

18.311 — MIT (Spring 2014)

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Problem Set # 03. Due: Fri. March 14.

Turn it in before 3:00 PM, in the boxes provided in Room E18-366.

IMPORTANT:

- Turn in the regular and the special problems **stapled in two SEPARATE** packages.
- **Print your name** in each page of your answers.
- In page one of each package **print the names** of the other members of your group.

Update: fixed a problem with problem 1.4 below.

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1 Regular Problems.

1.1 Statement: Haberman problem 61.03.

Assume that a velocity field, $u = u(x, t)$ exists. Show that the acceleration of an individual car is given by

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x}. \quad (1.1)$$

1.2 Statement: Haberman problem 63.06.

Assume that $u = u(\rho)$. If α denotes the car's acceleration, show that

$$\alpha = -\rho \frac{du}{d\rho} \frac{\partial u}{\partial x}. \quad (1.2)$$

Is the minus sign here reasonable?

1.3 Statement: Haberman problem 67.01.

Suppose that

$$u = u_{\max} \left(1 - \frac{\rho}{\rho_{\max}} \right). \quad (1.3)$$

- (a) What is the density wave velocity if the traffic density is nearly ρ_0 everywhere.
- (b) Show that this density wave velocity is less than the car's velocity.

1.4 Statement: Haberman problem 71.02.

Consider the flow-density relationship used by H. Greenberg¹ to match Lincoln Tunnel data

$$q = a \rho \ln \left(\frac{\rho_{\max}}{\rho} \right),$$

where $a > 0$ is a constant. **Show that** the density wave velocity relative to a moving car is the same constant no matter what the density.

¹Greenberg H., *An Analysis of Traffic Flow*, Operations Research, **7**:79–85, (1959).

1.5 Problem 6302a. State laws and road capacity.

If cars obey (exactly) the typical state law² on following distances: (1) What is the road capacity? (2) At what density and velocity does this maximum occur? (3) Will increasing the speed limit increase the road's capacity? (4) Draw a plot of the flux q versus the density ρ . (5) Draw a plot of the flux q versus the velocity u .

The relationship of the car velocity u to the car density ρ prescribed by the state laws has the form

$$u = \min \left\{ u_M, V \left(\frac{1}{\rho L} - 1 \right) \right\}. \quad (1.4)$$

Here L is the car length, V is the “trigger” velocity in the law,³ and u_M is the speed limit. Assume that $L = 16 \text{ ft} = (1/330) \text{ mile}$, $V = 10 \text{ m.p.h.}$, and $u_M = 50 \text{ m.p.h.}$ (80 k.p.h.).

1.6 Statement: Linear 1st order PDE # 05.

Consider the problem

$$x u_x + (x + y) u_y = 1, \quad \text{with} \quad u(1, y) = y \text{ for } 0 < y < 1. \quad (1.5)$$

Question 1. Write the solution $u = u(x, y)$ in the region where it is uniquely determined.

Question 2. Describe the region in the plane where the solution to (1.5) is uniquely determined.

Question 3. Write all the functions $u = u(x, y)$ that satisfy (1.5) on $x > 0$ and $-\infty < y < \infty$.

Question 4. Write all the functions $u = u(x, y)$ that satisfy the pde in (1.5) for $x < 0$.

Question 5 (*challenge/optional*). What happens along $x = 0$? Can you produce solutions to the pde that are continuous in the “punctured” plane (plane minus the origin)?

2 Special Problems.

2.1 Statement: TFPa14. Solve using dimensionless variables.

Consider the traffic flow equation

$$\rho_t + q_x = 0, \quad (2.6)$$

with the quadratic flow $q = \frac{4 q_m}{\rho_j^2} \rho (\rho_j - \rho)$, where $q_m = \text{road capacity}$ and $\rho_j = \text{jamming density}$.

Assume that either

² See equation (1.4).

³ As in: Keep the distance to the next car at one car length L per every V in speed.

$$(a) \rho(x, 0) = \begin{cases} \rho_0 & \text{for } x \leq 0 \\ \frac{\rho_0}{1 + (x/d)^2} & \text{for } x \geq 0 \end{cases} \quad \text{or} \quad (b) \rho(x, 0) = \begin{cases} \rho_0 & \text{for } x \leq 0 \\ \frac{\rho_0}{1 + (x/d)} & \text{for } x \geq 0 \end{cases}$$

where $0 < \rho_0 < \rho_j$ and $d > 0$ are constants. Then

1. Write everything using dimensionless variables.
2. Write the solution to (a) and (b) above using characteristics.
3. Show that there are no characteristic crossings. Thus (2) gives the complete solution.
4. In the case of (b), solve for $\xi = \xi(x, t)$, and write the solution explicitly.

Note: in (4) ξ denotes the label for the characteristics, defined as follows: $x = \xi$ at $t = 0$.

2.2 Statement: TFPa19.

Linear first order equation and characteristics.

Find the solution to

$$x \psi_x - y \psi_y + \psi = 2x, \quad \text{with } \psi = 0 \text{ on the curve } \Gamma \text{ given by } y = 1. \quad (2.7)$$

Draw the characteristics which touch Γ . Where is the solution defined?

THE END.