18.311 — MIT (Spring 2014)

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Problem Set # 02. Due: Tue. March 4.

Turn it in before 3:00 PM, in the boxes provided in Room E18-366.

IMPORTANT:

— Turn in the regular and the special problems **stapled in two SEPARATE** packages.

— **Print your name** in each page of your answers.

— In page one of each package **print the names** of the other members of your group.

Contents

1 Regular Problems.

1.1 Statement: Haberman problem 57.06.

Consider an infinite number of cars, each designated by a number β. Assume that the car labeled by β starts from $x = \beta$ ($\beta > 0$) with zero velocity, and also assume it has a constant acceleration β.

(a) Determine the position and velocity of each car as a function of time.

(b) Sketch the path of a typical car.
(c) Determine the velocity field $u = u(x,\, t)$.
(d) Sketch the curves along which $u = u(x,\, t)$ is constant.

1.2 Statement: Haberman problem 60.01.

Consider a semi-infinite highway $0 \leq x < \infty$ (with no entrances of exits other than at $x = 0$). Show that the number of cars on the highway at time t is:

$$N_0 + \int_0^t q(0,\, \tau)\, d\tau, \tag{1.1}$$

where N_0 is the number of cars in the highway at time $t = 0$. You may assume that $\rho(x,\, t) \to 0$ as $x \to \infty$. Justify the equation **both:** directly (by physical reasoning), as well as by using the equation $\rho_t + q_x = 0$.

1.3 Statement: Haberman problem 60.03.

(a) Without any mathematics, explain why $\boldsymbol{\int_{a(t)}^{b(t)} \rho(x,\, t)\, dx}$ is constant if a and b (not equal to each other) are moving with the traffic.

(b) Using part **(a)**, re-derive the equation

$$\frac{d}{dt} \int_a^b \rho\,(x,\, t)\, dx = q(a,\, t) - q(b,\, t), \tag{1.2}$$

where $a < b$ are any two points in the road.

(c) Assuming $\dfrac{\partial \rho}{\partial t} = -\dfrac{\partial}{\partial x}(\rho\, u)$, verify mathematically that part **(a)** is valid.

1.4 Statement: Haberman problem 60.04.

If the traffic flow is increasing as x increases ($\dfrac{\partial q}{\partial x} > 0$), explain physically[1] why the density must be decreasing in time ($\dfrac{\partial \rho}{\partial t} < 0$).

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[1] You do not need any equations.

1.5 Statement: Haberman problem 67.04.

Consider the equation

$$(\rho_1)_t + c\,(\rho_1)_x = 0. \tag{1.3}$$

Suppose that we observe ρ_1 in a coordinate system moving at velocity v. Show that

$$(\rho_1)_t + (c - v)\,(\rho_1)_x = 0. \tag{1.4}$$

Does the car density ρ stay constant moving at the car velocity?

1.6 Statement: Linear 1st order PDE # 08.

Solve the problem below, using the method of characteristics: **(a)** Compute the characteristics, as done in the lectures, starting from each point in the data set. **(b)** Next solve for the solution u along each characteristic. **(c)** Finally, eliminate the characteristic variables ζ and s from the expression[2] for u obtained in step (b) — using the result in step (a) — to obtain the solution as a function of x and y.

$$(x - y)\,u_x + (x + y)\,u_y = x^2 + y^2, \tag{1.5}$$

with data $u(x, 0) = (1/2)\,x^2$ for $1 \le x < \exp(2\pi)$.

Further question: *Where in the* $(x,\, y)$ *plane does the problem above define the solution* u*?* That is: *what is the region of the plane characterized by the property that: through each point in this region there is exactly one characteristic connecting it with the curve where the data is given?*

2 Special Problems.

2.1 Statement: Linear 1st order PDE # 02.

Consider the following problem

$$x\,u_x + y\,u_y = 1 + y^2, \qquad \text{with} \qquad u(x,\,1) = 1 + x \quad \text{for } -\infty < x < \infty. \tag{2.6}$$

Part 1. Use the method of characteristics to solve this problem. Write the solution $\boldsymbol{u = u(x,\, y)}$ **(explicitly!)** as a function of $\boldsymbol{x}$ and $\boldsymbol{y}$ on $\boldsymbol{y > 0}$. **Hint.** *Write the characteristic equations:* $\dfrac{dx}{ds} = \ldots$, $\dfrac{dy}{ds} = \ldots$, *and* $\dfrac{du}{ds} = \ldots$. *Then solve these equations using the initial data (for* $s = 0$*)* $x = \tau$, $y = 1$, *and* $u = 1 + \tau$, *for* $-\infty < \tau < \infty$. *Finally, eliminate* s *and* τ, *to get* u *as a function of* x *and* y.

Part 2. Explain why $\boldsymbol{u = u(x,\, y)}$ is not determined by the problem above for $\boldsymbol{y \le 0}$ (you may use a diagram). **Hint.** *Draw, in the x-y plane, the characteristic curves computed in part 1.*

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[2] Here ζ is the label for each characteristic, and s is a parameter along the characteristics.

2.2 Statement: Linear 1st order PDE # 09 (surface evolution).

The evolution of a material surface can (sometimes) be modeled by a pde. In evaporation dynamics, where the material evaporates into the surrounding environment, consider a surface described in terms of its "height" $h = h(x,\, y,\, t)$ relative to the $(x,\, y)$-plane of reference. Under appropriate conditions, a rather complicated pde can be written[3] for h. Here we consider a (drastically) simplified version of the problem, where the governing equation is

$$h_t = \frac{A}{r}\, h_r, \quad \text{for } r = \sqrt{x^2 + y^2} > 0 \text{ and } t > 0, \quad \text{where } A > 0 \text{ is a constant.} \tag{2.7}$$

Axial symmetry is assumed, so that $h = h(r,\, t)$. Obviously, ***h* should be an even function of** r. This is both evident from the symmetry, and necessary in the equation to avoid singular behavior at the origin. Assume now

$$h(r,\, 0) = H(r^2), \tag{2.8}$$

where H is a smooth function describing a localized bump. Specifically: **(i)** $H(0) > 0$, **(ii)** H is monotone decreasing. **(iii)** $H \to 0$ as $r \to \infty$. **Note that** $h(r,\, 0)$ *is an even function of* r.

1. Using the theory of characteristics, write an explicit formula for the solution of (2.7 – 2.8).
2. Do a sketch of the characteristics in space time — i.e.: $r > 0$ and $t > 0$.
3. What happens with the characteristic starting at $r = \zeta > 0$ and $t = 0$ when $t = \zeta^2/2\,A$?
4. Show that the resulting solution is an even function of r for all times.
5. Show that, as $t \to \infty$, the bump shrinks and vanishes. *Hint: pick some example function H with the properties above, and plot the solution for various times. This will help you figure out why the bump shrinks and vanishes.*

THE END.

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[3] From mass conservation, with the details of the physics going into modeling the flux and sink/source terms.