1 Release a bungee jumping cord hanging vertically

1.1 Statement: Release a bungee jumping cord hanging vertically

Consider a bungee jumping cord hanging vertically, attached at the top (at some fixed point), with a mass $M$ at the bottom. For simplicity we will neglect side motions (that is: we assume that the whole assembly, cord and mass, is hanging perfectly vertical at all times). The equations governing this situation (derived in another problem) are

$$u_{tt} - c^2 u_{ss} = g,$$  \hspace{1cm} (1.1)

where $c^2$ is a wave velocity (stretching waves along cord). The boundary conditions for this equation are

$$u = 0 \text{ at } s = 0 \quad \text{and} \quad M u_{tt} = g M - k (u_s - 1) \text{ at } s = L.$$  \hspace{1cm} (1.2)

In this equation $g$ is the acceleration of gravity, and

1. $x$ be the vertical coordinate, increasing downwards. The top end of the cord is attached at $x = 0$.
2. The cord is idealized as a curve (neglect thickness). Each point along the cord is assigned a label $s$, $0 \leq s \leq L$, defined as follows: When the cord is not under tension (nor compressed), $s$ is the distance (along the cord) from the point to the end of the cord attached at $x = 0$. The unstretched cord length is $L$.
3. The state of the cord is described by $u = u(s, t)$, where $u$ is defined as follows: $x = u(s, t)$ is the vertical coordinate of the point along the cord whose label is $s$.
4. The cord is assumed to satisfy Hooke’s law: an unstretched length of cord $\Delta s$, when stretched to length $\Delta x$, generates a tension $T = k \frac{\Delta x - \Delta s}{\Delta s}$, where $k > 0$ (a constant) is the cord coefficient of elasticity.
5. $c^2 = k/\rho$, where $\rho$ (a constant) is the mass of the cord per unit (unstretched) length.

Typical, order of magnitude, numbers: $L = 25\text{m}$, $\rho = 0.2\text{kg/m}$, $k = 10^3\text{N}$, $c \approx 70\text{m/s}$, and $M = 70\text{kg}$ (mks units)

Questions/tasks:
Q1. Find the steady state solution for 1.1-1.2, and the tension \( T = k (u_s - 1) \) at \( s = 0, L \).

Q2. After having made the jump over a bridge, a jumper is hanging placidly several meters over the surface of a warm tropical lake, admiring the colored fish below. At time \( t = 0 \) the cord attachment at \( x = 0 \) breaks.

A. What happens to the jumper during the time interval \( 0 < t < t_c \), some critical time \( t_c \)? What is \( t_c \)?

B. When the cord breaks at the top, the tension is released there and the cord contracts back to its natural length. But it cannot go beyond this: cords are not able to withstand compression well, and they bend instead. Hence, there is going to be a chunk of cord, \( 0 \leq s \leq \sigma_b(t) \), that coils as it falls (basically on free fall). Give a formula for \( \sigma_b(t) \).

Note: the solution to the equation is NOT needed to do this part Q2.

1.2 Answer: Release a bungee jumping cord hanging vertically

Q1. The steady state solution is

\[ u^* = -\frac{g}{2c^2}s^2 + \alpha s, \text{ where } \alpha = 1 + \frac{gM}{k} + \frac{gL}{c^2} = 1 + \frac{g(M + gL)}{k}. \] (1.3)

It follows that, at \( s = 0 \) the tension is: \( T = k (u_s^* - 1) = k (\alpha - 1) = g (M + gL) \). That is: exactly what is needed to support the weight of the cord plus the mass at the bottom. Similarly, \( T = gM \) at \( s = L \).

Q2. Equation (1.1) is the wave equation, hyperbolic with characteristic speeds \( \pm c \). Nothing propagates faster than \( c \). In particular, singularities propagate exactly at speed \( c \). When the cord breaks, the tension at the top drops to zero, which produces a discontinuity in \( T \) that propagates downward at speed \( c \). Thus

A. For \( 0 < t < t_c = L/c \) nothing happens to the jumper. In fact, the cord below \( s = \sigma_b(t) = ct \), and the jumper, just hang there as if nothing had happened.

B. The coiled portion of the cord, in free fall, is given by \( 0 \leq s \leq \sigma(t) = ct \). Beyond \( t_c \) both the jumper and the cord are in free fall.

Remarks

1. If the answer in A seems counter-intuitive, think of a far away star going super-nova. The event may have happened thousands of years ago, but we continue seeing the star as if nothing has happened till a time interval \( d/cL \) has passed, where \( d \) is the distance to the star and \( cL \) is the speed of light. Not sure this is more “intuitive”, but at least it should be more familiar.

The phenomena described in Q1-Q2 happens when you release a slinky. For striking videos, search the internet: “slinkies falling”, “slinky dropped from building”, “slinky drop slow motion”, and (particularly relevant to us) “slinky drop with tennis ball”. “Awesome HD Slinky Slow-Mo” in youtube is actually awesome.

Here are a few links:

https://www.youtube.com/watch?v=uiyMuHuCFo4 “Awesome HD Slinky Slow-Mo”
https://www.youtube.com/watch?v=8UimHnsWSBc “Slow motion slinky drop 1000fps”

These are some “tennis ball” videos:

https://www.youtube.com/watch?v=SqlHDh4STRc “Slinky with a Tennis ball vpython”
https://www.youtube.com/watch?v=okb2tChFyWU “Slinky Drop Extended”

Note that slinkies do not coil that easily [approximating them as a line, when they are not stretched, is not good]. Hence the “coiled cord” description above is replaced by “solid chunk of slinky”, which you will see bending and coiling only in the videos with a very long slinky.
2. How much coiling of the cord you actually see depends on how stiff the rope is [specifically: how large \( c \) is]. The top of the cord starts free fall immediately, and so will be at \( x = \frac{1}{2} g t^2 \). If the speed \( c \) is large, \( \sigma \) will reach the mass very quickly, while the top has barely fallen. Then rather little coiling will happen, with the rope just “snaking” a bit. At the other end you can have a situation where the free falling part overtakes that wave at \( x = u^*(\sigma(t)) \), at which point the simple picture in this problem no longer applies.

3. When the cord breaks at the top, transversal vibrations in the cord will be (generally) triggered. If they are small, it can be shown that they propagate at speed \( c \sqrt{(u^* - 1)/u^*} < c \). Thus these waves cannot reach the jumper before \( t = t_c \).

4. Actual bungee cords do not follow Hooke’s law, particularly for the extreme stretching involved in jumping. In the model here, this means that the equation has to be corrected, and becomes non-linear, with a wave speed that depends on the solution (specifically \( c = c(u_s) \)). The main qualitative picture does not change, but the quantitative details do. In particular, note: (i) If the cord behaves like a “hard spring” (\( c \) increases with stretching), then the wave triggered by the break at the top becomes a rarefaction wave [no longer localized at \( s = \sigma_0(t) \), as in the linear problem. (ii) If the cord behaves like a “soft spring” (\( c \) decreases with stretching), then the wave from the top becomes a shock wave.

5. Another effect neglected by (1.3) is dissipation, which “destroys” hyperbolicity. However, it is not important over the short time span involved in this problem.

### 2 Hodograph transformation #01

**Note #1 This problem is optional!** However, I strongly urge you to give it a try, as it is relevant to the material we will be covering next week [May 5-7].

**Note #2 Example #3 in the problem statement refers to the (inviscid and isentropic) Euler equations of Gas-Dynamics in mass-Lagrangian coordinates. We derived these equations in Eulerian coordinates, where they take the form

\[
\rho_t + (\rho u)_x = 0 \quad \text{and} \quad (\rho u)_t + (\rho u^2 + p)_x = 0. \tag{H}
\]

In mass-Lagrangian coordinates the space coordinate \( x \) is replaced by the mass, \( \zeta \), between some arbitrary (fixed) particle in the gas, and the current position. That is \( \zeta = \zeta(x, t) = \int_{x^*}^{x} \rho(s, t) \, ds \), where \( x = x^*(t) \) is the position of the fixed point in the gas — i.e.: \( \frac{dx^*}{dt} = u(x^*, t) \).

Then, as long as no vacuum state arises, the transformation from the Eulerian coordinates \( (x, t) \) to the (mass) Lagrangian coordinates \( (\zeta, t) \) has the inverse \( x = x(\zeta, t) = x^* + \int_{0}^{\zeta} v(z, t) \, dz \), where we think of \( v = 1/\rho \) (the specific volume) as a function of \( \zeta \) and \( t \) in the integral — i.e.: \( v = v(\zeta, t) \). Using that \( \zeta_t = -\rho u \) and \( \zeta_x = \rho \), it is easy to see that [H] above and (2.3) are the same equations. **Alternatively**, you can check directly that the equations in (2.3) correspond to the conservation of mass and momentum in Lagrangian coordinates.

#### 2.1 Statement: Hodograph transformation #01

The **hodograph transformation** reverses the roles of the independent and dependent variables. It is helpful when dealing with a quasi-linear first order nonlinear P.D.E. such that:

**a. The equation is homogeneous**: all the terms involve exactly one derivative.

\(^1\) So that \( \rho > 0 \) everywhere.
b. The coefficients do not involve the independent variables.
c. The number of dependent variables does not exceed the number of independent variables.

In such cases the transformation can be used to linearize the P.D.E. The transformation is, in some vague sense, a
generalization to P.D.E. of separation of variables. Namely, the solution of the nonlinear equation \( \frac{du}{dt} = f(y) g(t) \)
by separation of variables is equivalent to the following process: (1st) Introduce a new independent variable by
\( ds = g dt \), which transforms the equation into one with no dependence on the independent variable \( \frac{dy}{ds} = f(y) \).
(2nd) Invert the roles of the dependent and independent variables, to obtain the linear equation \( \frac{ds}{dy} = \frac{1}{f(y)} \).

**Example 1: scalar 1-st order evolution in one space dimension.**
Consider the following nonlinear equation, for the real valued function \( u = u(x, t) \),
\[
\begin{align*}
    u_t + c(u) u_x &= 0, & \text{where } c = c(u) \text{ is some given function.} \\
    \text{(2.1)}
\end{align*}
\]

**1-a.** Do a hodograph transformation, and write the equation for \( x = X(u, t) \). This should give you a trivial equation.
Write the general solution to this equation. How is this solution related to the solution to the initial value problem
\( u(x, 0) = f(x) \) to \( (2.1) \), obtained by characteristics?

**1-b.** Do the same for \( t = T(u, x) \).

**Example 2: scalar 1-st order evolution in two space dimensions.**
Find a transformation that linearizes the equation
\[
\begin{align*}
    u_t + a(u) u_x + b(u) u_y &= 0, & \text{where } a = a(u) \text{ and } b = b(u) \text{ are some given functions.} \\
    \text{(2.2)}
\end{align*}
\]

**Example 3: isentropic Gas Dynamics in one space dimension.**
The (inviscid) Euler equations of Gas Dynamics, in mass-Lagrangian coordinates, are as follows
\[
\begin{align*}
    v_t - u \zeta &= 0, & \text{and } u_t + p \zeta &= 0, \\
    \text{(2.3)}
\end{align*}
\]
where \( v \) is the specific volume, \( u \) is the flow velocity, and \( p = p(v) \) is the pressure. Furthermore: \( a^2 = - \frac{dp}{dv} > 0 \),
with \( a > 0 \) the sound speed in mass-Lagrangian coordinates.

**3-a.** Do a hodograph transformation, and write the equations that \( \zeta = Z(v, u) \) and \( t = T(v, u) \) satisfy. This should
give you a system of two, first order, linear equations in \( Z \) and \( T \).

**3-b.** Write the characteristic equations for the system that you obtained in \( 3-a \).
Recall that the system of equations in \( (2.3) \) is equivalent (for solutions without shocks) to the following Riemann
invariant form:
\[
    \begin{align*}
        u \mp b &= \text{constant} \quad \text{along the characteristic curves } \frac{d\zeta}{dt} = \pm a, \\
        \text{(2.4)}
    \end{align*}
\]
where \( b = b(v) = \int^v a(s) \, ds \). In particular, the solutions such that one of the Riemann variables \( u \mp b \) is identically
constant (not just constant along each characteristic) are called *simple waves*.

**3-c.** What is the relationship of the characteristics for the system that you found in \( 3-a \), with the Riemann variables
\( R_{\pm} = R_{\pm}(u, v) = u \mp b? \)

**3-d.** Assume a simple wave solution to the system in \( (2.3) \). What is peculiar about the hodograph map \( (\zeta, t) \rightarrow (u, v) \)
in this case — what is the image in the \( (u, v) \) plane? Is there an inverse map \( \zeta = Z(v, u) \) and \( t = T(v, u) \), as
assumed when doing the hodograph transformation?
2.2 Answer: Hodograph transformation #01

Example 1: scalar 1-st order evolution in one space dimension.

1-a. We have \( u_x = 1/X_u \) and \( u_t = -X_t/X_u \). Hence (2.1) becomes

\[
X_t = c(u). \tag{2.5}
\]

Of course, for this to work we need that \( X_u \) (equivalently \( u_x \)) be finite and non-zero.

The general solution to (2.5) is given by

\[
x = c(u) t + g(u). \tag{2.6}
\]

In particular, if \( g \) is the inverse function\(^2\) to the initial data \( u(x, 0) = f(x) \) — assuming that there is such an inverse, then (2.6) becomes:

\[
u = f(x - c(u) t). \tag{2.7}
\]

This is, of course, the implicit form of the solution that results from applying the method of characteristics to the initial value problem for (2.1).

1-b. We have \( u_t = 1/T_u \) and \( u_x = -T_x/T_u \). Hence (2.1) becomes

\[
T_x = 1/c(u). \tag{2.8}
\]

Of course, for this to work we need that \( T_u \) (equivalently \( u_t \)) be finite and non-zero. The general solution to (2.8) is

\[
t = x/c(u) + h(u), \]

which is the same as (2.6) — with \( g = -ch \).

Example 2: scalar 1-st order evolution in two space dimensions.

Let \( t = T(u, x, y) \). Then \( u_t = 1/T_u \), \( u_x = -T_x/T_u \) and \( u_y = -T_y/T_u \). Hence (2.1) becomes

\[
1 = a(u) T_x + b(u) T_y, \tag{2.9}
\]

which is linear. Again, this only works provided \( T_u \) is finite and non-zero.

Example 3: isentropic Gas Dynamics in one space dimension.

Since we will deal only with solutions without shocks, we rewrite the system of equations (2.3) in the form

\[
v_t - u_\zeta = 0, \quad \text{and} \quad u_t - a^2 v_\zeta = 0. \tag{2.10}
\]

3-a. By implicit differentiation of \( \zeta = Z(v, u) \) and \( t = T(v, u) \), it follows that

\[
J u_t = +Z_v, \quad J u_\zeta = -T_v, \quad J v_t = -Z_u, \quad \text{and} \quad J v_\zeta = +T_u, \tag{2.11}
\]

where \( J = Z_v T_u - Z_u T_v \) is the Jacobian of the transformation. If we assume that \( J \) is finite and nonzero, substitution of (2.11) into (2.10) then yields the system of linear first order equations:

\[
Z_u - T_v = 0, \quad \text{and} \quad Z_v - a^2(u) T_u = 0. \tag{2.12}
\]

Notice that elimination of \( Z \) (by taking cross derivatives) in this system yields the (linear) wave equation

\[
T_{vv} - a^2(v) T_{uu} = 0. \tag{2.13}
\]

3-b. The system in (2.12) has the following characteristic form

\[
(Z_v \pm a Z_u) \mp a (T_v \pm a T_u) = 0. \tag{2.14}
\]

Equivalently

\[
\frac{dZ}{dv} \mp a(v) \frac{dT}{dv} = 0 \quad \text{along the characteristic curves} \quad \frac{du}{dv} = \pm a(v). \tag{2.15}
\]

\(^2\) That is: \( g(f(x)) = x \).
3-c. Notice that the characteristic curves in (2.15) are given by $R_{\pm}(u, v) = u \mp b(v) = \text{constant}$, where $R_{\pm}$ are the Riemann variables for the system in (2.10) — see (2.4). On the other hand, the Riemann variables for (2.12) are not invariant along characteristics. Namely, their evolution is coupled:

$$\frac{d}{dv} (Z \mp a(v) dT) = \mp a'(v) T \quad \text{along the characteristic curves} \quad \frac{du}{dv} = \pm a(v). \quad (2.16)$$

This shows that, while (2.12) is linear, this does not mean that it is trivial. The hodograph transformation is not a “panacea”.

3-d. For a simple wave solution to the system in (2.3) — equivalently to (2.10), one of the Riemann invariants is identically constant, say $R_+ = u - b \equiv \text{constant}$. This means that $u$ is a function of $v$. Hence the map $(\zeta, t) \rightarrow (u, v)$ takes the whole $(\zeta, t)$ plane into just a curve in the $(u, v)$ plane. In particular, the Jacobian of the map vanishes identically $u_\zeta v_t - u_t v_\zeta \equiv 0$. Of course, in this case there is no inverse map $\zeta = Z(v, u)$ and $t = T(v, u)$, as assumed when doing the hodograph transformation.

THE END.