18.311 — MIT (Spring 2014)

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Problem Set # 04. Due: Friday April 4.

Turn it in before 3:00 PM, in the boxes provided in Room E18-366.

IMPORTANT:

— Turn in the regular and the special problems **stapled in two SEPARATE** packages.
— **PRINT your name** in each page of your answers. ***PRINT*** (the pencil will not break).
— In page one of each package **print the names** of the other members of your group.

Contents

1 Regular Problems.

1.1 Statement: Haberman problem 69.01.

Show that for an *observer moving with the traffic*, the rate of change of the measured density is

$$\frac{d\rho}{dt} = (u - c(\rho))\,\rho_x, \tag{1.1}$$

where $c = \dfrac{dq}{d\rho}$.

1.2 Statement: TFPa11. Longest queue through a light.

A traffic signal (at $x = 0$) is green for $0 \le t \le T$, and red for all other times. If $\rho(x,\,0) = \rho_j$ for $x \le 0$, $\rho(x,\,0) = 0$ for $x > 0$, and $q = (4\,q_m/\rho_j^2)\,\rho\,(\rho_j - \rho)$, determine the trajectory of the last car to make the light. What is the longest traffic queue that can pass through the intersection during the green light?

1.3 DiAn21 statement: Non-dimensional form.

Consider the problem (5^{th} order Linear KDV equation Initial Value Problem)

$$\frac{\partial \tilde{u}}{\partial \tilde{t}} + \alpha\,\frac{\partial \tilde{u}}{\partial \tilde{x}} + \beta\,\frac{\partial^5 \tilde{u}}{\partial \tilde{x}^5} = 0, \quad -\infty < x < \infty \;\text{ and }\; t > 0, \quad \tilde{u}(\tilde{x}, 0) = \frac{u_m\,d^2}{(d^2 + \tilde{x}^2)}, \tag{1.2}$$

where (a) $\alpha > 0$, $\beta > 0$, $u_m > 0$, and $d > 0$ are dimensional constants; (b) $\tilde{u}$ is density of some quantity (wiggies per unit length); and (c) tildes (i.e.: $\tilde{u}$, $\tilde{x}$, and $\tilde{t}$) denote dimensional variables.

1. **What are the dimensions of the constants $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{u_m}$, and $\boldsymbol{d}$?**
2. **Introduce a-dimensional variables**[1] $\boldsymbol{u}$, $\boldsymbol{x}$, **and $\boldsymbol{t}$, so that the equation takes the form** $$\boldsymbol{u_t + u_x + \gamma\, u_{xxxxx} = 0}, \tag{1.3}$$ where γ is a constant without dimensions, and the initial condition involves no free constants.[2]

1.4 Statement: TFPb09. Quasi-linear equation solution.

Find the solution to

$$x^2\,\psi_x - \psi^2\,\psi_y = 0\,, \tag{1.4}$$

with $\boldsymbol{\psi = x}$ on $\boldsymbol{y = x}$, for $\boldsymbol{x > 0}$. **Where is the solution defined, and why?**

1.5 Statement: TFPa21. Characteristics for 3D scalar quasi-linear equation.

Consider a p.d.e. of the form

$$P(x,\,y,\,z,\,\psi)\,\psi_x + Q(x,\,y,\,z,\,\psi)\,\psi_y + R(x,\,y,\,z,\,\psi)\,\psi_z = W(x,\,y,\,z,\,\psi)\,, \tag{1.5}$$

[1] That is, $x = \tilde{x}/L$, $t = \tilde{t}/T$, and $u = \tilde{u}/U$, for appropriate choices of a length L, a time T, and a density U.
[2] That is, no letter constants in it.

for the real valued function $\psi = \psi(x,\, y,\, z)$, for some given coefficient functions P, Q, R, and W. Assume that the values of ψ are given on some surface Γ. Specifically, let the surface be described (parametrically) by

$$x = X(u,\, v), \quad y = Y(u,\, v), \quad \text{and} \quad z = Z(u,\, v), \tag{1.6}$$

by some functions X, Y, and Z, defined in some region Γ_p of the u-v plane. Then

$$\psi = \Psi(u,\, v), \tag{1.7}$$

for some function Ψ.

(a) Give a detailed description of how you would solve the problem for ψ above, using the method of characteristics.

(b) Use your method to find the solution to

$$A\,\psi_x + B\,\psi_y + \psi_z = -\psi, \tag{1.8}$$

where A and B are constants, and $\boldsymbol{\psi = \Psi(x, y)}$ for $\boldsymbol{z = 0}$.

1.6 Statement: TFPa18. Envelopes for families of curves.

Find the envelope of each of the following family of curves:

- **a.** $y = c\,x - (1 + c^2)\,x^2$.
- **b.** $x^2 + a^2\,y^2 = a$.
- **c.** $(1 - c)\,x + c\,y = c - c^2$.

2 Special Problems.

2.1 DiAn26 statement: Non-dimensional form.

Consider the problem (Burgers' equation Boundary Value Problem)

$$\frac{\partial \tilde{u}}{\partial \tilde{t}} + \alpha\,\tilde{u}\,\frac{\partial \tilde{u}}{\partial \tilde{x}} - \beta\,\frac{\partial^2 \tilde{u}}{\partial \tilde{x}^2} = 0, \quad 0 < x < d \ \text{ and } \ t > 0, \quad \tilde{u}(0, \tilde{t}) = \tilde{u}(d, \tilde{t}) = 0, \tag{2.9}$$

with some initial data. Here (a) $\alpha > 0$, $\beta > 0$, and $d > 0$ are dimensional constants; (b) $\tilde{u}$ is density of some quantity (wiggies per unit length); and (c) tildes (i.e.: $\tilde{u}$, $\tilde{x}$, and $\tilde{t}$) denote dimensional variables.

1. **What are the dimensions of the constants $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, and $\boldsymbol{d}$?**
2. **Introduce a-dimensional variables[3] $\boldsymbol{u}$, $\boldsymbol{x}$, and $\boldsymbol{t}$, so that the equation takes the form** $$\boldsymbol{u_t + u\,u_x - u_{xx} = 0}, \tag{2.10}$$ where $0 < x < 1$.

[3] That is, $x = \tilde{x}/L$, $t = \tilde{t}/T$, and $u = \tilde{u}/U$, for appropriate choices of a length L, a time T, and a density U.

2.2 Statement: TFPa20. Semi-linear 1st order eqn. & characteristics.

Consider the equation

$$x^2\,\psi_x - x\,y\,\psi_y = \psi^2\,, \tag{2.11}$$

subject to $\boldsymbol{\psi = 1}$ on the curve $\boldsymbol{\Gamma}$ given by $\boldsymbol{x = y^2}$. This is a semi-linear problem (the terms involving derivatives of the solution are linear), that can be written in terms of characteristics.

A. Compute the characteristic curves that cross the curve Γ, as follows: (i) Parameterize the curve Γ, say: $\boldsymbol{x = \xi^2}$ and $\boldsymbol{y = \xi}$, for $\boldsymbol{-\infty < \xi < \infty}$. (ii) Write the ode for the characteristic curves, in terms of some parameter (say, s) along each curve. (iii) Solve the ode for the characteristics, with the condition that $\boldsymbol{x = \xi^2}$ and $\boldsymbol{y = \xi}$, for $\boldsymbol{s = 0}$.

B. Draw the characteristics, in the x-y plane, that you just computed. Which region of the plane do the curves cover? What happens with the characteristic corresponding to $\xi = 0$?

C. Solve the ode that ψ satisfies along each characteristic. Eliminate the parameters ξ and s in terms of x and y, and write an explicit formula for the solution $\psi = \psi(x,\, y)$ to (2.11).

D. Where is the solution ψ defined? **Hint.** *Be careful with your answer here! What happens with ψ along each characteristic, far enough from Γ? What happens with the $\xi = 0$ characteristic?*

THE END.