Take home exam # 2 (MIT 18.306, Spring 2007).

Rodolfo R. Rosales (Math. Dept., room 2-337, Cambridge, MA 02139).

May 7, 2007.

Course TA: Nikkos Savva, MIT, Dept. of Mathematics, Cambridge, MA 02139.

Due: Last lecture.

1 Statements for the assigned problems.

1.1 Statement: Gas Dynamics strong shock conditions.

Consider the 1-D Euler equations of Gas Dynamics, and a right shock wave. In a frame of reference moving with the wave, the equations governing the propagation of the shock are

$$\rho_0\, u_0 = \rho_1\, u_1 < 0 \qquad \text{(conservation of mass)}, \tag{1.1}$$

$$\rho_0\, u_0^2 + p_0 = \rho_1\, u_1^2 + p_1 \qquad \text{(conservation of momentum)}, \tag{1.2}$$

$$\rho_0\, u_0\, E_0 + p_0\, u_0 = \rho_1\, u_1\, E_1 + p_1\, u_1 \qquad \text{(conservation of energy)}, \tag{1.3}$$

$$\rho_0 < \rho_1 \qquad \text{(entropy)}, \tag{1.4}$$

where ρ is the gas density, u is the flow velocity, p is the pressure, $E = \dfrac{1}{2}\,u^2 + e$ is the energy per unit mass, e is the internal energy per unit mass, the subscript 0 (resp. 1) indicates values immediately ahead (resp. behind) the shock, and $\rho_0\, u_0 < 0$ because the flow is from right to left across the shock.

Equation (1.3) is equivalent to $\rho_0\, u_0 \left(\dfrac{1}{2}\,u_0^2 + h_0\right) = \rho_1\, u_1 \left(\dfrac{1}{2}\,u_1^2 + h_1\right)$ where $h = e + \dfrac{p}{\rho}$ is the enthalpy. Hence, using (1.1), we see that **equation (1.3) can be replaced by:**

$$\frac{1}{2}\,u_0^2 + h_0 = \frac{1}{2}\,u_1^2 + h_1. \tag{1.5}$$

Furthermore, assume a **strong shock in a polytropic gas**, so that

$$h = \frac{\gamma\, p}{(\gamma - 1)\,\rho} \iff a^2 = \gamma\,\frac{p}{\rho}, \qquad \text{and} \qquad M = -\frac{u_0}{a_0} \gg 1, \tag{1.6}$$

where $\gamma > 1$ is the ratio of specific heats, $a > 0$ is the sound speed, and M is the Mach number with respect to the flow ahead. Note that $M > 1$ is equivalent to equation (1.4).

DERIVE (approximate, leading order) expressions for u_1, ρ_1, and p_1 in terms of u_0, ρ_0, and p_0.

HINT: Use equation (1.6) to eliminate p and h — from equations (1.2) and (1.5) — in favor of a^2. Then use $M \gg 1$ to simplify the resulting equations. From then on, it should be smooth sailing.

PART II: rewrite the shock equations in terms of the coordinate frame where the fluid ahead of the shock is at rest, with $U > 0$ the shock velocity relative to the flow ahead.

1.2 Statement: Cerenkov radiation and Mach cone.

Cerenkov radiation is the electromagnetic radiation emitted when a charged particle passes through an insulator at speed greater than the light speed in the medium. It shows up as a glowing blue cone of light with the traveling particle at its tip. It is somewhat analogous to the sonic boom produced by a supersonic aircraft. Below a simple model that captures the essence of the phenomena.

Consider a point traveling along a straight line at speed v, forcing the wave equation in 3-D (for an homogeneous and isotropic media). Assume also that $\mathrm{v} > c$, where c is the wave speed. In appropriate non-dimensional units, the mathematical problem is

$$u_{tt} - \Delta\, u = \delta(x)\,\delta(y)\,\delta(z - \beta\, t), \tag{1.7}$$

where $\beta = \mathrm{v}/c > 1$ and $\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2$ is the Laplace operator in 3-D. We are *interested in the solution of this problem when the point moves into a media at rest,*[1] *in unbounded space,*[2] *and all the transients are gone* — i.e.: the motion started far in the past.

A. Show that the problem can be reduced to an initial value problem for the wave equation in 2-D, and use this to find the solution. What is the half angle of the Cerenkov cone?

B. What if $\boxed{0 \le \beta < 1?}$ Can the problem be reduced to a 2-D initial value problem? Why not?

HINT-1. (An 18.03 hint!) Example: how to reduce to an initial value problem the impulse problem for the harmonic oscillator $\ddot{u} + u = c\,\delta(t)$, with $u \equiv 0$ for $t < 0$. Look for a continuous solution where u solves the homogeneous problem for $t \neq 0$, and $\dot{u}$ has an appropriate jump at $t = 0$. Hence, for $t > 0$, u solves the homogeneous problem, with initial conditions $u(0) = 0$ and $\dot{u}(0) = c$.

HINT-2. In order to solve the initial value problem for the wave equation in 2-D, you will need the Green's functions for the equation. These are:

$$G_1 = \frac{1}{2\,\pi}\,\frac{\partial}{\partial t}\left(\frac{H(t-r)}{\sqrt{t^2 - r^2}}\right),$$

$$\text{so that, for } t = 0: \quad G_1 = \delta(x)\,\delta(y) \text{ and } (G_1)_t = 0, \tag{1.8}$$

[1] That is: $u \equiv 0$.

[2] That is: all of R^3.

$$G_2 = \frac{1}{2\pi}\left(\frac{H(t-r)}{\sqrt{t^2-r^2}}\right),$$

$$\text{so that, for } t=0: \quad G_2 = 0 \qquad \text{and} \quad (G_2)_t = \delta(x)\,\delta(y), \tag{1.9}$$

where $r = \sqrt{x^2+y^2}$, both G_1 and G_2 solve $u_{tt} = u_{xx} + u_{yy}$, and $H(\zeta) = \frac{1}{2}(1+\operatorname{sign}(\zeta))$ is the Heaviside step function.

HINT-3. Note that $\delta(z-\beta\,t) = \delta(\beta\,t - z) = \frac{1}{\sqrt{\beta^2-1}}\,\delta\left(\frac{\beta\,t - z}{\sqrt{\beta^2-1}}\right)$.

Remark 1.1 *You should note that, at the conic wave-front, the solution develops a very large amplitude (in fact: it is singular there). This is what triggers the blue "glow". In fact, the singularity is caused by the point source (i.e.: a delta forcing) approximation. For a small, but finite size, source the solution will develop a large amplitude at the wave front, but will not be singular.*
In the sonic boom case (supersonic propagation of a point source in, say, air) there is no infinite fields anywhere: at the wave front a shock wave appears[3] (which cuts off the infinities). Furthermore: the shock wave is a "robust" object: it does not disappear when a finite size source is used.

1.3 Statement: Steady State Shallow Water (problem 01).

The conservation form of the equations for 2-D shallow water waves over a flat bottom is

$$0 = h_t + (h\,u)_x + (h\,v)_y, \tag{1.10}$$

$$0 = (h\,u)_t + (h\,u^2 + \frac{1}{2}\,g\,h^2)_x + (h\,v\,u)_y, \tag{1.11}$$

$$0 = (h\,v)_t + (h\,u\,v)_x + (h\,v^2 + \frac{1}{2}\,g\,h^2)_y, \tag{1.12}$$

where h is the fluid depth, u is the x-flow velocity, v is the y-flow velocity, and g is the acceleration of gravity. The steady state (time independent) form of these equations is

$$0 = (h\,u)_x + (h\,v)_y, \tag{1.13}$$

$$0 = (h\,u^2 + \frac{1}{2}\,g\,h^2)_x + (h\,v\,u)_y, \tag{1.14}$$

$$0 = (h\,u\,v)_x + (h\,v^2 + \frac{1}{2}\,g\,h^2)_y. \tag{1.15}$$

[3]In some sense, the nonlinearity regularizes the solution: shocks do not involve infinite values of physical quantities.

Answer the following questions:

1. Under which conditions on (h, u, v) is (1.13 – 1.15) strictly[4] hyperbolic? Use the **Froude number** $\boxed{F = \sqrt{(u^2+v^2)/(g\,h)}}$ in your answer. You need $F > 0$ to even ask the question — **WHY?**
2. When the characteristic equation has a double root, the system is not hyperbolic.[5] Show this. **Hint:** The system is invariant under rotations. Hence, when computing the eigenvector(s), you can rotate the coordinate system so that $v = 0$ at the point of interest.
3. The system always has (at least) one characteristic, which has a Riemann invariant. Find it.

1.4 Statement: Moving point source in 1-D.

Situations where one has a moving source in the context of wave propagation are quite common. In particular — when the source is compact and one is only interested in the resulting wave pattern far away from the source[6] — one can often simplify the question by assuming a point source. Here we consider a very simple example of this type, in 1-D and for a scalar first order equation with constant coefficients (homogeneous media). We also assume "trivial" initial conditions.

When the equation is also linear, the problem is very simple, and the only (mildly) interesting effect that occurs is that of "resonance" when the source moves at the characteristic speed. The mathematical problem in this case is

$$u_t + c\,u_x = \delta(x - s\,t) \quad \text{and} \quad u(x,\,0) = 0, \tag{1.16}$$

where c is the wave speed, s is the source speed (both constants), and $\delta(\cdot)$ is Dirac's delta function. **Show that (1.16) is equivalent to**

$$u_t = \delta(x - v\,t) \quad \text{and} \quad u(x,\,0) = 0, \tag{1.17}$$

for some constant v. Then **solve (1.17) for all possible values of** v**.** What happens at resonance?

The situation becomes much more interesting when the equation is nonlinear. Then the source can produce (or not) a precursor shock moving ahead of it — depending on the source speed and strength, and the (unrealistic) growth of the linear response in the resonant case is suppressed. As an example, consider the problem for the conserved density u

$$u_t + \left(\frac{1}{2}\,u^2\right)_x = \delta(x - c\,t) \quad \text{and} \quad u(x,\,0) = 0, \tag{1.18}$$

---

[4] All the characteristic directions are distinct: the characteristic equation has three distinct roots.
[5] There is only one eigenvector associated with the double root.
[6] Distances much greater than the source size.

where c is a constant. **Solve this problem for all possible values of c.**

HINTS

H1. The solution responds to the delta-function forcing on the right with a discontinuity along $x = c\,t$. The discontinuity is such that the derivatives (interpreted in the weak sense) produce the delta function. Namely

$$-c\,[u] + \frac{1}{2}\,[u^2] = 1, \tag{1.19}$$

where $[\,]$ = jump across discontinuity (value ahead minus value behind). Specifically, if u_a is the value of u immediately ahead of the discontinuity, and u_b is the value immediately behind it, then: $[u] = u_a - u_b$ and $[u^2] = u_a^2 - u_b^2$.

Note that **not all the solutions to this equation are acceptable.** The next hint, and remark 1.2, deal with this issue.

H2. Characteristics converge into shocks. However, the discontinuity along $x = c\,t$ is not a shock, but the response to a point forcing: the characteristics enter on one side of $x = c\,t$, and exit on the other. The only exception is when they enter/exit on one side and are parallel on the other — see remark 1.2. But **the characteristics never converge on both sides of $x = c\,t$.**

H3. The characteristics for the un-forced equation are: $\dfrac{dx}{dt} = u$, along which $\dfrac{du}{dt} = 0$. Hence, *the initial value $u(x,\,0) = 0$ will persist at any given point x, till affected by something that makes the characteristic equations fail — namely: either a shock wave or the delta-function forcing.*

H4. The solution to (1.18) is rather simple. It is made up by constant strength/speed shocks, regions where u is constant, and rarefaction fans. Further, it is a function of x/t only — **why?**

H5. The shock conditions for (1.18) reduce to: (i) Shock speed is the average of u across the discontinuity. (ii) The value of u behind the shock is larger than the value ahead.

H6. The rarefaction fans for (1.18) are solutions where all the characteristics determining u emanate from a single point in space time and "fan" out.

Remark 1.2 *Here we elaborate on the subject matter of hints* **H1** *and* **H2**. *Specifically: what restrictions the solutions of the jump equation in (1.19) must satisfy — which is what* **H2** *is all about. Our objective is to understand the behavior of the characteristics for the equation in (1.18) at/near the location $x = c\,t$ of the delta function forcing. To do this, we consider (1.18) as the limit of*

$$u_t + \left(\frac{1}{2}\,u^2\right)_x = f_\epsilon(x - c\,t) \quad as \quad \epsilon \to 0, \tag{1.20}$$

where $f_\epsilon(z)$ is a smooth, positive function, with total unit area, vanishing outside $|z| < \epsilon$. The characteristic equations for this problem are

$$\frac{dx}{dt} = u, \quad along\ which \quad \frac{du}{dt} = f_\epsilon(x - c\,t). \tag{1.21}$$

Then, as long as the characteristics are outside the forcing region $ct - \epsilon < x < ct + \epsilon$, they are straight lines — along which u is constant. When they enter the forcing region, on the other hand, they accelerate (as u increases). Hence, the following situations (and nothing else) can arise:

1.2-a. Characteristic enters the forcing region from the left, with $u > c$.
Then u starts increasing, the characteristic speeds up and leaves on the right side of the forcing region, carrying a larger value of u. The $\epsilon \to 0$ limit of this situation is (1.22) below.

1.2-b. Characteristic enters the forcing region from the left, with u barely above c; i.e.: $u = c + O(\epsilon)$.
Similar to item **1.2-a**, *but the $\epsilon \to 0$ limit is: Immediately behind $x = ct$, $u = c$ and the characteristics are parallel to the path of the delta function. Immediately ahead of $x = ct$, $u > c$ and the characteristics exit (to the right) from the path of the delta function. See (1.23) below.*

1.2-c. Characteristic is overtaken by the forcing region, and enters it from the right, with $u < c$ and sufficiently far below.
Then, once inside the forcing region, u starts increasing and the characteristic speeds up. However, before the value of u along the characteristic reaches c, the characteristic reaches the back of the forcing region, and exits it. The $\epsilon \to 0$ limit of this situation is (1.24) below.

1.2-d. Same as item 1.2-c, but the value of u (when the characteristic enters the forcing region from the right) is just critical.
Then the characteristic just barely makes it out (from the back) of the forcing region. The $\epsilon \to 0$ limit of this situation is (1.25) below.

1.2-e. Same as item 1.2-c, but the value of u (when the characteristic enters the forcing region from the right) is too close to c.
Then, once inside the forcing region, u starts increasing, the characteristic speeds up, and grows beyond c. Thus the characteristic will end up exiting the forcing region from the same side it entered. In the $\epsilon \to 0$ limit of this, the characteristic "bounces back" (with a higher value of u) into the ahead of the path of the delta function. But this creates a multiple valued region for the solution ahead of the delta, which means that this is an inconsistent situation, and cannot occur.[7]

Thus, in terms of u_a and u_b (values immediately ahead — respectively behind – the discontinuity), **these are the (only) acceptable possibilities for the solutions to equation (1.19):**

$$\text{Case 1:} \quad u_a > u_b > c. \tag{1.22}$$

$$\text{Case 2:} \quad u_a > u_b = c. \tag{1.23}$$

$$\text{Case 3:} \quad u_a < u_b < c. \tag{1.24}$$

$$\text{Case 4:} \quad u_a < u_b = c. \tag{1.25}$$

[7] If you try to solve equation (1.19) with a value of $u_a < c$ that is too close to c, you will see that then u_b would have to be complex, with a nonzero imaginary part.

THE END.