Problem Set Number 5, 18.306
MIT (Winter-Spring 2017)

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Due last day of classes, Spring 2017.
Turn it in (by 3PM) at the Math. Problem Set Boxes, right outside ................. room 4-174. There is a box/slot there for 306. Be careful to use the right box (there are many slots).

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1 Green’s functions #03

Statement: Green’s functions #03

Find the Green’s function for the initial value problem for the heat equation with mixed (as stated below) boundary conditions in an interval. Namely, solve the problem

\[ G_t = G_{xx} \quad \text{for} \quad 0 < x < 1 \quad \text{and} \quad t > 0, \quad \text{with} \]

(a) Boundary conditions \( G_x = 0 \) at \( x = 0 \) and \( G = 0 \) at \( x = 1 \).
(b) Initial conditions \( G(x, 0) = \delta(x - y) \), where \( 0 < y < 1 \), and \( \delta \) = Dirac’s delta function.

This problem can be done by the method of images, using the Green’s function for the infinite line

\[
 G_\infty(x, y, t) = \frac{1}{2\sqrt{\pi t}} \exp \left(-\frac{(x - y)^2}{4t}\right).
\]

HINT: Use 1-5 to replace the problem in (1.1) by one on the infinite line, periodic (which period?) and with the appropriate odd/even properties. Then use (1.2) to solve this new problem.

Let \( u = u(x, t) \) solve the heat equation, for \(-\infty < x < \infty \) and \( t > 0 \), with initial data \( u_0 \).

1. If \( u \) is even with respect to a point \( x_0 \), then \( u_x = 0 \) at \( x = x_0 \).
2. If \( u \) is odd with respect to a point \( x_0 \), then \( u = 0 \) at \( x = 0 \).
3. If \( u_0 \) is even with respect to a point \( x_0 \), then \( u \) is even with respect to \( x_0 \).
4. If \( u_0 \) is odd with respect to a point \( x_0 \), then \( u \) is odd with respect to \( x_0 \).

5. If \( u_0 \) is periodic, then \( u \) is periodic.

Proof of 3–5: use uniqueness, and the symmetries of the equation and data. Note that, for uniqueness to apply, we need to restrict the solutions in some fashion, for example: require them to be bounded.

## 2 Green’s functions #05

**Statement:** Green’s functions #05

Find the Green’s function for the initial value problem for the heat equation with Robin (as stated below) boundary conditions in the semi-infinite line. Namely, solve the problem

\[
G_t = G_{xx} \quad \text{for} \quad x > 0 \quad \text{and} \quad t > 0,
\]

with

(a) Boundary condition \( G - G_x = 0 \) at \( x = 0 \).

(b) Initial conditions \( G(x, 0) = \delta(x - y) \), where \( 0 < y \) and \( \delta = \) Dirac’s delta function.

(c) \( G \) is bounded for any \( t > 0 \).

This problem can be done by the method of images, using the Green’s function for the infinite line

\[
G_\infty(x, y, t) = G_\infty(x - y, t) = \frac{1}{2\sqrt{\pi t}} \exp \left(-\frac{(x - y)^2}{4t}\right).
\]

**Remark 2.1** You have to be careful with Robin boundary conditions. For example: \( T_t = T_{xx} \) for \( x, t > 0 \), with \( T + T_x = 0 \) at \( x = 0 \), has the solution \( T = e^{-x^2+t} \). This is well behaved in space, but grows exponentially in time. The reason is that the condition \( T_x = -T \) leads to a run-away heating: the hotter it gets at the origin, the larger the heat flow across there is. Physically, this is non-sense.

**Hints**

Use the facts below to replace the problem for \( G \) by one solvable by the method of images.\(^\dagger\) Then use \((2.2)\) to solve the new problem.

1. Let \( u \) solve the heat equation for \( x, t > 0 \), with \( u - u_x = 0 \) at \( x = 0 \) and initial data \( u_0 \). Then \( v = u - u_x \) solves the heat equation for \( x, t > 0 \), with \( v \) at \( x = 0 \) and initial data \( v_0 = u_0 - u'_0 \).

2. Let \( v \) solve the heat equation for \( x, t > 0 \), with \( v \) at \( x = 0 \) and initial data \( v_0 = u_0 - u'_0 \) — where \( u_0 \) is bounded at infinity. Assume that \( v \) and its derivatives are bounded at infinity. Let \( u \) be the (bounded at infinity) solution to the ode \( u_x - u = -v \) (this defines \( u \) uniquely). Then

   2.1 The initial data for \( u \) is \( u_0 \), \( u - u_x \) vanishes at the origin, and \( u = e^{\alpha \int_0^\infty e^{-s} v(s, t) \, ds} \).

   2.2 \( u \) satisfies the heat equation.

   Proof. Substitute \( v = u - u_x \) into the heat equation. Thus \( (u_t - u_{xx}) - (u_t - u_{xx})_x = 0 \). Hence (for some \( \alpha \)) \( u_t - u_{xx} = \alpha(t) e^x \). But \( u_t - u_{xx} \) is bounded at infinity. Hence \( \alpha = 0 \).

\(^\dagger\) Use that: if \( v \) is a (bounded at infinity) solution to the heat equation with odd initial data, then \( v \) is odd — in particular \( v = 0 \) at the origin. This follows from uniqueness.
3 Laplace equation in a circle #01

Statement: Laplace equation in a circle #01

Green’s function: Laplace equation in a circle, with Dirichlet BC. Consider the question of determining the steady state temperature in a thin circular plate such that: (a) The temperature is prescribed at the edges of the plate, and (b) The facets of the plate (top and bottom) are insulated. This problem can be written — using polar coordinates and non-dimensional variables (selected so that the plate has radius 1) — in the form

\[
\frac{1}{r^2} \left( r \frac{d}{dr} \left( r \frac{dT}{dr} \right) + \frac{d^2 T}{d\theta^2} \right) = \Delta T = 0, \quad \text{for } 0 \leq \theta \leq 2\pi \quad \text{and } 0 \leq r \leq 1, \quad (3.1)
\]

with the boundary condition

\[
T(1, \theta) = h(\theta), \quad \text{for some given function } h. \quad (3.2)
\]

\[†\] and \( h \) are \( 2\pi \)-periodic in \( \theta \).

The solution to this problem, as follows from separation of variables, is

\[
T = \sum_{n=-\infty}^{\infty} h_n r^{|n|} e^{in\theta}, \quad \text{where } h_n = \frac{1}{2\pi} \int_{0}^{2\pi} h(\theta) e^{-in\theta} d\theta. \quad (3.3)
\]

For \( r < 1 \) this formula provides a solution to the equation even if \( h \) is not very smooth (or in fact, not even a function) and thus has a very poorly convergent Fourier series \( \sum h_n e^{in\theta} \). For example, assume that the sequence \( \{h_n\} \) is bounded. Then, in any disk of radius \( R < 1 \): (i) the series for \( T \) in (3.3) converges absolutely; and (ii) the series obtained by term by term differentiation, for any derivative of \( T \), also converges absolutely. It follows that \( T \) is smooth for \( r < 1 \). In a similar fashion, one can justify exchanging the order of integration and summation — as you are asked to do below.

1. Assume that \( h = \delta(\theta) \), and write an explicit formula for the solution \( T \).

2. In (3.3), substitute the formula for \( h_n \) into the series for \( T \), exchange the summation and integration order, and derive an equation of the form

\[
T = \int_{0}^{2\pi} G(r, \theta - \phi) h(\phi) d\phi. \quad (3.4)
\]

3. Show that:

(a) \( G > 0 \) for \( r < 1 \).

(b) \( G(r, \theta) \to 0 \) as \( r \to 1 \), if \( \theta \neq 0 \).

(c) \( \int_{0}^{2\pi} G(r, \theta) d\theta = 1 \) for \( r < 1 \).

Hint. Add the positive and negative indexes \( n \) separately. Then use that \( \sum_{n=0}^{\infty} z^n = \frac{1}{1-z} \) for \( |z| < 1 \).

4 Normal modes and a boundary layer

Statement: Normal modes and a boundary layer

Consider the situation where small particles in a liquid column of depth \( L \) are settling under the influence of gravity, while undergoing brownian motion. We will now make a very simple model for this (I stress the “simple” in simple model, do not take it too seriously).

Let \( 0 < x < L \) be the depth coordinate, with the surface at \( x = L \) and the bottom at \( x = 0 \). If the particles are very small, inertia is not important, and their motion is determined by the balance between the gravitational force on

\[\text{Check, by direct substitution, that this is indeed the solution. That (3.2) is satisfied should be "obvious".}\]
each particle, and the fluid drag. In addition, if the particles are not too close together,\(^1\) this leads to a constant downward velocity — which creates a flux \(\vec{q}_s = -a \, u \, \hat{n}\), where \(u\) is the particle density, \(a > 0\) is the settling velocity, and \(\hat{n}\) is the unit vector pointing up. The brownian motion creates an additional flux \(\vec{q}_b = -\nu \, \text{grad}(u)\), where \(\nu > 0\) is the diffusivity. Thus we obtain an advection-diffusion equation, which in 1-D is

\[
    u_t = \nu \, u_{xx} + a \, u_x \quad \text{for} \quad x > 0,
\]

where the total flux is \(q = -\nu \, u_x - a \, u\). In addition, we impose the boundary conditions:

- The flux vanishes at the bottom, \(0 = q(0, t) = -\nu \, u_x(0, t) - a \, u(0, t)\). (4.2)
- The flux vanishes at the surface, \(0 = q(L, t) = -\nu \, u_x(L, t) - a \, u(L, t)\). (4.3)

**Remark 4.1** In the absence of diffusion, \(\nu = 0\), the solutions to (4.1) take the simple form \(u = f(x + at)\), which results because all the particles are going down at the same velocity. In this case the boundary condition in (4.2) does not make sense, since it leads to \(0 = f(at)\) for all times. That is, only \(u \equiv 0\) satisfies it! The reason for this problem is clear: there is nothing in the equation that can stop the particles as they approach the bottom, so how can \(q = 0\) happen there?

A reasonable modeling assumption when \(\nu = 0\) (particles are too large for the brownian motion to matter) is to posit that, as they arrive to the bottom, they stop and accumulate there at some maximum density \(u_{\text{max}}\), with the effective position of the bottom moving up as particles arrive. If \(x = \sigma(t)\) is this position (accumulation front), conservation of particles leads to the equation

\[
    \frac{d\sigma}{dt} = a \frac{u(\sigma, t)}{u_{\text{max}} - u(\sigma, t)}, \quad \text{where} \quad u(\sigma, t) = f(\sigma + at)
\]

and \(u = f(x + at)\) for \(x > \sigma\). Of course, we must also assume that \(0 \leq f < u_{\text{max}}\) — which is OK, since the “constant speed downward” assumption requires \(u \ll u_{\text{max}}\) anyway.

The problem in remark 4.1 does not arise when \(\nu > 0\), because diffusion can balance the advection term \(a \, u_x\), provided that the density gradient is large enough near the bottom. In this problem we will explore the behavior of the solutions near the bottom, particularly in the case when \(0 < \nu \ll 1\).

1. **Justify (4.4).**

2. The system in (4.1-4.3), as written, does not appear to be self-adjoint — which would be a not very desirable situation. However, it turns out that this is false: the system is actually self-adjoint. **Show this** by exhibiting a change of variables, \(u \to v\), for which the equation becomes the diffusion equation, \(v_t = \nu \, v_{xx}\) for \(0 < x < L\), with Robin boundary conditions.
   
   *Hint.* Try \(v = u \, e^{\beta \, x + \alpha \, t}\), for some appropriately selected constants \(\beta\) and \(\alpha\).

3. **Find all the normal mode solutions** of the problem for \(v\) — that is: non-trivial solutions of the form \(v = e^{\lambda \, t} \, \phi(x)\), where \(\lambda\) is a constant. Make sure that you get all of them, else you will miss the point of items 5 and 6.

   The associated eigenvalue problem (i.e.: \(\lambda \, \phi = \nu \, \phi_{xx}\), with the appropriate boundary conditions) is self-adjoint. Hence the spectrum is real, and the eigenfunctions can be used to represent “arbitrary” functions. In this case the spectrum is a discrete set of eigenvalues \(\{\lambda_n\}\), and the eigenfunctions form an orthogonal basis.

4. Given initial conditions \(u(x, 0) = u_0(x)\), use the results in items 1–3 to **write the solution to (4.1–4.3) as a series** involving the normal modes

\[
    u = \sum_n a_n \, e^{(\lambda_n - \alpha) \, t} \, \phi_n(x) \, e^{-\beta \, x}.
\]

**Write explicit expressions\(^4\)** for the coefficients \(a_n\).

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\(^1\) Hence we can ignore their interactions via the fluid.

\(^4\) They should be given by integrals involving \(u_0\) and the eigenfunctions \(\phi_n\).
5. What happens with the solution in (4.5) as $t \to \infty$?

6. Examine your answer to item 5 when $0 < \nu / a$ becomes small.
   — Can you explain/interpret the result using physical arguments?
   — At what point would you stop trusting the model predictions? *Hint. See remark 4.1.*

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### 5 Random walk in 2-D #01

**Statement:** Random walk in 2-D #01

Consider a particle executing a discrete random walk in the plane, with time step $k > 0$, according to the following rule:

Let $(x, y) = (x_1, y_1)$ be the position of the particle at time $t = t_1$. Then, during the time interval $t_1 \leq t \leq t_1 + k$, the particle moves to one of the four positions $(x_1 \pm h_x, y_1)$ and $(x_1, y_1 \pm h_y)$

— with equal probability $p = 1 / 4$. Here $h_x > 0$ and $h_y > 0$ are some given constants.

Let now $\rho = \rho(x, y, t)$ be the probability distribution function for the particle position. That is, the probability that the particle is in some set $\Omega$ at time $t$ is given by

$$ \mathcal{P}\{\text{particle position } \in \Omega\} = \int_{\Omega} \rho \, dx \, dy. \tag{5.2} $$

Perform now the following tasks

1. Write the equation that gives $\rho$ at time $t + k$ in terms of $\rho$ at time $t$.

2. Write $h_x = \mu_x h$ and $h_y = \mu_y h$ — where $\mu_x > 0$ and $\mu_y > 0$ are some constants, and *consider the limit in which $k$ and $h$ both vanish*. Find a scaling relating $h$ and $k$ such that, in this limit, the equation you wrote for item 1 becomes a diffusion equation for $\rho$. Write this equation.

3. Write the fundamental solution for the equation derived in item 2. That is, the solution $\rho_f$ corresponding to the initial value $\rho_f(x, y, 0) = \delta(x) \delta(y)$.

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*Assume that $\rho$ is twice continuously differentiable in space, and continuously differentiable in time.*