1 Equations for a bungee jumping cord hanging vertically

Derive the equations for a bungee jumping cord hanging vertically with a mass at the lower end.

1. Let $x$ be the vertical coordinate, increasing downwards. Further: the top end of the cord is attached at $x = 0$.

2. Neglect the thickness of the cord, so you can idealize it as a curve, and label each point along the cord by a coordinate $s$, $0 \leq s \leq L$, defined as follows: When the cord is not under tension (nor compressed), $s$ is the distance (along the cord) from the point to one end of the cord — specifically: the end that is attached at $x = 0$. Hence the unstretched length of the cord is $L$.

3. Let $\rho = \text{constant}$ be the mass of the cord per unit length — that is, the mass between $s = a$ and $s = b > a$ is: $(b - a) \rho$. Furthermore, let $g = \text{acceleration of gravity}$.

4. Describe the state of the cord, at any time, by $u = u(s, t)$, where $u$ is defined as follows: $x = u(s, t)$ is the vertical coordinate of the point along the cord whose label is $s$. For example: in the absence of motion and zero gravity, the cord is described by $u = s$. 

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1 Equations for a bungee jumping cord hanging vertically

Statement: Equations for a bungee jumping cord hanging vertically

Consider a bungee jumping cord hanging vertically, attached at the top (at some fixed point), with a mass $M$ at the bottom. For simplicity we will neglect side motions (that is: we assume that the whole assembly, cord and mass, is hanging perfectly vertical at all times), but not vertical motion. Your task is to is to derive equations for the dynamics of the system, under the following assumptions:

1. Let $x$ be the vertical coordinate, increasing downwards. Further: the top end of the cord is attached at $x = 0$.

2. Neglect the thickness of the cord, so you can idealize it as a curve, and label each point along the cord by a coordinate $s$, $0 \leq s \leq L$, defined as follows: When the cord is not under tension (nor compressed), $s$ is the distance (along the cord) from the point to one end of the cord — specifically: the end that is attached at $x = 0$. Hence the unstretched length of the cord is $L$.

3. Let $\rho = \text{constant}$ be the mass of the cord per unit length — that is, the mass between $s = a$ and $s = b > a$ is: $(b - a) \rho$. Furthermore, let $g = \text{acceleration of gravity}$.

4. Describe the state of the cord, at any time, by $u = u(s, t)$, where $u$ is defined as follows: $x = u(s, t)$ is the vertical coordinate of the point along the cord whose label is $s$. For example: in the absence of motion and zero gravity, the cord is described by $u = s$. 

5. Assume that the cord is perfectly elastic and satisfies Hooke’s law. Thus an unstretched length of cord $s$, when stretched to length $s$, generates a tension $T = k \frac{\Delta x}{\Delta s}$, where $k > 0$ is the cord coefficient of elasticity. Recall that the tension, at any point along the cord, is the force that one side exerts on the other — if you were to cut the cord at some point, the tension is the force that you would have to apply to each side of the cut to keep it from retracting.

6. Cords are not able to withstand compression well, and they bend when under compression. Thus, assume that the vertical motion along the cord is small enough to keep it always under tension, everywhere — this is where having a mass at the bottom helps.

Hints.

#1 You need to derive an equation for $u$, which applies for $0 < s < L$. For this purpose use the conservation of the vertical momentum (you can calculate the momentum density, and flux, in terms of $u$ and the constants $\rho$ and $k$). Note that there is also a momentum source, due to gravity. Write this equation in both its integral, as well as its differential form.

To write the integral form you will need to assume that $u$ has continuous partial derivatives; to write the differential form you will need to assume that the second derivatives are continuous as well.

#2 The equation for $u$ needs boundary conditions (BC) at the ends of the cord. This is where the mass $M$ comes into play: note that the mass position is $x = u(L, t)$.

2 Duhamel’s principle for a damped wave equation #01

Statement: Duhamel’s principle for a damped wave equation #01

Consider the dissipatively damped wave equation with a forcing

$$L u = u_{tt} - c^2 u_{xx} - \mu u_{xxt} = f(x, t),$$

where $c > 0$ and $\mu > 0$ are constants, and $f$ is some applied external force. Assume now that the equation is valid in the interval $a < x < b$, with some homogeneous boundary conditions (see remark 2.1), and initial values $u(x, 0) = u_t(x, 0) = 0$. Show that the solution to this problem can be written in the form

$$u = \int_0^t U(x, t, \tau) d\tau,$$

where (for each $\tau \geq 0$) $U$ satisfies the homogeneous problem $LU = 0$, with the same boundary conditions as $u$, and suitably selected initial conditions at $t = \tau$. What are these initial conditions?

Note. Assume the solutions involved are smooth enough to justify differentiation under the integral sign, and similar.

Hint. There are many ways in which an expression like (2.2) can solve (2.1). We are looking for a very simple one, where $U$, as a function of $x$ and $t$, satisfies $LU = 0$ for each $\tau$, and $t > \tau$, with initial conditions at $t = \tau$. That is: $U(x, \tau, \tau)$ and $U_t(x, \tau, \tau)$ are given. Both $u$ and $U$ satisfy the same homogeneous boundary conditions.

Remark 2.1 Equation (2.1) describes, for example, the small vibrations of a string under tension, which resists bending via dissipative forces (i.e., forces proportional to the rate of bending).

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1 During an actual bungee jump, the cord experiences extensions over a huge range, and Hooke’s law is not valid. However, here we are looking at a post-jump scenario, with the cord already stretched and experiencing small variations in the range of stretching.

2 Actually, these hypothesis are too strong — e.g.: the conservation form is OK if $u$ has “corners”. But do not worry about this here.
Exactly what the boundary conditions are does not matter here, as long as they are linear and homogeneous. For example, at \( x = a \), it could be: \( u = 0 \) (tied string), \( u_x = 0 \) (string with free end), or \( T u_x - bu_t = 0 \) (string end sliding along a rod, with friction force \(-bu_t\) balancing the transversal component of the tension force. Similarly, at \( x = b \), it could be: \( u = 0 \), or \( u_x = 0 \), or \( T u_x + bu_t = 0 \).

3 Eikonal equation #03

Statement: Eikonal equation #03

Consider the Eikonal equation, which describes the propagation of a wave-front which moves normal to itself at a given speed \( c \) (which may depend on position \( c = c(\vec{x}) \)). In 2-D, when \( c = \text{constant} \), the equation takes the non-dimensional form

\[
\phi_x^2 + \phi_y^2 = 1,
\]

(3.1)

where units have been selected such that \( c = 1 \). The relation of the solution, \( \phi = \phi(x, y) \), with the wave front is that: the wave front at time \( t \) is given by the level curve \( \phi = \phi(x, y) = t \). The equation is then to be solved with knowledge of the initial wave front: a given curve \( \Gamma \), with unit normal \( \hat{n} \) pointing in the direction of propagation. \(^3\)

Equation (3.1) has characteristics, called rays. The rays are the lines defined by

\[
\frac{dx}{dt} = \phi_x \quad \text{and} \quad \frac{dy}{dt} = \phi_y.
\]

(3.2)

It is easy to see that, with this definition, equation (3.1) — plus the equations that result from taking the \( x \) and \( y \) partial derivatives of the equation: \( \phi_x \phi_{xx} + \phi_y \phi_{xy} = 0 \) and \( \phi_x \phi_{yx} + \phi_y \phi_{yy} = 0 \) — implies that

\[
\frac{d\phi_x}{dt} = 0, \quad \frac{d\phi_y}{dt} = 0, \quad \text{and} \quad \frac{d\phi}{dt} = 1.
\]

(3.3)

The set of five equations in (3.2–3.3) is called the characteristic form of the Eikonal equation. It is a closed system of ODE, that can be solved (for each ray) given the initial wave front — see remark 3.1. It is, in fact, equivalent to (3.1).

Remark 3.1 For every point \((x_0, y_0)\) \(\in \Gamma\) there is a ray, which is obtained by solving the system of ODE with the initial conditions (at \( t = 0 \))

\[
x = x_0, \quad y = y_0, \quad \phi = 0, \quad \phi_x = n_1, \quad \text{and} \quad \phi_y = n_2,
\]

where \( n_1 \) and \( n_2 \) are the components of the unit normal \( \hat{n}_0 \) to \( \Gamma \) at \((x_0, y_0)\). Solving the ODE yields \( x = X(x_0, y_0, t), \) \( y = Y(x_0, y_0, t), \) \( \phi = \Phi(x_0, y_0, t), \) etc. The PDE solution follows, in principle, by solving for \( x_0 \) and \( y_0 \) as functions of \( (x, y, t) \) (using \( X \) and \( Y \)), and substituting into \( \Phi \).

Your tasks are:

1. Find an equation for the evolution along the rays of the Hessian of \( \phi \). Namely, the matrix

\[
M = \begin{bmatrix}
\phi_{xx} & \phi_{xy} \\
\phi_{yx} & \phi_{yy}
\end{bmatrix}
\]

(3.4)

2. Solve the equation for \( M \) derived in item 1, and write a formula giving the front curvature \( \kappa = \phi_{xx} + \phi_{yy} \) along each ray, as a function of \( t \), and the front curvature \( \kappa_0 \) on the ray at the wave front corresponding to \( t = 0 \). Note: the formula for \( \kappa \) involves \( t \) and \( \kappa_0 \) only.

\(^3\)A context in which the Eikonal equation applies is for high frequency, monochromatic, solutions to the wave equation \( u_{tt} - c^2 \Delta u = 0 \). The rays then correspond to the concept of “light rays” in Geometrical Optics.
Hints: (i) Consider the second order derivatives (i.e. \( \partial^2_{xx} \), \( \partial^2_{xy} \), and \( \partial^2_{yy} \)) of equation (3.1). Then use that, for any \( f = f(x, y) \), its derivative along the rays is given by \( \frac{\partial}{\partial t} f = \phi_x f_x + \phi_y f_y \). (ii) To solve the equation for \( M \), proceed as follows: Let \( M_0 \) be the value of \( M \) at \( t = 0 \). Define \( W = (1 + M_0 t) M \), and write the equation \( W \) satisfies. Using that \( W = M_0 \) at \( t = 0 \), you should now be able to solve this equation by inspection. (iii) Once you have solved the equation for \( M \), use it to get the behavior of the eigenvalues of \( M \)—note that \( \kappa \) is the sum of the eigenvalues. (iv) Finally, inspect the gradient of equation (3.1). What does it tell you about the eigenvalues of \( M \)?

4 First order PDE Riemann problem #01

Statement: First order PDE Riemann problem #01

Consider the following conservation law (in one-dimensional variables)

\[
    u_t + \left( \frac{1}{2} u^2 \right)_x = 1, \quad \text{for} \quad -\infty < x < \infty \quad \text{and} \quad t > 0,
\]

where \( u \) is conserved, and shocks are used to avoid multiple-valued solutions.

Find the solution to the Riemann problem for this equation. That is: for the initial values

\[
    u(x, 0) = a \quad \text{for} \quad x < 0 \quad \text{and} \quad u(x, 0) = b \quad \text{for} \quad x > 0,
\]

where \( a \) and \( b \) are arbitrary real constants \( -\infty < a, b < \infty \). Hint: The solution involves shocks, expansion fans, and regions where \( u \) depends on time only. Expansion fans are regions where all the characteristics emanate from a single point in space-time.

4.1 Extras: Justification of quadratic fluxes

Here we justify the use of conservation laws of the form

\[
    u_t + \left( \frac{1}{2} u^2 \right)_x = S, \quad \text{for} \quad -\infty < x < \infty \quad \text{and} \quad t > 0,
\]

where \( S \) is some source term, \( u \) is conserved and can be both positive or negative, and shocks are used to avoid multiple-values in the solution.

Consider a scalar conservation law in 1-D, with a source term, \( \rho_t + q_x = S \),

\[
    \rho = \rho(x, t) \quad \text{is the density of some conserved quantity (hence } \rho \geq 0), \quad q = Q(\rho) \quad \text{is the corresponding flux, and} \quad S = S(\rho, x) \quad \text{is the density of sources/sinks. Assume now that } S \text{ is “small” — this is made precise below in item 2.}
\]

Then solutions where \( \rho \) is close to a constant should be possible. Hence let \( \rho_0 > 0 \) be some fixed (constant) density value, and proceed as follows:

1. Expand \( Q \) near \( \rho_0 \) using Taylor’s theorem

\[
    Q = q_0 + c_0 (\rho - \rho_0) + \frac{s_1}{2 \rho_0} (\rho - \rho_0)^2 + \ldots,
\]

where \( q_0 \) is the flux for \( \rho = \rho_0 \), \( c_0 \) is the corresponding characteristic speed, and \( s_1 \) is a constant with the dimensions of a velocity. Assume that \( s_1 \neq 0 \); in fact, \( s_1 > 0 \).

2. Let \( S_1 > 0 \) be some “typical” value for the source term size, and let \( L > 0 \) be some “typical” length scale. Then the source term is small in the sense that

\[
    0 \ll \epsilon^2 = \frac{L S_1}{s_1 \rho_0} \ll 1.
\]

\text{\footnote{If } s_1 < 0, \text{ a similar analysis is possible. Note that } s_1 \text{ is a measure of how nonlinear equation (4.4) is. The further away for zero } s_1 \text{ is, the stronger the leading order nonlinear term in the equation is.}}
Introduce the a-dimensional variables
\[ \tilde{x} = \frac{x - c_0 t}{L} \quad \text{and} \quad \tilde{t} = \frac{\epsilon s_1}{L} t, \]
with \( \rho = \rho_0 (1 + \epsilon u) \) and \( S = S_1 \tilde{S} \). (4.6)

Then (4.4) becomes
\[ u_{\tilde{t}} + \left( \frac{1}{2} u^2 + O(\epsilon) \right)_{\tilde{x}} = \tilde{S}. \] (4.7)

Upon neglecting the \( O(\epsilon) \) term, this has the form in (4.3).

5 Simple finite differences for a 1st order PDE

Statement: Simple finite differences for a 1st order PDE

Consider the linear PDE
\[ u_t + (c(x) u)_x = 0, \quad \text{where} \quad c = 1 + \frac{1}{4} \cos(x), \] (5.1)
and \( u \) is periodic of period \( 2 \pi \) — i.e.: \( u(x + 2 \pi, t) = u(x, t) \). Assume now that you are asked to calculate the solution of this p.d.e. for \( 0 \leq t \leq T = 6 \), with initial condition given by
\[ u(x, 0) = u_0(x) = \exp(-x^2) \quad \text{for} \quad -\pi \leq x \leq \pi, \] (5.2)
extended periodically outside the interval \([-\pi, \pi]\).

You can extract a lot of information from the solution by characteristics of the problem above, but actual numerical values are not easy to access from it. For this, the best thing to do is to integrate the problem numerically. Here we will consider a few naive numerical algorithms for this purpose.

First, introduce a numerical grid, as follows:
\[ x_n = -\pi + nh \quad \text{for} \quad 1 \leq n \leq N, \quad \text{and} \quad t_m = mk \quad \text{for} \quad 0 \leq n \leq M, \] (5.3)
where \( N \) and \( M \) are “large” integers, \( h = 2 \pi/N \), and \( k = T/M \). Let \( u^m_n \) be the numerical solution’s grid point values. The expectation is that these values will be related to the exact solution \( u(x, t) \) by \( u^m_n = u(x_n, t_m) + \text{small error} \), with the error vanishing as \( N \) and \( M \) grow.

Next, consider the following numerical discretizations of the problem, which arise upon replacing the derivatives in the equation by finite differences that approximate them up to errors of some positive order in \( h \) or \( k \).

A. \( 0 = \frac{1}{k} (u^m_{n+1} - u^m_n) + \frac{1}{h} \left( c(x_n) u^m_n - c(x_{n-1}) u^m_{n-1} \right). \)

B. \( 0 = \frac{1}{k} (u^m_{n+1} - u^m_n) + \frac{1}{h} \left( c(x_{n+1}) u^m_{n+1} - c(x_n) u^m_n \right). \)

C. \( 0 = \frac{1}{k} (u^m_{n+1} - u^m_n) + \frac{1}{2h} \left( c(x_{n+1}) u^m_{n+1} - c(x_{n-1}) u^m_{n-1} \right). \)

In all cases, once \( u^m_n \) is known for some \( m \) and all \( n \), \( u^m_{n+1} \) can be explicitly computed, for all \( n \). \textbf{Note:} when a formula above in \( A \), \( B \) or \( C \), calls for a value \( u^m_n \) outside the range \( 1 \leq n \leq N \), the periodic boundary conditions, which translate into \( u^m_{n+N} = u^m_n \), must be used.

The tasks in this problem are:
1. **Causality and numerics.** Using arguments based solely on how the information propagates in the exact solution (characteristics), versus how it propagates in the numerical schemes above, **argue that:** (1.1) One of the schemes above cannot possibly work. (1.2) A necessary condition for the other two to work is that a restriction of the form \( k \leq \lambda h \) be imposed on the time step — where \( \lambda > 0 \) is a constant that depends on \( c = c(x) \).

2. Implement the schemes and try them out with various space resolutions,\(^5\) and a corresponding time resolution. Do you see convergence? Do the results agree with your analysis in item 1? **Report what you see, and illustrate it with plots** — a few well selected plots should be enough!

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### 6 Steady State Shallow Water #01

**Statement: Steady State Shallow Water #01**

The conservation form of the equations for 2-D shallow water waves over a flat bottom is

\[
0 = h_t + (hu)_x + (hv)_y, \tag{6.1}
\]

\[
0 = (hu)_t + (hu^2 + \frac{1}{2}gh^2)_x + (hv)_y, \tag{6.2}
\]

\[
0 = (hv)_t + (hu)_x + (hv^2 + \frac{1}{2}gh^2)_y, \tag{6.3}
\]

where \( h \) is the fluid depth, \( u \) is the \( x \)-flow velocity, \( v \) is the \( y \)-flow velocity, and \( g \) is the acceleration of gravity. The steady state (time independent) form of these equations is

\[
0 = (hu)_x + (hv)_y, \tag{6.4}
\]

\[
0 = (hu^2 + \frac{1}{2}gh^2)_x + (hv)_y, \tag{6.5}
\]

\[
0 = (hu)_x + (hv^2 + \frac{1}{2}gh^2)_y. \tag{6.6}
\]

**Answer the following questions:**

1. Under which conditions on \( (h, u, v) \) is (6.4 – 6.6) strictly\(^6\) hyperbolic? Use the **Froude number**

\[
F = \sqrt{\frac{u^2 + v^2}{gh}}
\]

in your answer. You need \( F > 0 \) to even ask the question — **WHY?**

2. When the characteristic equation has a double root, the system is not hyperbolic.\(^7\) **Show this.**

   **Hint:** The system is invariant under rotations. Hence, when computing the eigenvector(s), you can rotate the coordinate system so that \( v = 0 \) at the point of interest.

3. The system always has (at least) one characteristic, which has a Riemann invariant. Find it.

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\(^5\) \( N \) in the range \( 20 \leq N \leq 200 \) should be more than enough to see what happens.

\(^6\) All the characteristic directions are distinct: the characteristic equation has three distinct roots.

\(^7\) There is only one eigenvector associated with the double root.
7 Wave equations #01

Statement: Wave equations #01

Consider an elastic (homogeneous) string under tension, tied at one end, initially at rest, and forced by a (small amplitude) harmonic shaking of the other end. To simplify the situation, assume that all the motion is restricted to happen in a plane.

After a proper adimensionalization, the situation is modeled by the mathematical problem below for the wave equation in 1-D — where \( u = u(x, t) \) is the displacement from equilibrium of the string,

\[
\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0, \quad \text{for} \quad 0 < x < 1, \quad \text{and} \quad t > 0, \tag{7.1}
\]

with initial data \( u(x, 0) = u_t(x, 0) = 0 \), and boundary conditions

\[
u(0, t) = 1 - \cos(\omega t) \quad \text{and} \quad u(1, t) = 0. \tag{7.2}
\]

Find the solution to this problem, for the times \( 0 < t \leq 4 \). Furthermore: note that the solution, while making sense in the classical sense (no need to invoke generalized function derivatives), is not infinitely differentiable. There are certain lines along which “singularities” occur. Find these lines of singularity, and describe what the situation is along them (nature of the singularities) — the lines are, of course, characteristics. Finally: what is special about the cases \( \omega = \pi \) and \( \omega = \pi/2 \)?

THE END

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\(^8\) This formulation neglects dissipation in the string.