Due Fri. April 17, Spring 2020.

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1 IVP for a kinematic wave equation #03

Statement: IVP for a kinematic wave equation #03

Let $u$ be the density of some conserved quantity, governed by the equation

$$u_t + \left( \frac{1}{2} \ u^2 \right)_x = 0, \quad -\infty < x < \infty,$$

(1.1)

where shocks should be used to avoid multiple values. Consider now the following initial value $u(x, 0) = -\sin(x)$, and answer/perform the following questions/tasks:

1. **When and where do shocks form?** Find the places in space-time at which a shock has to be started to prevent the characteristics from crossing, and stop multiple values from arising.

2. **Where are the shocks located, for all times? Explicit formula required.** Find the shock path, for each shock identified in item 1. Hint: Do not attempt to explicitly compute $u$ on each side of the shock, to then solve the shock equations. Instead, show (e.g.; using the characteristic equations) that a certain symmetry applies to the solution. Then use it to get the shock position.

3. **What does the solution look like as $t \to \infty$?** Give a complete, and explicit, description — valid with errors much smaller than $1/t$. Hint: The characteristics end when they encounter a shock. Thus, argue\(^1\) that: as $t \to \infty$, only the characteristics starting near certain points remain in the solution. Use this to get an approximate description of the solution in this limit.

4. **Extra credit:** Calculate the first correction to the approximation in item 1.

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\(^1\) When I say argue, I really mean, argue. Using this just because I said so is not acceptable.
5. Find (a parametric description of) the envelope of the characteristics for \( t > 0 \), and plot it. Furthermore:

(5.1) Note that the envelope has many branches, describe them all. (5.2) Describe the asymptotic behavior of each branch for \( t \to \infty \).

**Hint.** The initial value is periodic of period \( 2\pi \). Argue that this period should carry over to the characteristics — hence to the solution, shocks, envelope of the characteristics, etc. Thus you only have to describe one copy only of each of the objects that you are asked to analyze in this problem. **Important:** When I say “argue” above, I mean it. Using this just because I said so here is not acceptable.

## 2 Smooth kinematic waves have infinite conservation laws

**Statement:** Smooth kinematic waves have infinite conservation laws

Consider the (kinematic wave) equation for some conserved scalar density \( \rho = \rho(x, t) \) in one dimension

\[
\rho_t + q_x = 0, \quad \text{where} \quad q = Q(\rho)
\]

(2.1)

is the flow function — which we assume is sufficiently nice (say, it has a continuous first derivative). **SHOW THAT:** if \( f = f(\rho) \) is some arbitrary (sufficiently nice) function, then there exists a function \( g = g(\rho) \) such that for any classical solution of (2.1), we have

\[
f_t + g_x = 0.
\]

(2.2)

In other words, (formally) \( f \) behaves as a “conserved” quantity, with flow function \( g \). In particular:

\[
\frac{d}{dt} \int_a^b f dx = g|_{x=a} - g|_{x=b}
\]

(2.3)

expresses the “conservation” of \( f \) for any interval \( a < x < b \).

**Very important:** it is crucial that the solution \( \rho \) have derivatives in the classical sense. When shocks are present, this result is false. As you will be asked to show in another problem, when shocks are present (2.2) has to be replaced by

\[
f_t + g_x = \sum_{\text{shocks}} c_s \delta(x - x_s(t)),
\]

(2.4)

where \( x = x_s(t) \) are the shock positions, \( \delta \) is the Dirac delta function, and the \( c_s = c_s(t) \) are some coefficients that are generally not zero. Thus the shocks act as sources (or sinks) for \( f \).

## 3 Entropy conditions for scalar (non convex) problem #01

**Statement:** Entropy conditions for scalar (non convex) problem #01

Consider the following conservation law, for the single scalar function \( v = v(x, t) \),

\[
v_t + p_x = 0, \quad \text{where} \quad p = v(v^2 - 1),
\]

(3.1)

\(^2\) That is: no shocks, so no generalized derivatives are involved.
v is the density for some conserved quantity, and \( p \) is the flux. Shocks for this equation, if allowed, should satisfy the Rankine-Hugoniot jump conditions

\[ -s[v] + [p] = 0, \quad \iff \quad s = \frac{[p]}{[v]} = v_a^2 + v_a v_b + v_b^2 - 1, \quad (3.2) \]

where \([\cdot]\) denotes the jump across the discontinuity of the enclosed quantity, \( v_a \) (resp. \( v_b \)) is the value of \( v \) immediately ahead of (resp. behind) the discontinuity, and \( v_a \neq v_b \).

The objective of this exercise is to find what additional restrictions (“entropy” conditions) the solutions to (3.2) must satisfy to produce acceptable as shocks.

We will try two approaches.

(A) “STABILITY” ANALYSIS: Let \( v \) be a steady state “solution” to (3.1), with a shock satisfying (3.2). That is, let

\[ v(x, t) = a \text{ for } x \geq st, \quad \text{and} \quad v(x, t) = b \text{ for } x \leq st, \quad (3.3) \]

where \( a \neq b \) are constants and \( s = a^2 + ab + b^2 - 1 \). Then:

\[ \text{Declare (3.3) an “acceptable” solution if and only if the linear problem that results when infinitesimal perturbations to (3.3) are considered is well posed.} \] \quad (3.4)

In other words: consider solutions to (3.1 – 3.2) of the following form:

\[ v(x, t) = a + A(x, t) \text{ for } x \geq st + r(t), \quad \text{and} \quad v(x, t) = b + B(x, t) \text{ for } x \leq st + r(t), \quad (3.5) \]

where \( A, B, \) and \( r \) are infinitesimal. This will lead to a system of linear equations for \( A, B, \) and \( r \) — with initial conditions

\[ \begin{aligned} A(x, 0) &= A_0(x) \text{ defined for } x \geq 0 \text{ only,} \\ B(x, 0) &= B_0(x) \text{ defined for } x \leq 0 \text{ only,} \\ r(0) &= r_0. \end{aligned} \quad (3.6) \]

Then the solution in (3.3) is accepted if and only if the initial value problem for \( A, B, \) and \( r \) is well posed (solutions exist and are unique).

Your task #1: Derive the equations satisfied by \( A, B, \) and \( r \), and find which conditions must be imposed on \( a \) and \( b \) so that the initial value problem for \( A, B, \) and \( r \) is well posed.

HINT-1. Existence will be obvious.\(^3\) However: be careful that you check uniqueness!

HINT-2. Interpret the conditions on \( a \) and \( b \) graphically. In the \( p-v \) plane,\(^4\) consider the two curves: \( p = v(v^2 - 1) \), and the secant line going through \( (v = a, p = p(a)) \) and \( (v = b, p = p(b)) \) — note that \( \text{the shock speed } s \text{ is the slope of this secant line!} \)

What do the conditions on \( a \) and \( b \) mean in terms of this picture?

(B) ZERO VISCOSITY LIMIT: Consider equation (3.1) as the \( \epsilon \downarrow 0 \) of the equation

\[ v_t + p_x = \epsilon v_{xx}. \quad (3.7) \]

Then the solution in (3.3) is accepted if and only there is a solution of this equation whose limit is (3.3). To be more precise: a solution to (3.7) of the form

\[ v = y \left( \frac{x - st}{\epsilon} \right) \quad (3.8) \]

is sought such that \( y(z) \to a \) as \( z \to \infty \) and \( y(z) \to b \) as \( z \to -\infty \) (notice that \( y \) satisfies a second order O.D.E., which can be easily integrated once). If such a solution exists, then (3.3) is accepted.

Your task #2: Carry out the calculation above, and find out under what restrictions on \( a \) and \( b \) a solution such as in (3.8) can be found. Compare your answers with those from the task #1.

\(^3\) The problem is very simple, and the solutions can be written explicitly.

\(^4\) Let the horizontal axis be \( v \), and the vertical be \( p \).
**HINT-3.** You will find yourself having to inspect an O.D.E. of the form $y' = F(y)$. You do not need to be able to solve this equation explicitly to answer the question. Just notice that the solutions to $y' = F(y)$ connect consecutive zeros of $F$ from $z = -\infty$ to $z = +\infty$, with the direction of the connection determined by the sign of $F$ between the two zeros. As in hint-2, a geometrical approach helps: think of $F$ as the difference between $p(y) = y(y^2 - 1)$ and a straight line.

**Your task #3:** Imagine that, in equation (3.1), we replace $p = v(v^2 - 1)$ by $p = (v^2 - 1)^2$. How will this change the answers? In particular — **note that these are the ONLY questions whose answer is required, but you must justify your answer:** (i) Will the conditions resulting from a stability analysis, and those resulting from a zero viscosity limit, be the same? (ii) If not, which set of conditions is better?

**HINT-4.** If you followed the advice of the prior hints, and interpreted your analysis for the prior two tasks geometrically, you should be able to perform this last task with nearly zero algebra. If not, you probably missed some key issue with tasks #1 or #2. Go back and take a second look.

What your prior analysis should have shown you is that the zero viscosity limit is a “global” criteria — depending on the values of $p(v)$ for the whole range between $a$ and $b$, while the stability analysis is “local” criteria — depending on the behavior of $p$ near $a$ and $b$ only.

**Remark 3.1.** Equation (3.1) is a not-too unrealistic (qualitative “toy” model) for some of the problems that arise when shock waves and phase transitions are (simultaneously) involved. Just to give you a little bit of the flavor of the connection: for a polytropic gas the equation of state $e = p v/ (\gamma - 1)$ gives isothermals and isentropes that are convex curves in the $p$-$v$ plane. However, when van der Waals forces are added, the curves become non-convex and develop local maximums and minimums like $p = p(v)$ above in (3.1), which lead to complicated restrictions on what solutions of the Rankine-Hugoniot conditions should be allowed, etc. Of course, (3.1) is far too simple to capture anymore than a few of the issues that arise in the “real” problem.

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**4 Information loss and traffic flow shocks**

**Statement:** Information loss and traffic flow shocks

In this problem we will show that shocks in traffic flow are associated with information loss, and will derive a formula that quantifies this loss. For simplicity, consider periodic solutions to the traffic flow equation

$$\rho_t + q_x = 0, \quad \text{periodic of period } T > 0,$$

where the traffic flux is given by $q = Q(\rho)$. Assume that there is a single shock per period, at $x = \sigma(t)$. The shock velocity, $s$, is given by the Rankine-Hugoniot jump condition

$$\frac{d\sigma}{dt} = s = \frac{q_R - q_L}{\rho_R - \rho_L},$$

where $\rho_R$ (resp.: $\rho_L$) is the value of $\rho$ immediately to the right (resp.: the left) of the shock discontinuity, $q_R = Q(\rho_R)$, and $q_L = Q(\rho_L)$. In addition, the shock satisfies the *entropy condition*

$$c_L > s > c_R,$$

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5 The nature of $\lim_{z \to \pm \infty} y$ depends on the zero: simple (higher order) zeros give exponential (algebraic) behavior.

6 Actually, the important curves to consider are the Hugoniot curves, which are also convex.
where \( c = \frac{dQ}{dp} \) is the characteristic speed, \( c_R = c(\rho_R) \), and \( c_L = c(\rho_L) \). Now, let
\[
\mathcal{I} = \mathcal{I}(t) = \int_{\text{period}} \frac{1}{2} \rho^2(x, t) \, dx.
\] (4.4)

1. **Show that:**
\[
\frac{d\mathcal{I}}{dt} = \int_{\text{period}} \frac{1}{2} (q_L - q_R)(\rho_L + \rho_R) + \int_{\rho_L}^{\rho_R} \rho \, c(\rho) \, d\rho.
\] (4.5)

**Hint.** For any short enough time period, a constant “\( a \)” exists such that \( a < \sigma(t) < a + T \) during the time interval. Given this, write \( \mathcal{I} \) in the form \( \mathcal{I} = \int_a^b \frac{1}{2} \rho^2 \, dx + \int_{\sigma}^{\sigma+T} \frac{1}{2} \rho^2 \, dx \), before taking the time derivative. Then use the fact that, in each of the two intervals \( \rho \) satisfies
\[
\rho_t = -c(\rho) \rho_x = -\frac{1}{\rho} f_x \quad \text{where} \quad f = \int_0^\rho s \, c(s) \, ds.
\] (4.6)

In addition, use (4.2) to eliminate \( \sigma \) from the resulting formulas.

2. **Show that:**
\[
\frac{dJ}{dt} < 0
\] (4.7)

by interpreting the right hand side in (4.5) geometrically, as (the negative of) the area between two curves — such area known to be positive. **Hint.** Use that \( \rho \, c = \frac{dQ}{dp}(\rho \, q) - q \), and (4.9) below.

**Remark 4.1** Recall that, in traffic flow, \( q = Q(\rho) \) is a concave function,\(^7\) vanishing at both \( \rho = 0 \) and the jamming density \( \rho_J > 0 \), positive for \( 0 < \rho < \rho_J \), with a maximum (the road capacity \( q_m \)) at some value \( 0 < \rho_m < \rho_J \). Because of this, the conditions (4.2–4.3) are equivalent to: \(^8\)

The shock velocity is given by the slope of the secant line joining \((\rho_L, q_L)\) to \((\rho_R, q_R)\). Further \( \rho_L < \rho_R \).

Note that **For \( \rho_L < \rho < \rho_R \), the curve \( q = Q(\rho) \) is above the secant line,** because \( Q \) is concave.

**Remark 4.2** How does \( \mathcal{I} \) in (4.4) related to “information”? Basically, we argue that a function contains the least amount of information when it is a constant, and that the more “oscillations” it has, the more information it carries. A (rough) measure of this is given by
\[
\mathcal{J} = \int_{\text{period}} \frac{1}{2} (\rho - \overline{\rho})^2 \, dx,
\] (4.10)
where \( \overline{\rho} \) is the average value of \( \rho \) (note that conservation guarantees that \( \overline{\rho} \) is a constant in time). Alternatively, write \( \rho \) the Fourier series \( \rho = \sum_n \rho_n(t) \, e^{i n \pi x/T} \) for \( \rho \). Then the amount of “oscillation” in \( \rho \) can be characterized by \( \sum_{n \neq 0} \frac{1}{2} |\rho_n|^2 \), which is the same as (4.10). It is easy to see that
\[
\mathcal{J} = \mathcal{I} - T \overline{\rho}^2.
\] (4.11)
Hence \( \dot{\mathcal{J}} = \dot{\mathcal{I}} \), so that (4.7) indicates that the shock is driving \( \rho \) towards a constant value.

The notion of “information” that we use here is not the same as that in “information theory” (which requires a stochastic process of some kind). Neither is it the same as the related notion of “entropy” of statistical mechanics (for this we would need a statistical theory\(^9\) for traffic flow). Nevertheless, \( S = -\mathcal{J} \) plays the same role for (4.1) that entropy plays for gas dynamics.

Finally, we point out that it can be shown that: for any convex function \( h = h(\rho) \) (i.e.: \( h'' > 0 \))
\[
\frac{d}{dt} \int_{\text{period}} h(\rho) \, dx < 0 \quad \text{when shocks are present},
\]
though showing this is a bit more complicated than for the special case \( h = \frac{1}{2} \rho^2 \).

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\(^7\) Concave means that \( \frac{d^2 Q}{d\rho^2} < 0 \). Note also that only solutions satisfying \( 0 \leq \rho \leq \rho_J \) are acceptable.

\(^8\) Thus shocks move slower than cars, and the density a driver sees increases as the car goes through a shock.

\(^9\) Such theories exist.
5 The zero viscosity limit for scalar convex conservation laws

Statement: The zero viscosity limit for scalar convex conservation laws

Consider a scalar convex conservation law, with a small amount of "viscosity" added. Namely:

\[ \rho_t + q_x = \nu \rho_{xx}, \quad \text{with} \quad q = q(\rho) \quad \text{smooth and convex:} \quad \frac{\partial^2 q}{\partial \rho^2} \geq C_q > 0, \quad \text{and} \quad \rho(x, 0) = f(x), \quad (5.1) \]

where \( C_q \) is some constant, \( 0 < \nu \ll 1 \), \( f \) is smooth, and \( f \) and all its derivatives vanish (rapidly) as \( |x| \to \infty \). We want to investigate the behavior, as \( \nu \to 0 \), of the solution to this problem. In particular, we want to compute

\[ I = I(t) = \int_{-\infty}^{\infty} \Psi(\rho(x, t)) \, dx, \quad \text{as} \quad \nu \to 0, \quad (5.2) \]

where \( \Psi \) is a smooth convex function such that \( \Psi(0) = 0 \). Furthermore, let \( h = h(\rho) \) be defined by

\[ \frac{dh}{d\rho} = c(\rho) \frac{d\Psi}{d\rho} \quad \text{and} \quad h(0) = 0, \quad \text{where} \quad c = \frac{dq}{d\rho}. \quad (5.3) \]

Note 1. We will assume that the solution \( \rho = \rho(x, t) \) to (5.1) exists, that it is smooth, that \( \rho \) and all its derivatives vanish (rapidly) as \( |x| \to \infty \), and that \( \rho \) is unique (within this class).

Note 2. Here we will use concepts introduced in the problem Lax Entropy condition for scalar convex conservation laws and information loss — you should read the statement for this exercise. This will also serve the purpose of giving meaning to \( I \) as a measure of the "wavy-ness" of the solution.

Multiplying by \( \frac{d\Psi}{d\rho} \) the equation \( \rho_t + c(\rho) \rho_x = \nu \rho_{xx} \) satisfied by \( \rho \), and using (5.3), we obtain

\[ \Psi(\rho)_t + h(\rho)_x = \nu \Psi'(\rho) \rho_{xx}, \quad (5.4) \]

where the prime denotes differentiation with respect to \( \rho \). Integrating this equation then yields

\[ \frac{dI}{dt} = \nu \int_{-\infty}^{\infty} \Psi'(\rho) \rho_{xx} \, dx = -\nu \int_{-\infty}^{\infty} \Psi''(\rho) \rho_x^2 \, dx < 0, \quad (5.5) \]

which shows that \( I \) is, always, a decreasing function of time.

From (5.5) is seems natural to conclude that: in the limit \( \nu \to 0 \), \( I \) becomes constant. However, this is false. As \( \nu \to 0 \), the solution to (5.1) develops thin transition layers (shocks) where the derivatives become large, so that the right hand side in (5.5) does not vanish as \( \nu \to 0 \). Your task here is to check this fact. Proceed as follows.

First, consider the case where the solution to (5.1) develops a single shock as \( \nu \to 0 \), along some path \( x = \sigma(t) \). Then,

For \( 0 < \nu \ll 1 \), the solution near \( x = \sigma \) (for any fixed time \( t \)) can be described by a traveling wave solution to the equation in (5.1), of the form

\[ \rho \sim R(z), \quad \text{where} \quad z = \frac{x - s \tau}{\nu} \quad \text{and} \quad s = \frac{d\sigma}{dt}, \quad (5.6) \]

with limits \( \rho_R \) as \( z \to \infty \) and \( \rho_L \) as \( z \to -\infty \). On the other hand, away from \( x = \sigma \), the derivatives of \( \rho \) remain bounded as \( \nu \to 0 \).

Use (5.6) and (5.5) to compute the \( \nu \to 0 \) limit of \( \frac{dI}{dt} \). You should be able to show that

\[ \frac{dI}{dt} \to h_R - h_L - s (\Psi_R - \Psi_L), \quad (5.7) \]

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\(^{10}\) Hence \( \frac{d^2 \Psi}{d\rho^2} \geq C_\Psi > 0 \), for some constant \( C_\Psi \).
where the subscript $R$ indicates evaluation at $\rho_R$, and the subscript $L$ indicates evaluation at $\rho_L$. Notice that this is the same formula for the contribution to the rate of change of $I$ by a shock listed in the statement for the problem Lax Entropy condition for scalar convex conservation laws and information loss — see the 9th equation there.

**Hint.** Consider the first integral in (5.5). Away from $x = \sigma$, the derivatives of the solution remain bounded, and the contribution of these regions to the integral vanishes as $\nu \to 0$. Hence we can limit the integration to a small region near the shock, namely

$$\frac{dI}{dt} \approx \nu \int_{\sigma-\epsilon}^{\sigma+\epsilon} \Psi'(\rho) \rho_{xx} \, dx, \quad 0 < \nu \ll \epsilon \ll 1,$$

(5.8)

where the traveling wave approximation $\rho \sim R(z)$ in (5.6) can be used. From this, using the o.d.e. that $R$ satisfies, the result in (5.7) follows.

**Second**, if the solution to (5.1) develops more than one shock as $\nu \to 0$, then a calculation as the one above can be done near each of the shock locations $x = \sigma_t(t)$, obtaining as the result that the right hand side in (5.7) is replaced by a sum including the contributions for each one of the shocks.

**Remark 5.1** The result in this problem illustrates a persistent phenomena in nonlinear p.d.e. that incorporate “small" perturbations involving the highest derivatives in the equation. Then, as the perturbations vanish, their effects on the solution do not. For example: in high Reynolds number flows thin boundary layers can form (mostly near walls), where viscous effects dominate, and contribute a finite amount (that does not go away as the Reynolds number goes to infinity) to the flow behavior. In other cases high frequency (thin) structures appear over large regions of the solution, which again do not go away as the perturbations vanish. This second type of situation is quite often associated with open problems where a good mathematical theory is lacking — e.g.: collision-less shocks in plasmas, turbulence, etc.

6 Discontinuous Coefficients in Linear 1st order pde #04m

**Statement:** Discontinuous Coefficients in Linear 1st order pde #04m

Consider the following initial value problem

$$u_t + \text{sign}(x) u_x = a \delta(x) u \quad \text{for} \quad t > 0 \quad \text{and} \quad -\infty < x < \infty, \quad \text{with} \quad u(x, 0) = U(x),$$

(6.1)

where $U$ is an “arbitrary" smooth function, $\delta(\cdot)$ is the Dirac's delta function, and $a$ is a constant.

Because of $a$ the $\delta(x)$ is the right hand side, we expect the solution (if any) to this problem to be singular at $x = 0$. But then $u_x$ would not be defined at $x = 0$, so what would be the meaning of $\text{sign}(x) u_x$? For that matter: what does $\delta(x) u$ mean when $u$ is singular at $x = 0$? Worse still: the characteristics (away from $x = 0$) of the problem, $\frac{dx}{dt} = \text{sign}(x)$, are all straight lines emanating from $x = 0$ — hence any lack of meaning at $x = 0$ is carried out to the whole real line by the characteristics.

Because of the problems pointed out in the prior paragraph, in this generality this problem has little or no chance of being meaningful. However, consider the special case when $a = -2$. Then, at least formally (since $d\text{sign}(x)/dx = 2 \delta(x)$) equation (6.1) is equivalent to

$$u_t + (\text{sign}(x) u)_x = 0 \quad \text{for} \quad t > 0 \quad \text{and} \quad -\infty < x < \infty, \quad \text{with} \quad u(x, 0) = U(x).$$

(6.2)

Assume that $u$ is the density of some conserved quantity (in particular, $u$ must be non-negative: $u \geq 0$) with flux given by $Q = \text{sign}(x) u$. Then show that (6.2) has a unique, un-ambiguous meaning, and write an explicit formula for the solution.

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11 Note that you do not need to have an explicit formula for $R$. Knowing the o.d.e. that $R$ satisfies, and knowing the limits as $z \to \pm \infty$ for $R(z)$, should be enough.
Hint: The equation has a clear meaning in each of the two regions \( \{ x, t > 0 \} \) and \( \{ x < 0 < t \} \). Thus pose the problems in each of these regions, in terms of the (unknown) values of the solution along the sides of the time axis: “\( u(\pm 0, t) = V_{\pm}(t) \)”. Then use the fact that \( u \) is a non-negative conserved density to find \( V_{\pm}(t) \) — you will need to invoke the integral form of the conservation law. Furthermore, assume that \( u \) is an integrable function, so that “stuff” cannot be concentrated at one point. That is: \( \int_a^b u \, dx \) vanishes as \( |b - a| \to 0 \).

7 Weak solutions (problem #02)

Statement: Weak solutions (problem #02)

Let \( \rho = \rho(x, t) \) be a piece-wise \( C^1 \) real valued function defined on space-time. Specifically: assume that there is a smooth curve \( x = x_s(t) \) such that \( \rho \) is defined everywhere but on this curve, and

1. \( \rho = \rho(x, t) \) has a continuous (partial) derivatives for \( x < x_s \) and \( x > x_s \).

2. \( \rho = \rho(x, t) \) and its partial derivatives have both left and right (finite and continuous) limits along the curve \( x = x_s(t) \).

For any continuous function \( h = h(\rho) \), define \( h^- \) and \( h^+ \) as the left and right limits, respectively, of \( h \) along the curve \( x = x_s(t) \). Namely: \( h^- = \lim_{x \to x_s^-, x < x_s} h(\rho) \) and \( h^+ = \lim_{x \to x_s^+, x > x_s} h(\rho) \). Furthermore, let \( |h| = h^+ - h^- \) be the jump in \( h \) across the curve.

Using the definition of generalized function derivatives, show that:

For any continuously differentiable functions \( f = f(\rho) \) and \( g = g(\rho) \),

\[
  f_t + g_x = (f_t + g_x)_f + (-\sigma [f] + [g]) \delta(x - x_s(t)),
\]

(7.1)

where \((\cdot)_f\) is used to indicate the standard derivatives — defined for \( x \neq x_s \) only, \( \delta \) is the Dirac delta function, and \( \sigma = dx_s/dt \).

Note 1: Assume that your test functions \( \phi = \phi(x, t) \) have infinitely many derivatives and vanish outside some bounded region in space-time. That is \( \phi \in C_0^\infty \).

Note 2: The curve given by \( x = x_s(t) \) need not be monotone. In other words, \( \sigma = \sigma(t) \) can vanish, switch signs, etc.