Take home exam # 1 (MIT 18.306, Spring 2007).

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1 Statements for the assigned problems.

1.1 Statement: Eikonal equation (problem 01).

Consider the Eikonal equation (for the wave equation in 2-D) in a context where the wave speed is a constant (homogeneous media), so that we can set (upon non-dimensionalization) $c = 1$. Then

$$\phi_x^2 + \phi_y^2 = 1. \tag{1.1}$$

Consider now the situation where the wave-front $\phi = 0$ is a parabola. Specifically:

$$\phi = 0 \quad \text{on} \quad y = x^2, \tag{1.2}$$

with propagation direction towards y increasing. For this problem, **this is what you should do:**

1. Find the family of all the rays (characteristics) for $t > 0$. The easiest way is to describe it is parametrically: $x = x(s,\, t)$ and $y = y(s,\, t)$, where $x(s,\, 0) = s$, $y(s,\, 0) = s^2$, and t is time of travel along the ray (for the wave-fronts) starting from the initial wave-front — i.e.: $\phi = t$.

2. Find the caustic. The caustic is the envelope of the family of rays = the locus of the intersections of infinitely close neighbors in the family of rays[1] = a curve such that each point in it belongs to one of the rays, and it is tangent to the ray there.[2] In this case of constant wave speed, where the rays are straight lines, the caustic is also the locus of the centers of curvature of the wave-fronts (all the wave-fronts have the same set of centers of curvature).

From the parametric description of the family of rays in item **1**, you should be able to obtain the caustic parametrically in terms of s. However, you should also be able to find a very simple formula — of the form $(y - y_a)^\alpha = \text{const.}\,(x - x_a)^2$ — for the caustic. Do so.

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[1] $(x_*,\, y_*) \in$ caustic $\iff$ $x_* = x(s,\, t) = x(s + ds,\, t + dt)$ and $y_* = y(s,\, t) = y(s + ds,\, t + dt)$, for some s and t.

[2] $(x_*,\, y_*) \in$ caustic $\iff$ for some s and t: $x_* = x(s,\, t)$, $y_* = y(s,\, t)$, and $(x_t(s,\, t),\, y_t(s,\, t))$ is tangent to the caustic at $(x_*,\, y_*)$.

3. Do a sketch of the wave-front $\phi = 0$, and of the caustic. Indicate the region of the plane where the rays cross and give rise to multiple values in the solution to the equation.
4. The earliest time at which a ray crossing occurs corresponds to the singular point in the caustic (the arête). Find the position of the arête in space, the ray, and the time (or wave-front, as $\phi = t$) it correspond to. Let these parameters be x_a, y_a, s_a, and t_a. Explicitly show that t_a is the earliest time at which a crossing of rays occurs.
5. Add to the sketch in item **3** the wave-front $\phi = t_a$. This wave-front is singular at the arête; describe the nature of this singularity. In particular, show that the wave-front satisfies (at leading order) a formula of the form $(y - y_a) \sim \text{const.}\,(x - x_a)^\mu$ near the arête.

1.2 Statement: Eikonal equation (problem 02).

Consider the Eikonal equation (for the wave equation in 2-D) in the context where the wave speed is not a constant (non-homogeneous media). In particular, consider the following situation

$$c^2\,(\phi_x^2 + \phi_y^2) = 1, \tag{1.3}$$

where $c = c_1 > 0$ for $y > 0$, and $c = c_2 > 0$ for $y < 0$, with $c_1 \neq c_2$. Of course, in a situation like this, we must worry about what is the meaning of the equation for $y = 0$. From the derivation in the lectures, where we saw that ϕ is a phase, it is easy to see that what we want to require in that ϕ be continuous across $y = 0$, with the equation satisfied on each side. On the other hand, *the gradient of ϕ will, most definitely, not be continuous.* In fact, investigating what happens with $\nabla\phi$ across $y = 0$ is the purpose of this problem.

Assume that $\nabla\phi$ is continuous on each side of $y = 0$, with continuous limits on each side as $y \to 0$. Let $\nabla^+\phi = \lim\limits_{y\to 0,\, y>0} \nabla\phi$ and $\nabla^-\phi = \lim\limits_{y\to 0,\, y<0} \nabla\phi$. **Find a relationship between $\nabla^+\phi$ and $\nabla^-\phi$.**

Note: since $\nabla\phi$ is the direction of the rays — given by $\dfrac{d\vec{r}}{dt} = c^2\,\nabla\phi$ — the relationship you find should be equivalent to Snell's law. **Show** that this is, indeed, the case.

1.3 Statement: Initial Values for a Kinematic Wave (problem 02).

(Here we explore the issue of dissipation at shocks). In *Initial Values for a Kinematic Wave (problem 01)* we considered the following question: Solve the Kinematic Wave equation (for the conserved quantity u)

$$u_t + \left(\frac{1}{2}\,u^2\right)_x = 0, \tag{1.4}$$

using the initial values

$$u(x,\,0) = \begin{cases} \phantom{1+{}}0 & \text{for} \quad -\infty < x \le -1, \\ 1+x & \text{for} \quad -1 \le x \le 0, \\ 1-x & \text{for} \quad 0 \le x \le 1, \\ \phantom{1+{}}0 & \text{for} \quad 1 \le x < \infty, \end{cases} \tag{1.5}$$

and introducing shocks in the solution (whenever needed to eliminate multiple values due to crossing of the characteristics). **Let the solution to this particular problem be** $\boxed{u = U(x,\,t)}$ — which is given explicitly in the answer to the *Initial Values for a Kinematic Wave (problem 01).* Then, **DO THE FOLLOWING:**

1. **VERIFY** directly (using the explicit form for U) that $A = \int_{-\infty}^{\infty} U(x,\,t)\,dx = 1$ for all times, so that the total amount of u is conserved, as it should.
2. **VERIFY** directly (using the explicit form for U) that the "energy" $E = \int_{-\infty}^{\infty} \frac{1}{2}\,U^2(x,\,t)\,dx$ is constant for $0 \le t \le 1$, and decreases for $t > 1$ — time derivative strictly less than zero. Since $t = 1$ is the time when a shock in the solution forms, this provides an explicit example showing that *shocks dissipate "energy" — even though (1.4) formally has no dissipation!* **The purpose of this problem is to understand a little of why and how this happens.**
3. Consider an arbitrary solution u to equation (1.4). **SHOW THAT,** as long as u has no shocks,[3] u also satisfies the equation

$$\left(\frac{1}{2}\,u^2\right)_t + \left(\frac{1}{3}\,u^3\right)_x = 0. \tag{1.6}$$

   Thus, if $u \to 0$ as $|x| \to \infty$, the "energy" $E = \int_{-\infty}^{\infty} \frac{1}{2}\,u^2(x,\,t)\,dx$ will be a constant — this shows that the first part of the result in item **2** is generic.
4. At a shock $x = x_s(t)$ for a solution u to equation (1.4), equation (1.6) does not hold. In fact, from the rules governing weak derivatives, it can be shown that[4]

$$\left(\frac{1}{2}\,u^2\right)_t + \left(\frac{1}{3}\,u^3\right)_x = \sum_{\text{shocks}} \left[-\frac{dx_s}{dt}\,\frac{1}{2}\,u^2 + \frac{1}{3}\,u^3\right]\,\delta(x - x_s(t)), \tag{1.7}$$

   where $\delta(\cdot)$ is the Dirac delta function, and the brackets $[\cdot]$ denote the jump in the enclosed function across the shock — specifically: value immediately ahead of the shock minus value immediately behind. Thus, **SHOW THAT:**

$$\left(\frac{1}{2}\,u^2\right)_t + \left(\frac{1}{3}\,u^3\right)_x = \sum_{\text{shocks}} \frac{1}{12}\,[u]^3\,\delta(x - x_s(t)). \tag{1.8}$$

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[3] Thus u has derivatives, and satisfies the equation in the usual sense.

[4] You will be asked to show this in another problem.

Since $[u] < 0$ at shocks for (1.4) — entropy condition **(SHOW THIS)**— the right hand side in (1.8) is negative, so that energy is dissipated. **INTEGRATE** this last equation from $x = -\infty$ to $x = \infty$, assuming that u vanishes as $|x| \to \infty$, and **OBTAIN AN EQUATION** for the time derivative of the energy E. **VERIFY** that E, as calculated in item **2**, satisfies this equation.

Note 1: Assume that (for each shock) along the curve $x = x_s(t)$ the function u has a discontinuity such that u, u_t, and u_x exist and are continuous on each side of the curve, and have left and right limits as the curve is approached. Furthermore, assume that dx_s/dt exists.

Note 2: Remember that the discontinuity across a shock must satisfy the Rankine-Hugoniot jump condition, as well as the entropy condition.

5. *How is it that the solutions to (1.4) end up with dissipation at shocks, when the equation itself has no explicit dissipation parameter?* The reason has to do with the fact that shocks arise in a singular limit as the dissipation parameter vanishes, as you will be asked to show here.

As explained in the lectures, adding shocks to the solutions of (1.4) — in order to resolve multiple values issues, as well as infinities in the derivatives — is not a mathematical step, but a (physical) *modeling issue*: the equation plus shock conditions includes further physical assumptions than the original model without them. Specifically: there is a diffusion-like process that "fights" (and stops) the steepening caused by the nonlinearity when the derivatives become large enough, stabilizing a transition in the solution from one value to another over a very thin layer. This layer is then modeled as being infinitely thin, with the solution being discontinuous across it: a shock.

For example, in the case of equation (1.4), a more complete model would include a small "viscosity" coefficient $0 < \nu \ll 1$, with the equation modified to

$$u_t + \left(\frac{1}{2}\,u^2\right)_x = \nu\, u_{x\,x}. \tag{1.9}$$

Then the solutions of (1.4) are obtained from the solutions of (1.9) in the limit $\nu \to 0$. As we saw in the lectures, for equations like this, the shocks can be modeled as traveling waves connecting two values of u over a layer of width $O(\nu)$ — i.e.: $u = F\left(\dfrac{x - s\,t}{\nu}\right)$, where s is the shock velocity and F is the shock structure function. The important conclusion from this that we need here is that:

$$\left.\begin{array}{l} \textit{Each shock layer has an } x\textit{-width of size } O(\nu). \\ \textit{In the shock layer: } u = O(1),\ u_x = O(\nu^{-1}),\ u_t = O(\nu^{-1}), \text{ etc.} \end{array}\right\} \tag{1.10}$$

Consider now a solution of (1.9) that vanishes (fast enough) as $|x| \to \infty$ and **SHOW THAT**

$$\frac{dE}{dt} = \frac{d}{dt}\int_{-\infty}^{\infty} \frac{1}{2}\, u^2(x,\, t)\, dx = -\nu \int_{-\infty}^{\infty} u_x^2(x,\, t)\, dx. \tag{1.11}$$

Hence E is decreasing. Furthermore[5] **ARGUE** that, as $\nu \to 0$, $\dfrac{dE}{dt}$ has a nonzero negative value if the solution has shocks — this limiting value of $\dfrac{dE}{dt}$ is, of course, the one that you were asked to derive in item **4**.

The results above illustrate how it is that the solutions to equation (1.4) dissipate energy, even though the dissipation parameter — the viscosity ν in (1.9) — vanishes. What happens is that (for small, but finite ν) *the amount of dissipation produced by each shock is (basically) independent of the value of* ν. This follows because the amount of dissipation is not just proportional to ν, but is also a function of the gradients involved — and the nonlinearity in the equation pushes these gradients up till they have size $O(\nu^{-1})$.

1.4 Statement: Laplace equation (problem 01).

Consider a thin, homogeneous, heat conducting sheet, insulated on the top and the bottom. If $T = T(x,\, y,\, t)$ is the temperature in the sheet, then the conservation of heat (and Fick's law) leads to the heat equation — which in non-dimensional units has the form

$$T_t = \Delta\, T = T_{xx} + T_{yy}. \tag{1.12}$$

Let Ω be the region of space occupied by the sheet, and assume that along the boundary $\partial\Omega$ of this region the heat flux is known and given by some function, say: $F = F(s)$ per unit length (where s is the arc-length along $\partial\Omega$).

The problem to be solved is then (1.12) inside Ω, with the boundary conditions on $\partial\Omega$

$$\partial_n T = \hat{n} \cdot \nabla T = F(s), \tag{1.13}$$

where $\hat{n}$ is the unit outside normal to $\partial\Omega$. In particular, for *steady state*, we have Laplace's equation in Ω

$$0 = \Delta\, T = T_{xx} + T_{yy}, \tag{1.14}$$

with the Neumann conditions in (1.13).

1. Show that there is an integral condition that F must satisfy if the problem (1.13 – 1.14) has a solution. **Hint:** Gauss theorem.

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[5] Hint: use (1.10).

2. Give a physical interpretation to the condition in **1.** Why do you need it, and what happens when it is not satisfied?

2. The solution to (1.13 – 1.14), if there is one, is determined only up to an arbitrary additive constant. How would you determine this constant, and what is it related to — i.e.: knowledge of what physical quantity gives it to you?

1.5 Statement: Wave equations (problem 03).

This problem investigates the issue of the characteristics as the places where "weak" singularities of the solutions can occur — where by "weak" singularities we mean lack of smoothness in the solutions which is not strong enough to destroy their meaning as classical solutions.

EXAMPLE: consider the linear first order scalar equation for $u = u(x,\, y)$

$$a\, u_x + b\, u_y + c\, u = d, \tag{1.15}$$

where $a = a(x,\, y)$, $b = b(x,\, y)$, $c = c(x,\, y)$, and $d = d(x,\, y)$, are some given smooth functions, with $a^2 + b^2 \neq 0$. Consider now a function $u = u(x,\, y)$ and a curve $\phi(x,\, y) = 0$ — where ϕ is smooth and has a non-zero gradient, such that $u = u(x,\, y)$ is continuous and

1. u has continuous partial derivatives where $\phi \neq 0$, which have a continuous limit on each side of the curve — in other words: the graph of u is a "nice" surface, except that it has a crease along the given curve. *Derivatives in directions parallel to the curve exist and are continuous everywhere, while derivatives in directions that cross the curve have a simple discontinuity as the curve is crossed.*

2. On each side of the curve, u satisfies equation (1.15). Because of item **1**, the limits of the solution (as the curve is approached on each side) on the curve, also satisfy the equations.

Such a u is a solution to equation (1.15) in the "classical" sense, but it has a lack of smoothness across the curve $\phi = 0$, which is as strong as it can while still allowing a solution in the classical sense. *Question: are there any restrictions on what the curve $\phi = 0$ can be?*

In order to answer the question in the prior paragraph, we introduce a local (curvilinear) coordinate system such that ϕ is one of the coordinate functions — this can always be done. So, let $\psi = \psi(x,\, y)$ be a smooth function with non-zero gradient such that $\nabla\phi$ and $\nabla\psi$ are not co-lineal, and re-write the equation using ϕ and ψ as independent variables. Then

$$d = (a\, \phi_x + b\, \phi_y)\; u_\phi + (a\, \psi_x + b\, \psi_y)\; u_\psi + c\, u, \tag{1.16}$$

which should apply for $\phi > 0$ and $\phi < 0$, with continuous limits as $\phi \to 0$ from each side. Furthermore, from item **1** it follows that u_ψ is continuous everywhere, while u_ϕ has a simple discontinuity at $\phi = 0$. Thus taking the limit (from both sides $\phi > 0$ and $\phi < 0$) as $\phi \to 0$ of the equation, and then taking the difference of these two limits, we obtain

$$0 = (a\,\phi_x + b\,\phi_y)\;[u_\phi] \quad \text{along} \quad \phi = 0, \tag{1.17}$$

where $[u_\phi]$ denotes the (non-zero) jump in u_ϕ across the curve. Hence we obtain the following equation that must be satisfied by the curve

$$0 = a\,\phi_x + b\,\phi_y \quad \text{along} \quad \phi = 0. \tag{1.18}$$

Since $\phi_x\,dx + \phi_y\,dy = 0$ along the curve, it follows that

$$a\,dy = b\,dx, \tag{1.19}$$

which is equivalent to the (parametric) equation for the characteristics obtained in the lectures (by other means); namely: $\dfrac{dx}{ds} = a$ and $\dfrac{dy}{ds} = b$.

THESE ARE YOUR TASKS IN THIS PROBLEM:

PART I . . . .

The arguments above appear to impose no restrictions on the curve $\phi = 0$ if the singularities in u appear at higher order. This is **not true.** For example, assume that u has continuous derivatives up to second order, except that the second derivatives have simple discontinuities across $\phi = 0$. Thus, *it is the graph of (say)* u_x *that has a crease along the curve.* **SHOW** then that *the curve must be a characteristic.* **Hint:** Consider the equation that u_x satisfies.

PART II . . . .

Consider the linear second order scalar equation for $u = u(x,\,y)$

$$a\,u_{xx} + 2\,b\,u_{xy} + c\,u_{yy} = d, \tag{1.20}$$

where $a = a(x,\,y)$, $b = b(x,\,y)$, $c = c(x,\,y)$, and $d = d(x,\,y)$, are some given smooth functions, with $a\,c - b^2 \neq 0$. Using an argument similar to the one in (1.15 – 1.19), **FIND** the curves across which the solutions to the equation can have "weak" discontinuities — i.e.: the **characteristics.** In particular:

- **II-1.** Under which conditions on the coefficients $a,\,b,\ldots$ do such curves exist?
- **II-2.** What are the curves in the case $u_{xx} - u_{yy} = 0$?
- **II-3.** What happens in the case $u_{xx} + u_{yy} = 0$?

Hint: In this case you have to assume that it is the second partial derivatives of u that have simple discontinuities across the curve $\phi = 0$.

THE END.