Lecture topics for 18306.

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These are brief summaries of the 18.306 lectures. Further details can be found in the course web page notes, books, etc. These summaries will be updated from time to time. Check the date.

Contents

1 The heat/diffusion equation.

1.1 Random walks and Brownian motion in 1-D [S13].

For an alternative presentation of this topic, see the book by Salsa listed in the syllabus.

Consider a symmetric discrete random walk on the line, described by:

If the walker is at position x at time t, then at time $t+k$ it is at $x \pm h$, with probability $\frac{1}{2}$ each, (1.1.1)

where $k = \Delta t > 0$ and $h = \Delta x > 0$ are the discrete time and space steps, respectively. Hence the probability distribution function (pdf) for the process satisfies

$$p(x,\, t+k) = \frac{1}{2}\, p(x-h,\, t) + \frac{1}{2}\, p(x+h,\, t). \tag{1.1.2}$$

Note 1.1.1 *Strictly speaking (1.1.1) does not represent a single random walk, but infinitely many separate ones: one in each of the space-time grids $\{x_m = x_0 + m\,h,\ t_n = t_0 + n\,k\}$, where $0 \le x_0 < h$ and $0 \le t_0 < k$. Further: each choice of k and h produces a different set of walks. However, here we treat all these walks as a single one (with a single pdf) and justify it later — see remark 1.1.2.*

Note 1.1.2 *Note that (1.1.1) is a (rather coarse) simplification of the actual physics of real particles undergoing Brownian motion. For them the motion does not occur in steps of fixed length over fixed time intervals. A better model would be one in which the steps occur at random times, and have random lengths (the main approximation now being that we consider each jump as occurring instantaneously). However, as long as the jumps are independent, the details do not matter in the limit taken below — because of the Central Limit Theorem.*

Assume now that p in (1.1.2) is continuously differentiable (twice in time, thrice in space), write the equation in the form $p(x,\,t+k)-p(x,\,t)=\dfrac{1}{2}\left(p(x-h,\,t)+\dfrac{1}{2}\,p(x+h,\,t)-2\,p(x,\,t)\right)$, and expand in the limit $k, h \ll 1$. This yields

$$p_t = \frac{1}{2}\,\frac{h^2}{k}\,p_{x\,x} + O\left(\frac{h^3}{k},\,k\right). \tag{1.1.3}$$

If we now take the limit

$$h \downarrow 0 \quad \text{and} \quad h^2 = 2\,\nu\,k, \quad \text{where} \quad \nu > 0 \tag{1.1.4}$$

is a constant, we obtain the *diffusion equation* for p, with *diffusion coefficient* ν

$$p_t = \nu\,p_{x\,x}. \tag{1.1.5}$$

Note 1.1.3 *The limit in (1.1.4) is, of course, only an approximation in terms of the physics of Brownian motion. For example: consider the molecules in a gas. Then we interpret h and k as the mean distance and time between collisions. But then $v = h/k$ is related to the mean velocity of the molecules while in free flight between collisions. Clearly, $v \neq \infty$, as (1.1.4) implies — however, as long as it is large enough compared with the other scales in the problem, $v \approx \infty$ is acceptable.*

The particular situation where the walker position, say $x = 0$, is known at time $t = 0$ corresponds to the initial data, and solution,

$$p(x,\,0) = \delta(x) \quad \Longrightarrow \quad p = p_* = \frac{1}{\sqrt{4\,\pi\,\nu\,t}}\,\exp\left(-\frac{x^2}{4\,\nu\,t}\right). \tag{1.1.6}$$

Remark 1.1.1 Expected values. *The Fourier transform of the pdf can be used to easily compute expected values.*[1] *For example:*

$$P_*(\kappa,\,t) = \int_{-\infty}^{\infty} p_*(x,\,t)\,e^{i\,\kappa\,x}\,dx = \frac{1}{\sqrt{2\,\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\,s^2 + i\,\kappa\,\sqrt{2\,\nu\,t}\,s}\,ds = \exp\left(-\kappa^2\,\nu\,t\right). \tag{1.1.7}$$

Hence

$$E(x) = \int_{-\infty}^{\infty} x\,p_*(x,\,t)\,dx = -i\,\partial_\kappa P_*(0,\,t) = 0 \tag{1.1.8}$$

and

$$E(x^2) = \int_{-\infty}^{\infty} x^2\,p_*(x,\,t)\,dx = -\partial_\kappa^2 P_*(0,\,t) = 2\,\nu\,t. \tag{1.1.9}$$

That is: in unit time the particle diffuses, on average, a distance $\sqrt{2\,\nu}$. ♣

Remark 1.1.2 Justification of the limit taken. *As follows from the points in note 1.1.1, the expansions leading to (1.1.3) are not quite justified. For example, assume that $p_0 = p(x,\,0)$ is given at time $t = 0$ — with p_0 smooth. Then (1.1.2) defines p for all $t = n\,k$, but not values of t other*

[1] It plays the same role, for stochastic variables, that the generating function plays for discrete random variables.

than these. Furthermore, even for these values p is not just a function of x and t, as we assumed to get (1.1.3). That is:

$$p(x,\,k)=\frac{1}{2}\,p_0(x+h)+\frac{1}{2}\,p_0(x-h),\quad p(x,\,2\,k)=\frac{1}{4}\,p_0(x+2\,h)+\frac{1}{2}\,p_0(x)+\frac{1}{4}\,p_0(x-2\,h),$$

and so on. This means that, in general, it should be $p=p(x,t,h,\,k)$ *... thus one may ask: where are the partial derivatives*[2] *with respect to h and k in (1.1.3)?*

Rather than try to "fix" the derivation of (1.1.5) given by (1.1.3–1.1.4), we present here an independent justification — which, unfortunately, lacks the simplicity of (1.1.3–1.1.4). We start by writing the exact solution to (1.1.2) for the times $t_n=n\,k$*, given some initial data* $p_0=p(x,\,0)$*. This we can do by finding the normal modes*[3] *for the discrete-time equation*

$$p_{n+1}(x)=\frac{1}{2}\,p_n(x-h)+\frac{1}{2}\,p_n(x+h), \tag{1.1.10}$$

where $p_n(x)=p(x,\,t_n)$*. That is*

$$p_n(x)=\frac{1}{2\,\pi}\int_{-\infty}^{\infty}(\cos\omega\,h)^n\,e^{-i\,\omega\,x}\,P_0(\omega)\,d\,\omega, \tag{1.1.11}$$

where P_0 *is the Fourier transform of* p_0 *— which we assume is smooth enough to guarantee rapid decay of* P_0 *as* $|\omega|\to\infty$*. Next we consider the limit — note that this is the same as (1.1.4) —*

$$T>0 \text{ fixed, with } k=\frac{T}{N},\quad h=\sqrt{2\,\nu\,k},\quad \text{and } N\to\infty. \tag{1.1.12}$$

Thus, for any $t\geq 0$ *fixed and bounded* ω

$$(\cos\omega\,h)^n=\left(1-\frac{\nu}{n}\,t_n\,\omega^2+O\left(\frac{\omega^4}{N^2}\right)\right)^n\quad\longrightarrow\quad\exp(-\nu\,\omega^2\,t), \tag{1.1.13}$$

where n is the integer part of $\frac{t}{T}\,N$ *(so that* $t_n\to t$*). In addition* $\left|(\cos\omega\,h)^n\,e^{-i\,\omega\,x}\,P_0(\omega)\right|\leq|P_0(\omega)|$*. It follows that*

$$p_n\quad\longrightarrow\quad\frac{1}{2\,\pi}\int_{-\infty}^{\infty}e^{-i\,\omega\,x-\nu\,\omega^2\,t}\,P_0(\omega)\,d\,\omega, \tag{1.1.14}$$

which is exactly the solution to (1.1.5) with the given initial data. ♣

Remark 1.1.3 Useful integrals: $\displaystyle\int_{-\infty}^{\infty}e^{-x^2}\,dx=\sqrt{\pi}$ *and* $\displaystyle\int_{-\infty}^{\infty}e^{i\,k\,x-\frac{1}{2}\,x^2}\,dx=\sqrt{2\,\pi}\,e^{-\frac{1}{2}\,k^2}$.

Proof. The first integral follows because $\displaystyle\left(\int_{-\infty}^{\infty}e^{-x^2}\,dx\right)=\iint e^{-x^2-y^2}\,dx\,dy=\int_0^{2\,\pi}d\theta\int_0^{\infty}e^{-r^2}r\,dr=\pi.$

The second follows by moving the contour in the complex plane from $z=x$ to $z=i\,k+x$. ♣

[2] Even worse: how is p defined for $n\,k<t<(n+1)\,k$, in such a way that the result is smooth and satisfies (1.1.2)?

[3] The ansatz $p_n(x)=G^n\,\phi(x)$ leads to the **self-adjoint** eigenvalue problem $G\,\phi(x)=\frac{1}{2}\,\phi(x+h)+\frac{1}{2}\,\phi(x-h)$.

1.2 Duhamel's principle [S16].

For an alternative presentation of this topic, see the book by Salsa listed in the syllabus.

Consider a nonhomogeneous (forced) linear problem of the form

$$Y_t = \mathcal{L}\,Y + F \;\text{ for }\; t > 0, \quad \text{with initial data }\; Y(0) = 0. \tag{1.2.1}$$

Here $\mathcal{L}$ is a (time independent) linear operator in some vector space (see note 1.2.1), where the forcing function $F = F(t)$ and the solution $Y = Y(t)$ take values in. Define now $W = W(t,\, s)$ by

$$W_t = \mathcal{L}\,W \;\text{ for }\; t > s, \quad \text{with initial data }\; W(s,\, s) = F(s). \tag{1.2.2}$$

Then **the solution to (1.2.1) is**

$$\boldsymbol{Y(t) = \int_0^t W(t,\, s)\, ds}. \tag{1.2.3}$$

Proof. Take the time derivative of (1.2.3), and assume that $\mathcal{L}$ commutes with the integration in s.

Note 1.2.1 *The vector space to which the solutions to (1.2.1) belong may be finite dimensional, in which case (1.2.1) is an ode, or not. When the space is not finite dimensional (e.g.: example 1.2.1) we will assume that the space (and the operator $\mathcal{L}$) have sufficient extra structure to guarantee that the various initial value problems above are all well-posed.*

Example 1.2.1 *Consider the forced heat equation problem (here $\nu > 0$ is a constant)*

$$u_t - \nu\, u_{xx} = f(x,\, t) \;\textit{ for }\; t > 0 \;\textit{ and }\; -\infty < x < \infty, \quad \textit{with initial data }\; u(x,\, 0) = 0. \tag{1.2.4}$$

Then $\boldsymbol{u(x,\, t) = \int_0^t w(x,\, t,\, s)\, ds,}$ *where w is defined by*

$$w_t - \nu\, w_{xx} = 0 \;\textit{ for }\; t > s \;\textit{ and }\; -\infty < x < \infty, \quad \textit{with initial data }\; w(x,\, s,\, s) = f(x,\, s). \tag{1.2.5}$$

In particular, using the fundamental solution for the heat equation

$$G_f(x,\, t) = \frac{1}{\sqrt{4\,\pi\,\nu\,t}}\, \exp\left(-\frac{x^2}{4\,\nu\,t}\right) \tag{1.2.6}$$

we can write

$$u = \int_0^t ds \int_{-\infty}^{\infty} dy\; G_f(x-y,\, t-s)\, f(y,\, s). \tag{1.2.7}$$

This last formula shows that **the Green function for the problem in (1.2.4) is**

$$\boldsymbol{G_*(x,\, t) = 0} \;\textit{ for }\; \boldsymbol{t < s} \quad \textit{and} \quad \boldsymbol{G_*(x,\, t) = G_f(x-y,\, t-s)} \;\textit{ for }\; \boldsymbol{t > s.} \tag{1.2.8}$$

That is $\boldsymbol{u = G_*}$ *satisfies*

$$\boldsymbol{u_t - \nu\, u_{xx} = \delta(x-y)\,\delta(t-s)} \;\textit{ for }\; \boldsymbol{t > 0} \;\textit{ and }\; \boldsymbol{-\infty < x < \infty,} \tag{1.2.9}$$

with initial data $\boldsymbol{u(x,\, 0) = 0}$ *— where* $\boldsymbol{s > 0}$ *and* $\boldsymbol{-\infty < y < \infty}$ *are constants.*

Next we show that ***(1.2.8) is, precisely, Duhamel's principle (1.2.3) applied to (1.2.9).***

In this case Duhamel's principle says that
$$\boldsymbol{G_*(x,\, t) = \int_0^t w(x,\, t,\, \tau)\, d\tau,} \tag{1.2.10}$$
with w *is defined by*
$$w_t - \nu\, w_{x\,x} = 0 \quad \textit{for } t > \tau \; \textit{ and } \; -\infty < x < \infty, \;\; \textit{with } w(x,\, \tau,\, \tau) = \delta(x-y)\,\delta(\tau - s). \tag{1.2.11}$$
The solution to this is
$$w = G_f(x-y,\, t-\tau)\,\delta(\tau - s), \tag{1.2.12}$$
which upon substitution into (1.2.10) yields (1.2.8).

However, (1.2.12) involves a Dirac delta in the solution, so a bit of caution is needed.[4] *The approach in (1.2.4–1.2.7) avoids these difficulties.*

1.3 Tychonov's example [S17].

For an alternative presentation of this topic, see the book by Salsa listed in the syllabus.

It is a bit surprising that, unless extra restrictions on the solutions are imposed, the initial value problem for the heat equation on unbounded domains does not have a unique solution. Either of the following two restrictions is enough to guarantee uniqueness

1. The solution is **non-negative.**
2. There are constants a and b such that $|\boldsymbol{T}| \leq \boldsymbol{a\, e^{b\,x^2}}$. (1.3.1)

A proof of these results is beyond the scope of this course. Item **2** is a corollary of the:

Global maximum principle:

If $u_t - u_{x\,x} \leq 0$ for $t > 0$ and $-\infty < x < \infty$, and $u \leq a\, e^{b\,x^2}$ for some constants a and b, then the supremum of u occurs for $t = 0$. (1.3.2)

Tychonov's solution provides an (in fact, many) examples showing that uniqueness is lost if either **1** or **2** fails.[5] The solution is given by
$$T = \sum_0^\infty \frac{1}{(2\,n)!}\, x^{2\,n}\, h^{(n)}(t), \tag{1.3.3}$$
where $h^{(n)}$ is the n^{th} derivative of $\boldsymbol{h(t) = \exp(-1/t^p)}$, for some $p > 1$. It can be shown that

A. $\boldsymbol{T}$, as defined by (1.3.3), satisfies $\boldsymbol{T_t = T_{x\,x}}$. This is easy to do by differentiating the series term by term — justified because the series, and the series of term-by-term derivatives, converge absolutely and uniformly in any set where, for some constant M, $x^2 \leq M\,t$ [see (1.3.6)].

[4] Before taking it seriously, we should check it by formulating everything in terms of test functions. And we also ought to check the proof of Duhamel's principle in this context. Turns out everything checks, but we have not done it here.

[5] Clearly, a situation with lack of uniqueness must violate both **1** and **2**.

B. $\boldsymbol{T}$, as defined by (1.3.3), satisfies $\boldsymbol{T(x,0) = 0}$, because $h^{(n)}(0) = 0$ for all n. In fact, it can be shown that all the partial derivatives of $\boldsymbol{T}$ vanish for $\boldsymbol{t = 0}$! On the other hand $\boldsymbol{T_{2nx}(0,\, t) = h^{(n)}(t) \neq 0}$. This shows that uniqueness for the initial value problem for the heat equation in the line fails if no restrictions are imposed — any linear combination of solutions of the form in (1.3.3) can be added to the solution.

The main fact that allows a proof convergence is: There exists constants $\boldsymbol{0 < \mu,\, K < 1}$ such that

$$\left|h^{(n)}(t)\right| \leq \frac{n!}{(\mu\, t)^n}\, \exp\left(-\frac{K}{t^p}\right) \quad \text{for } \textbf{all} \quad t > 0 \quad \text{and } \textbf{all} \quad n = 0,\, 1,\, 2, \ldots \tag{1.3.4}$$

This is not too hard to show using the following consequence of Cauchy's theorem (applied to the function e^{-1/z^p}, analytic for $\mathrm{Re}(z) > 0$)

$$h^{(n)}(t) = \frac{n!}{2\,\pi\, i} \int_\Gamma \frac{e^{-1/z^p}}{(z-t)^{n+1}}\, dz \quad \text{for any} \quad t > 0 \;\text{ and }\; 0 < \mu < 1, \tag{1.3.5}$$

where Γ is a circle of radius $r = \mu\, t$, centered at $z = t$, tracked in the counter-clockwise direction — i.e.: $z = t + \mu\, t\, e^{i\,\phi}$, with $-\pi \leq \phi \leq \pi$.

From (1.3.4) it follows that

$$\begin{aligned} \sum_0^\infty \left|\frac{1}{(2\,n)!}\, x^{2\,n}\, h^{(n)}(t)\right| \;&\leq\; \sum_0^\infty \frac{n!}{(2\,n)!}\, \frac{x^{2\,n}}{(\mu\, t)^n}\, \exp\left(-\frac{K}{t^p}\right) \\ &\leq\; \sum_0^\infty \frac{1}{n!}\, \frac{x^{2\,n}}{(2\,\mu\, t)^n}\, \exp\left(-\frac{K}{t^p}\right) \;=\; \exp\left(\frac{x^2}{2\,\mu\, t} - \frac{K}{t^p}\right). \end{aligned} \tag{1.3.6}$$

which shows the absolute convergence of the series for T. Similar calculations can be done for the series of term-by-term derivatives of T.

1.4 Nonlinear diffusion equation and finite propagation speed. [S18].

For an alternative presentation of this topic, see the book by Salsa listed in the syllabus.

Porous media equation.

Consider a gas flowing through a porous media, with density ρ and flow velocity $\vec{u}$. Then the **conservation of mass** gives the equation

$$\rho_t + \mathrm{div}(\rho\, \vec{u}) = 0. \tag{1.4.1}$$

On the other hand, Darcy's law for porous media says that

$$\vec{u} = -\frac{\mu}{\nu}\, \mathrm{grad}\, p, \tag{1.4.2}$$

where $0 < \mu =$ permeability of the media, $0 < \nu =$ gas viscosity, and $p =$ gas pressure. Finally, if the flow is adiabatic, the **equation of state** $\boldsymbol{p = p(\rho)}$ follows (e.g.: $p \propto \rho^\gamma$). Putting this all together yields the nonlinear diffusion equation

$$\rho_t = \operatorname{div}\left(\frac{\mu}{\nu}\,\rho\,a^2\,\operatorname{grad}\rho\right), \quad \text{where } \; a^2 = a^2(\rho) = \frac{dp}{d\rho} > 0. \tag{1.4.3}$$

We will investigate some of the consequences of a nonlinear diffusion coefficient in the problems *Nonlinear diffusion from a point source #nn* and *Nonlinear diffusion from a point seed* of the problem series: *Point Sources and Green's functions.*

The End.