18.325 Vortex dynamics

Equation of motion of a line vortex

In most textbooks, the law of motion of a line vortex is not derived systematically. Indeed, it is usually stated as being obvious (for example, as a consequence of the Helmholtz laws) that a line vortex moves with the local "non-self-induced" velocity field – that is, the non-singular part of the velocity field once the line vortex singularity has been subtracted. However, since the velocity and vorticity are not strictly defined at the vortex singularity, it is difficult to argue that the line vortex corresponds to a vortex line that gets convected with the (non-self-induced) flow. It is satisfying to see a more precise derivation of this important result based on an invocation of Newton's second law. The monograph by Saffman (1992) presents a similar argument (using real analysis).

Let a point vortex of (constant) circulation Γ be at position $\underline{\alpha}(t)$ in an ideal fluid with unit density. Let C_ϵ denote a circular contour, centred at $\underline{\alpha}(t)$, with radius $\epsilon \ll 1$. Of course, the centre of the circle C_ϵ is moving with (complex) speed $d\underline{\alpha}/dt$.

Euler's equation was derived by insisting that Newton's second law holds at all points of the flow; if there are isolated singularities in the flow, it is necessary to enforce that the same law holds at those points too.

To do this, we insist that, as $\epsilon \to 0$, the force exerted by the fluid across C_ϵ on the fluid inside C_ϵ must equal the net momentum flux into the region enclosed by C_ϵ. Mathematically, this means that

$$\lim_{\epsilon \to 0} \left[-\oint_{C_\epsilon} p\mathbf{n}\, ds \right] = \lim_{\epsilon \to 0} \left[\oint_{C_\epsilon} \mathbf{u}\; (\mathbf{u} - d\underline{\alpha}/dt)\,.\mathbf{n}\, ds \right] \qquad (1)$$

where p is the fluid pressure and ds denotes an element of arclength. The left side of this equation is the force exerted across C_ϵ on the fluid inside the contour; the right side is the flux of momentum into the interior of C_ϵ. It should be noted that, on the right side, we have taken into account the fact that C_ϵ is itself moving at speed $d\underline{\alpha}/dt$.

We will now rewrite (1) in complex form. Let the complex number $\alpha(t)$ now denote the complex number corresponding to the vector $\underline{\alpha}(t)$. Let the complex potential for the flow be $w(z,t)$. The complex normal is given by

$n = idz/ds$ so that $\mathbf{n}\ ds$ takes the form idz in complex form. Thus (1) assumes the form

$$-\oint_{C_\epsilon} pidz = \oint_{C_\epsilon} \overline{\left(\frac{dw}{dz}\right)} \mathrm{Re}\left[\left(\frac{dw}{dz} - \frac{d\bar{\alpha}}{dt}\right) idz\right] \tag{2}$$

where we have used the fact that

$$\underline{a}.\underline{b} = \mathrm{Re}[a\overline{b}] \tag{3}$$

and, in obvious notation, a is the complex number corresponding to the vector $\underline{a}$. The unsteady version of Bernoulli's theorem, written in terms of the complex potential, asserts that the fluid pressure is given by the relation

$$p + \frac{\partial}{\partial t}\left(\frac{w + \overline{w}}{2}\right) + \frac{1}{2}\left|\frac{dw}{dz}\right|^2 = H(t) \tag{4}$$

where $H(t)$ is a function of time, but not space.
It follows that (2) is

$$\begin{aligned} &\oint_{C_\epsilon} \left[\frac{\partial}{\partial t}\left(\frac{w+\overline{w}}{2}\right) + \frac{1}{2}\left|\frac{dw}{dz}\right|^2 - H(t)\right] idz \\ &\quad = \oint_{C_\epsilon} \overline{\left(\frac{dw}{dz}\right)} \left[\left(\frac{dw}{dz} - \frac{d\bar{\alpha}}{dt}\right)\frac{idz}{2} - \left(\overline{\left(\frac{dw}{dz}\right)} - \frac{d\alpha}{dt}\right)\frac{id\bar{z}}{2}\right] \end{aligned} \tag{5}$$

The key equation (1) therefore simplifies to

$$\lim_{\epsilon\to 0}\left[\oint_{C_\epsilon} \frac{\partial}{\partial t}\left(\frac{w+\overline{w}}{2}\right) idz\right] = \frac{i}{2}\frac{d\alpha}{dt}\overline{I_1} - \frac{i}{2}\frac{d\bar{\alpha}}{dt}\overline{I_2} - \frac{i}{2}\overline{I_3} \tag{6}$$

where

$$\begin{aligned} I_1 &= \lim_{\epsilon\to 0}\oint_{C_\epsilon} \frac{dw}{dz} dz, \\ I_2 &= \lim_{\epsilon\to 0}\oint_{C_\epsilon} \frac{dw}{dz} d\bar{z}, \\ I_3 &= \lim_{\epsilon\to 0}\oint_{C_\epsilon} \left(\frac{dw}{dz}\right)^2 dz. \end{aligned} \tag{7}$$

The complex potential has the form

$$w(z,t) = -\frac{i\Gamma}{2\pi}\log(z-\alpha(t)) + \mathcal{W}(z,t) \tag{8}$$

where $\mathcal{W}(z,t)$ is analytic at $z=\alpha(t)$. It follows that

$$\frac{\partial w}{\partial t} = \frac{i\Gamma}{2\pi}\frac{d\alpha/dt}{z-\alpha} + \frac{\partial \mathcal{W}}{\partial t}, \quad \frac{dw}{dz} = -\frac{i\Gamma}{2\pi}\frac{1}{z-\alpha} + \frac{d\mathcal{W}}{dz}. \tag{9}$$

Now observe that

$$\lim_{\epsilon\to 0}\oint_{C_\epsilon}\frac{\partial w}{\partial t}dz = \left(\frac{i\Gamma d\alpha/dt}{2\pi}\right)2\pi i \tag{10}$$

by the residue theorem and the fact that

$$\lim_{\epsilon\to 0}\oint_{C_\epsilon}\frac{\partial \mathcal{W}}{\partial t}dz = 0 \tag{11}$$

owing to the fact that the integrand is uniformly bounded on C_ϵ. Note also that

$$\lim_{\epsilon\to 0}\oint_{C_\epsilon}\frac{\partial w}{\partial t}d\bar{z} = \lim_{\epsilon\to 0}\oint_{C_\epsilon}\left[\frac{i\Gamma}{2\pi}\frac{d\alpha/dt}{z-\alpha} + \frac{\partial \mathcal{W}}{\partial t}\right]\frac{-\epsilon^2 dz}{(z-\alpha)^2} \tag{12}$$

where we have used the fact that, on C_ϵ,

$$d\bar{z} = \frac{-\epsilon^2 dz}{(z-\alpha)^2} \tag{13}$$

since C_ϵ is given by the equation $(z-\alpha)(\bar{z}-\bar{\alpha}) = \epsilon^2$. Therefore

$$\lim_{\epsilon\to 0}\oint_{C_\epsilon}\frac{\partial w}{\partial t}d\bar{z} = \lim_{\epsilon\to 0}\left(-2\pi i\epsilon^2\left.\frac{\partial^2 \mathcal{W}}{\partial t\partial z}\right|_{\alpha}\right) = 0. \tag{14}$$

Now consider the evaluation of I_1, I_2 and I_3; all of which can be performed using the residue theorem. Direct application of the residue theorem leads to

$$I_1 = \Gamma. \tag{15}$$

Exploiting relation (13), the residue theorem is then applicable to the evaluation of I_2 and leads to the result $I_2 = 0$. Finally, direct use of the residue theorem shows that

$$I_3 = 2\Gamma\frac{d\mathcal{W}}{dz}(\alpha). \tag{16}$$

Substituting these results into (6) gives the final result

$$\frac{d\alpha}{dt} = \overline{\left(\frac{d\mathcal{W}}{dz}\right)\bigg|_{\alpha}} \tag{17}$$

or, on taking the complex conjugate,

$$\frac{d\bar{\alpha}}{dt} = \frac{d\mathcal{W}}{dz}\bigg|_{\alpha} \tag{18}$$

This says that the line vortex moves with the local non-self-induced velocity (the finite part of the velocity field at the point vortex position).