, 0, \omega(r, \theta, t))$$

where

$$\omega(r, \theta, t) = -\frac{1}{r\sin\theta}\left(\frac{\partial^2\Psi}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2\Psi}{\partial\theta^2} - \frac{\cot\theta}{r^2}\frac{\partial\Psi}{\partial\theta}\right)$$

Hence, by seeking separable solutions, determine the streamfunction Ψ associated with steady *irrotational* flow of speed U past a sphere of radius a.

7. Using the method of Fourier transforms, verify that the three-dimensional solution of

$$\nabla^2\psi = m\delta(\mathbf{x} - \mathbf{x_0})$$

which decays at infinity is given by

$$\psi = -\frac{m}{4\pi r}$$

where $r = |\mathbf{x} - \mathbf{x_0}|$.

Note: This is, of course, the Green's function used in constructing the 3-d Biot-Savart integral.

8. For an unbounded flow in three dimensions which vanishes at infinity, the Biot-Savart integral is given by

$$\underline{u}(\mathbf{x}) = -\frac{1}{4\pi}\int_{Flow}\frac{1}{r^3}(\mathbf{x} - \mathbf{x}') \wedge \underline{\omega}(\mathbf{x}')\ dV(\mathbf{x}')$$

where $r = |\mathbf{x} - \mathbf{x}'|$. Using this as a starting point, derive the Biot-Savart integral for an unbounded two-dimensional flow (vanishing at infinity) in the plane $z =$ constant.

Hint: Standard tables show that

$$\int_{-\infty}^{\infty}(x^2 + y^2 + z^2)^{-3/2}dz = \frac{2}{x^2 + y^2}$$