

18.311 — MIT (Spring 2014)

Rodolfo R. Rosales (MIT, Math. Dept., E17-410, Cambridge, MA 02139).

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Problem Set # 05. Due: Tuesday April 15.

Turn it in before 3:00 PM, in the boxes provided in Room E18-366.

IMPORTANT:

- Turn in the regular and the special problems **stapled in two SEPARATE** packages.
- **Print your name** in each page of your answers.
- In page one of each package **print the names** of the other members of your group.

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1 Regular Problems.

1.1 Statement: Haberman problem 77.06.

Consider Burger's equation, as derived in exercise 77.5:

$$\rho_t + u_{\max} \left(1 - \frac{2\rho}{\rho_{\max}} \right) \rho_x = \nu \rho_{xx}. \quad (1.1)$$

Suppose that a solution exists as a density wave, moving without change of shape at velocity V

$$\rho = f(x - Vt). \quad (1.2)$$

- (a) What O.D.E. is satisfied by f ?
- (b) Integrate this differential equation once. By graphical techniques show that a solution exists, such that $\rho \rightarrow \rho_2$ as $x \rightarrow \infty$ and $\rho \rightarrow \rho_1$ as $x \rightarrow -\infty$, only if $\rho_2 > \rho_1$. Roughly sketch this solution, and give a physical interpretation of this result.
- (c) Show that the velocity of wave propagation, V , is the same as the shock velocity separating $\rho = \rho_1$ from $\rho = \rho_2$ (occurring if $\nu = 0$).

1.2 Statement: Ideal gases thermodynamics.

Consider a gas in a container, at equilibrium. The gas can then be described in terms of the thermodynamic variables: ρ (density), $\mathbf{v} = \mathbf{1}/\rho$ (specific volume), \mathbf{p} (pressure), \mathbf{T} (absolute temperature), e (internal energy per unit mass), \mathbf{S} (entropy), etc. These quantities are not independent, in fact: given any two of them, the others follow via equations of state. In particular, e is a function of T

$$e = e(T) \tag{1.3}$$

and

$$T dS = de + p dv. \tag{1.4}$$

Note that the second law of thermodynamics states that $dS \geq 0$ for any allowed physical process.¹

In particular, for an *ideal gas*

$$p \mathbf{v} = R_s T, \tag{1.5}$$

where $R_s > 0$ is the specific gas constant. A *polytropic gas* is an ideal gas such that

$$e = c_v T, \tag{1.6}$$

where $c_v > 0$ is a constant — the *specific heat at constant volume*. In addition, we introduce

$$c_p = R_s + c_v \quad \text{and} \quad \gamma = \frac{c_p}{c_v}, \tag{1.7}$$

where c_p is the *specific heat at constant pressure* and γ is the ratio of specific heats — which generally satisfies $1 < \gamma < 2$. Note that $\mathbf{h} = \mathbf{e} + \mathbf{p} \mathbf{v}$ (the enthalpy) satisfies²

$$T dS = dh - \mathbf{v} dp \quad \text{and} \quad h = c_p T. \tag{1.8}$$

These are the **tasks for this problem**: *Assume a polytropic gas*, and

¹Equation (1.4) indicates that the temperature is an integrating factor for any “work” done on the system $\delta w = de + p dv$. This generalizes to more complicated systems. For example, in the presence of chemical reactions: $T dS = de + p dv - \sum_n \mu_n dC_n$, where the C_n are the concentrations and the μ_n are the chemical potentials.

²The names for c_v and c_p follow because $\frac{\delta w}{\delta T} = c_v$ if \mathbf{v} is kept constant, while $\frac{\delta w}{\delta T} = c_p$ if p is kept constant.

1. Write the internal energy in terms of the pressure and the specific volume: $e = e(\mathbf{p}, \mathbf{v})$.
2. Write the entropy in terms of the pressure and the specific volume: $S = S(\mathbf{p}, \mathbf{v})$.
Equation (1.4) determines S up to a constant. Let S_0 be the value of the entropy for $p = p_0$ and $\mathbf{v} = \mathbf{v}_0$. This determines the constant. *Warning: your answer should not involve things like the logarithm of a volume — only the log of a number makes sense!*
3. Write the pressure in terms of the entropy and the specific volume: $\mathbf{p} = \mathbf{p}(S, \mathbf{v})$.
4. Write the internal energy in terms of the entropy and the specific volume: $e = e(S, \mathbf{v})$.
5. Write the temperature in terms of the entropy and the specific volume: $T = T(S, \mathbf{v})$.

1.3 Statement for problem GaDy03.

Rankine Hugoniot conditions for Isentropic Gas Dynamics.

The one dimensional isentropic (constant entropy) Euler equations of Gas Dynamics are given by

$$\left. \begin{aligned} \rho_t + (\rho u)_x &= 0, \\ (\rho u)_t + (\rho u^2 + p)_x &= 0, \end{aligned} \right\} \quad (1.9)$$

where $\rho = \rho(x, t)$, $u = u(x, t)$, and $p = p(x, t)$ are the gas mass density, flow velocity, and pressure, respectively. The first equation implements the conservation of mass, and the second the conservation of momentum. These equations are complemented by an **equation of state**, relating the pressure to the density. This takes the form

$$p = P(\rho), \quad \text{where } P \text{ is a function satisfying } \frac{dP}{d\rho} > 0. \quad (1.10)$$

For example, for an ideal gas $P = \kappa \rho^\gamma$, where $\kappa > 0$ and $1 < \gamma < 2$ are constants.

Use conservation to derive the Rankine-Hugoniot jump conditions that shocks for the equations above must satisfy. In other words, consider a solution of the equations of the form

$$\left. \begin{aligned} \rho &= \rho_L \text{ and } u = u_L \text{ for } x < st, \\ \rho &= \rho_R \text{ and } u = u_R \text{ for } x > st, \end{aligned} \right\} \quad (1.11)$$

where $\rho_L > 0$, $\rho_R > 0$, u_L , u_R , and s are constants. Then find conditions that these constants have to satisfy so that mass and momentum are conserved.

Hint 1: In a frame of reference moving with the shock, the fluxes of mass and momentum on each side of the shock must be equal.

Hint 2: Alternatively: use the integral form of the conservation laws across an interval containing the shock; i.e.: $a < st < b$.

Remark 1.1 *One may question whether it makes sense to consider shock wave solutions within the context of the constant entropy assumption.³ However, it can be shown (though we will not do it here) that the amount of entropy produced within a shock is proportional to the cube of the shock strength for weak shocks. Hence, as long as we use the equations in situations where only weak shocks arise, it does make sense to consider shocks within the context of a constant entropy assumption.*

2 Special Problems.

2.1 Introduction: Eikonal equation.

Consider situations like the ones described in the examples below

- A.** Imagine a wave-front of some sort, propagating into a medium. For example, a combustion front, across which some chemical reactions in a solid combustible media are taking place. Then the fresh (unburnt) media is ahead of the front, and the burnt media is behind it.

If the chemical reactions happen fast enough, it is reasonable to approximate the burning region as infinitely thin, and model the front by a surface moving in space. This surface, and its evolution in time, can then be described by a function $\Phi = \Phi(\mathbf{r})$ such that the level surface

$$\Phi(\mathbf{r}) = t \tag{2.12}$$

is the wave-front at time t . **Notation:** here \mathbf{r} is the position vector in space. Thus, in 2-D $\mathbf{r} = (x, y)$, and in 3-D $\mathbf{r} = (x, y, z)$.

Of course, in order for this description to be useful, we need some equation for Φ , so that (for example) given that we know the position of the wave-front at some time t_0 (that is, we know the level surface $\Phi(\mathbf{r}) = t_0$), we can predict it for future times. We do this next.

If the media is isotropic, it is reasonable to assume that, at each point in space, the wave-front propagates normal to itself at some known velocity $c = c(\mathbf{r}) > 0$, which we can either get by direct measurement or (in principle, at least) from an analysis of the detailed physics at the burning front. With this assumption, it can be shown that **the phase function** satisfies the

$$\mathbf{Eikonal equation:} \quad (c \nabla \Phi)^2 = 1. \tag{2.13}$$

Note: We have assumed here the simplest of situations, where the speed of propagation depends only on the properties of the unburnt media ahead of the front (so that $c = c(\mathbf{r})$). More complicated situations can arise, where the speed of propagation depends also on other factors

³Constant entropy presumes adiabatic conditions, while changes at a shock are anything but slow. Hence shocks must produce entropy.

(such as the local curvature of the front), or other variables (such as the local temperature) for which extra equations are needed.

- B.** Another circumstance under which the Eikonal equation (2.13) arises is the one where high frequency waves (say, light or sound) propagate into some media. The idea is as follows: suppose that the governing equation is the wave equation

$$u_{tt} = \operatorname{div}(c^2 \nabla u), \quad \text{where } c = c(\mathbf{r}) > 0. \quad (2.14)$$

Consider now a stationary wave situation, where the field u is vibrating everywhere with the same frequency, and all the wave-fronts move along the same path (think of the light coming from some source, which does not change in time, nor moves, and is illuminating a region of space where nothing moves or changes either). If the wave-length is short enough, then near each point in space (and time) the wave-fronts will be nearly plane and will move normal to themselves at speed c . In this case, if we (again) introduce a **phase function** Φ to describe the wave-fronts, equation (2.13) will apply.

Notice that in this second situation there is not a single wave-front, but many (the light source emits as many wave-fronts per unit of time as its frequency). However, all the wave-fronts behave in the same way, taking the same sequence of positions in time (with only a time delay differentiating one wave-front from the other). That is, the wave-fronts are given by the equation

$$\Phi(\mathbf{r}) = t + \zeta, \quad (2.15)$$

where ζ is the time delay.

Remark 2.2 *The situation described in **B** above is the mathematical formulation of the idea that light (or high frequency sound waves) propagates along **rays** (which are straight on an homogeneous media, and curved otherwise). The Eikonal equation can be written in terms of characteristics (see **Part II** below), with the characteristics giving the rays along which light propagates.*

Furthermore, it can be shown (though we will not do this here) that the energy is carried along characteristics. Thus, when the rays diverge, the energy density (light intensity) in the waves goes down. Similarly, the light intensity goes up when the rays converge. In particular, when the rays cross, an infinite energy density is predicted. Of course, we cannot take this at face value (since when rays cross, the wavefronts develop infinite curvature, and the approximations made in deriving the Eikonal equation break down). Nevertheless, it is still true that the energy density becomes large at the locations where the rays cross.

*The envelope of the rays (see **Part II** below) for the Eikonal equation is thus very important, since it gives the location in space where we should expect to find large wave amplitudes. It is very easy to locate these places when dealing with light, since they correspond to very bright curves and surfaces*

in space; known as caustics. You can easily see caustics if you put a glass filled with water, on a white surface, under a bright light.

2.2 Statement: Eikonal equation (part 1).

Derive the Eikonal equation (2.13) to describe the situation where a moving surface, given by $\Phi(\mathbf{r}) = t$, moves at speed $c = c(\mathbf{r})$ (normal to itself) at each point in space.

Hint 2.1 Consider the wave-front at time t , and let $\hat{\mathbf{n}}$ be the unit normal to the wave-front at any place along it (pointing into the direction of propagation.⁴) Then because the wave-front propagates normal to itself at velocity c , it should be that:

\mathbf{r} is in the wave-front at time t if and only if $\mathbf{r} + \hat{\mathbf{n}} c dt$ is in the wave-front at time $t + dt$.

In other words

$$\Phi(\mathbf{r}) = t \iff \Phi(\mathbf{r} + \hat{\mathbf{n}} c dt) = t + dt. \quad (2.16)$$

2.3 Statement: Eikonal equation (part 2).

Consider the case of an homogeneous two dimensional media, where the speed of propagation c is a constant. In this case we can choose non-dimensional variables so that $\mathbf{c} \equiv \mathbf{1}$ and the Eikonal equation reduces to

$$(\nabla\Phi)^2 = 1, \quad (2.17)$$

where $\Phi = \Phi(x, y)$ and the wavefronts are the curves $\Phi(x, y) = \text{constant}$. This last equation has a characteristic form, given by:

$$\frac{d\mathbf{r}}{dt} = \mathbf{k} \quad \text{and} \quad \frac{d\mathbf{k}}{dt} = 0, \quad \text{along which} \quad \frac{d\Phi}{dt} = 1. \quad (2.18)$$

Here $\mathbf{r} = (x, y)$ is the position vector in space and $\mathbf{k} = \nabla\Phi$ is the wavenumber vector, which is normal to the wavefronts and (from (2.17)) also a unit vector. A solution of the Eikonal equation is determined once we give a wavefront, which can then be used to provide initial values for these characteristic equations.

JARGON: $\left\{ \begin{array}{l} \text{The curves in space } \mathbf{r} = \mathbf{r}(t) \text{ — given by the solution of} \\ \text{the characteristic equations (2.18) — are called } \mathbf{rays}. \end{array} \right.$

Consider now the situation where the wave front $\Phi = 0$ is the parabola $y = x^2$. Then

(a) Compute the characteristics for this situation.

⁴You can easily get $\hat{\mathbf{n}}$ from $\nabla\Phi$.

- (b) Consider the family of curves in space given by the rays just computed as part of doing item (a). Find the envelope of this family and express it in the form $y = y(x)$.
- (c) Plot the envelope computed in part (b).

Note that this problem has relatively simple algebra and you should not have big hassles with it, **provided you are careful, organized, and keep your notation straight.**

SOME HINTS AND NOTATION:

The initial wavefront $\Phi = 0$ can be described by the parametric equations $x = s$ and $y = s^2$, where $-\infty < s < \infty$ is the parameter. The unit tangent and unit normals to this curve are

$$\hat{\mathbf{t}} = \left(\frac{1}{\sqrt{1+4s^2}}, \frac{2s}{\sqrt{1+4s^2}} \right) \quad \text{and} \quad \hat{\mathbf{n}} = \left(\frac{-2s}{\sqrt{1+4s^2}}, \frac{1}{\sqrt{1+4s^2}} \right). \quad (2.19)$$

Then for each s we have a characteristic, where the formula for $\hat{\mathbf{n}}$ defines the initial value for \mathbf{k} .

After you solve the characteristic equations, you will end up with formulas for the rays of the form $x = x(s, t)$ and $y = y(s, t)$, where each s defines an individual ray. The rays are given parametrically, with t the parameter along each ray. You should find it rather easy to eliminate the parameter t from the formula for the rays, so that you can write the ray labeled by s as the solution to an equation of the form

$$F(x, y, s) = 0. \quad (2.20)$$

Once you do this, you'll have the standard form used to compute envelopes. From this, getting the envelope in the form $x = x(s)$ and $y = y(s)$ is rather straightforward. Eliminating s from these two last expressions should now be easy, and it will give you the requested form $y = y(x)$ for the envelope. This will end up being a rather simple formula, that you can plot without trouble.

THE END.