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1 Equations for a bungee jumping cord hanging vertically

1.1 Statement: Equations for a bungee jumping cord hanging vertically

Consider a bungee jumping cord hanging vertically, attached at the top (at some fixed point), with a mass \( M \) at the bottom. For simplicity we will neglect side motions (that is: we assume that the whole assembly, cord and mass, is hanging perfectly vertical at all times), but not vertical motion. **Your task is to derive equations for the dynamics of the system,** under the following assumptions:

1. Let \( x \) be the vertical coordinate, increasing downwards. Further: the top end of the cord is attached at \( x = 0 \).
2. **Neglect the thickness of the cord,** so you can idealize it as a curve, and label each point along the cord by a coordinate \( s \), \( 0 \leq s \leq L \), defined as follows: When the cord is not under tension (nor compressed), \( s \) is the distance (along the cord) from the point to one end of the cord — specifically: the end that is attached at \( x = 0 \). Hence the unstretched length of the cord is \( L \).
3. Let \( \rho = \text{constant} \) be the mass of the cord per unit length — that is, the mass between \( s = a \) and \( s = b > a \) is: \((b - a)\rho\). Furthermore, let \( g = \text{acceleration of gravity} \).
4. Describe the state of the cord, at any time, by \( u = u(s, t) \), where \( u \) is defined as follows: \( x = u(s, t) \) is the vertical coordinate of the point along the cord whose label is \( s \). For example: in the absence of motion and zero gravity, the cord is described by \( u = s \).
5. Assume that the cord is perfectly elastic and **satisfies Hooke’s law.**\(^1\) Thus an unstretched length of cord \( \Delta s \), when stretched to length \( \Delta x \), generates a tension \( T = k \frac{\Delta x - \Delta s}{\Delta s} \), where \( k > 0 \) is the cord coefficient of elasticity. Recall that the tension, at any point along the cord, is the force that one side exerts on the other — if you were to cut the cord at some point, the tension is the force that you would have to apply to each side of the cut to keep it from retracting.
6. Cords are not able to withstand compression well, and they bend when under compression. Thus, **assume that the vertical motion along the cord is small enough to keep it always under tension, everywhere** — this is where having a mass at the bottom helps.

**Hints.**

#1 You need to derive an equation for \( u \), which applies for \( 0 < s < L \). For this purpose use the conservation of the vertical momentum (you can calculate the momentum density, and flux, in terms of \( u \) and the constants \( \rho \) and \( k \)). Note that there is also a momentum source, due to gravity.

**Write this equation in both its integral, as well as its differential form.**

To write the integral form you will need to assume that \( u \) has continuous partial derivatives; to write the differential form you will need to assume that the second derivatives are continuous as well.\(^2\)

#2 The equation for \( u \) needs boundary conditions (BC) at the ends of the cord. This is where the mass \( M \) comes into play: note that the mass position is \( x = u(L, t) \).

1.2 Answer: Equations for a bungee jumping cord hanging vertically

The vertical momentum density per unit unstretched length of cord is \( \rho u_t \), the vertical momentum flux is given by the tension \( T \) in the cord, and there is a source of momentum provided by the force of gravity on the cord. Thus,

---

\(^1\) During an actual bungee jump, the cord experiences extensions over a huge range, and Hooke’s law is not valid. However, here we are looking at a post-jump scenario, with the cord already stretched and experiencing small variations in the range of stretching.

\(^2\) Actually, these hypothesis are too strong — e.g.: the conservation form is OK if \( u \) has “corners”. But do not worry about this here.
the conservation of vertical momentum leads to
\[ \frac{d}{dt} \int_a^b \rho u_x(s, t) \, ds = T(b, t) - T(a, t) + g \rho (b - a) = k (u_s(b, t) - u_s(a, t) + g \rho (b - a) \quad (1.1) \]
for any \( 0 \leq a < b \leq L \), where we have used that \( T = k(u_s - 1) \) — as follows from Hooke’s law. This is the integral form of the conservation of vertical momentum. If \( u \) has continuous second derivatives, this then yields the differential form of the conservation of vertical momentum
\[ u_{tt} - c^2 u_{xx} = g, \quad \text{for } 0 < s < L, \quad (1.2) \]
where \( c^2 = k/\rho \) is a wave velocity (stretching waves along cord). The boundary conditions for this equation are
\[ u = 0 \quad \text{at} \quad s = 0 \quad \text{and} \quad M u_{tt} = g M - k (u_s - 1) \quad \text{at} \quad s = L. \quad (1.3) \]
The first BC is obvious. The second arises from Newton’s law applied to the mass, which is being pulled up by the tension at the end of the cord.

There is a large variety of bungee cords, designed for different user weights, etc. A bit of search in the internet shows that the following number have the correct order of magnitude, in mks units: \( L = 25 \text{m}, \rho = 0.2 \text{kg/m, and } k = 10^3 \text{N} \). With these numbers, \( c \approx 70 \text{m/s} \).

### 2 Duhamel’s principle for a damped wave equation \#01

#### 2.1 Statement: Duhamel’s principle for a damped wave equation \#01

Consider the dissipatively damped wave equation with a forcing
\[ \mathcal{L} u = u_{tt} - c^2 u_{xx} - \mu u_{xxt} = f(x, t), \quad (2.1) \]
where \( c > 0 \) and \( \mu > 0 \) are constants, and \( f \) is some applied external force. Assume now that the equation is valid in the interval \( a < x < b \), with some homogeneous boundary conditions (see remark 2.1), and initial values \( u(x, 0) = u_t(x, 0) = 0 \). Show that the solution to this problem can be written in the form
\[ u = \int_0^t U(x, t, \tau) \, d\tau, \quad (2.2) \]
where (for each \( \tau \geq 0 \) \( U \) satisfies the homogeneous problem \( \mathcal{L} U = 0 \), with the same boundary conditions as \( u \), and suitably selected initial conditions at \( t = \tau \). What are these initial conditions?

Note. Assume the solutions involved are smooth enough to justify differentiation under the integral sign, and similar.

Hint. There are many ways in which an expression like (2.2) can solve (2.1). We are looking for a very simple one, where \( U \), as a function of \( x \) and \( t \), satisfies \( \mathcal{L} U = 0 \) for each \( \tau \), and \( t > \tau \), with initial conditions at \( t = \tau \). That is: \( U(x, \tau, \tau) \) and \( U_t(x, \tau, \tau) \) are given. Both \( u \) and \( U \) satisfy the same homogeneous boundary conditions.

Remark 2.1 Equation (2.1) describes, for example, the small vibrations of a string under tension, which resists bending via dissipative forces (i.e., forces proportional to the rate of bending). Exactly what the boundary conditions are does not matter here, as long as they are linear and homogeneous. For example, at \( x = a \), it could be: \( u = 0 \) (tied string), \( u_x = 0 \) (string with free end), or \( T u_x - b u_t = 0 \) (string end sliding along a rod, with friction force \( -b u_t \) balancing the transversal component of the tension force. Similarly, at \( x = b \), it could be: \( u = 0 \), or \( u_x = 0 \), or \( T u_x + b u_t = 0 \).
2.2 Answer: Duhamel’s principle for a damped wave equation 

Assume that \( u \) has the form in (2.2). Then

1. \( u_t = U(x, t, t) + \int_0^t U_t(x, t, \tau) d\tau \),
2. \( u_{tt} = \frac{\partial}{\partial t} U(x, t, t) + U_t(x, t, t) + \int_0^t U_{tt}(x, t, \tau) d\tau \),

where \( U_x, U_t, U_{\tau} \) indicates derivatives with respect to the corresponding argument slots in \( U \), while \( \frac{\partial}{\partial t} U(x, t, t) = U_t(x, t, t) + U_{\tau}(x, t, t) \). It follows that

\[
\mathcal{L} u = \frac{\partial}{\partial t} U(x, t, t) + U_t(x, t, t) - \mu U_{xx}(x, t, t) + \int_0^t \mathcal{L} U_{tt}(x, t, \tau) d\tau
\]

(2.3)

where we have used the fact that \( \mathcal{L} U = 0 \). It is then easy to see that, if the initial conditions

\[
U(x, \tau, \tau) = 0 \quad \text{and} \quad U_t(x, \tau, \tau) = f(x, \tau)
\]

are selected for \( U \), then (2.2) solves the problem.

Taking \( U(x, \tau, \tau) = v(x, \tau) \), where \( v(x, t) \) solves the heat equation \( v_t = \mu v_{xx} \) also works, but it is not as simple.

Remark 2.2 Physical interpretation (think of the equation as describing a string). The solution to (2.1) can be written as a linear combination (integral) of the solutions to the following problems

\[
\mathcal{L} v = v_{tt} - c^2 v_{xx} - \mu v_{xt} = \delta(t - \tau) f(x, \tau),
\]

(2.5)

one for each \( \tau \geq 0 \), with initial values \( v(x, 0; \tau) = v_t(x, 0; \tau) = 0 \) (here \( \delta(\cdot) \) is Dirac’s delta). Now, the solution to (2.5) should vanish up to \( t = \tau \) (string in equilibrium). At \( t = \tau \) an impulsive force of magnitude \( f \) per unit length is applied to the string. This sets the string to move, with velocity \( v_t = f(x, \tau) \) for \( t = \tau + 0 \). But this means that \( v \), for \( t \geq \tau \), satisfies the same problem as \( U \). Thus \( v \equiv 0 \) for \( t < \tau \), and \( v = U \) for \( t > \tau \). Hence

\[
u = \int_0^\infty v(x, t; \tau) d\tau = \int_0^t U(x, t, \tau) d\tau.
\]

(2.6)

3 Eikonal equation 

3.1 Statement: Eikonal equation 

Consider the Eikonal equation, which describes the propagation of a wave-front which moves normal to itself at a given speed \( c \) (which may depend on position \( c = c(x) \)). In 2-D, when \( c = \text{constant} \), the equation takes the non-dimensional form

\[
\phi_x^2 + \phi_y^2 = 1,
\]

(3.1)

where units have been selected such that \( c = 1 \). The relation of the solution, \( \phi = \phi(x, y) \), with the wave front is that: the wave front at time \( t \) is given by the level curve \( \phi = \phi(x, y) = t \). The equation is then to be solved with knowledge of the initial wave front: a given curve \( \Gamma \), with unit normal \( \hat{n} \) pointing in the direction of propagation. \(^3\)

\(^3\) A context in which the Eikonal equation applies is for high frequency, monochromatic, solutions to the wave equation \( u_{tt} - c^2 \Delta u = 0 \). The rays then correspond to the concept of “light rays” in Geometrical Optics.
Equation (3.1) has characteristics, called rays. The rays are the lines defined by

\[
\frac{dx}{dt} = \phi_x \quad \text{and} \quad \frac{dy}{dt} = \phi_y. \tag{3.2}
\]

It is easy to see that, with this definition, equation (3.1) — plus the equations that result from taking the \(x\) and \(y\) partial derivatives of the equation: \(\phi_x \phi_{xx} + \phi_y \phi_{xy} = 0\) and \(\phi_x \phi_{yx} + \phi_y \phi_{yy} = 0\) — implies that

\[
\frac{d\phi_x}{dt} = 0, \quad \frac{d\phi_y}{dt} = 0, \quad \text{and} \quad \frac{d\phi}{dt} = 1. \tag{3.3}
\]

The set of five equations in (3.2–3.3) is called the characteristic form of the Eikonal equation. It is a closed system of ODE, that can be solved (for each ray) given the initial wave front — see remark 3.1. It is, in fact, equivalent to (3.1).

**Remark 3.1** For every point \((x_0, y_0) \in \Gamma\) there is a ray, which is obtained by solving the system of ODE with the initial conditions \((at t = 0)\)

\[
x = x_0, \quad y = y_0, \quad \phi = 0, \quad \phi_x = n_1, \quad \text{and} \quad \phi_y = n_2,
\]

where \(n_1\) and \(n_2\) are the components of the unit normal \(\hat{n}_0\) to \(\Gamma\) at \((x_0, y_0)\). Solving the ODE yields \(x = X(x_0, y_0, t),\) \(y = Y(x_0, y_0, t),\) \(\phi = \Phi(x_0, y_0, t),\) etc. The PDE solution follows, in principle, by solving for \(x_0\) and \(y_0\) as functions of \((x, y, t)\) (using \(X\) and \(Y\)), and substituting into \(\Phi\).

**Your tasks are:**

1. Find an equation for the evolution along the rays of the Hessian of \(\phi\). Namely, the matrix

\[
M = \begin{bmatrix} \phi_{xx} & \phi_{xy} \\ \phi_{yx} & \phi_{yy} \end{bmatrix} \tag{3.4}
\]

2. Solve the equation for \(M\) derived in item 1, and write a formula giving the front curvature \(\kappa = \phi_{xx} + \phi_{yy}\) along each ray, as a function of \(t\), and the front curvature \(\kappa_0\) on the ray at the wave front corresponding to \(t = 0\). Note: the formula for \(\kappa\) involves \(t\) and \(\kappa_0\) only.

**Hints:** (i) Consider the second order derivatives (i.e. \(\phi_{xx}^2, \phi_{xy}^2, \text{and} \ \phi_{yy}^2\)) of equation (3.1). Then use that, for any \(f = f(x, y)\), its derivative along the rays is given by \(\frac{df}{dt} f = \phi_x f_x + \phi_y f_y\). (ii) To solve the equation for \(M\), proceed as follows: Let \(M_0\) be the value of \(M\) at \(t = 0\). Define \(W = (1 + M_0 t) M\), and write the equation \(W\) satisfies. Using that \(W = M_0\) at \(t = 0\), you should now be able to solve this equation by inspection. (iii) Once you have solved the equation for \(M\), use it to get the behavior of the eigenvalues of \(M\) — note that \(\kappa\) is the sum of the eigenvalues. (iv) Finally, inspect the gradient of equation (3.1). What does it tell you about the eigenvalues of \(M\)?

### 3.2 Answer: Eikonal equation #03

From the Eikonal equation (3.1), it follows that

\[
\begin{align*}
0 &= \phi_x \phi_{xxx} + \phi_{xx}^2 + \phi_y \phi_{yxx}, \\
0 &= \phi_x \phi_{xyx} + \phi_{xy} \phi_{xx} + \phi_{xx} \phi_{xyy} + \phi_y \phi_{yxy}, \\
0 &= \phi_x \phi_{yyx} + \phi_{xy} \phi_{yy} + \phi_{yy} \phi_{xyx}.
\end{align*} \tag{3.5-3.7}
\]

Thus

\[
0 = \phi_{xx}^2 + \phi_{yy}^2 + \frac{d}{dt} \phi_{xx} = \phi_{xy} \phi_{xx} + \phi_{xx} \phi_{yy} + \frac{d}{dt} \phi_{xy} = \phi_{xy}^2 + \phi_{yy}^2 + \frac{d}{dt} \phi_{yy}. \tag{3.8}
\]

Equivalently, in matrix form

\[
0 = \frac{d}{dt} M + M^2 \quad \implies \quad M = (I + M_0 t)^{-1} M_0, \tag{3.9}
\]
where $M_0$ is the value of $M$ for $t = 0$, and $I$ is the identity.

Proof: For any matrix $U = U(t)$, $\frac{d}{dt} U^{-1} = -U^{-1} \dot{U} U^{-1}$. This follows from differentiating $U U^{-1} = I$.

Let now $\lambda_0^0$ and $\lambda_0^2$ be the eigenvalues of $M_0$, with corresponding eigenvectors $\vec{v}_1$ and $\vec{v}_2$. Then, from (3.9), it follows that

$$M \vec{v}_j = \frac{\lambda_0^j}{1 + \lambda_0^j t} \vec{v}_j \quad \text{for} \quad j = 1, 2.$$  

(3.10)

Hence $M$ has the eigenvalues

$$\lambda_j = \frac{\lambda_0^j}{1 + \lambda_0^j t} \quad \text{for} \quad j = 1, 2,$$

(3.11)

with corresponding eigenvectors $\vec{v}_1$ and $\vec{v}_2$. Furthermore, by taking the gradient of equation (3.1), we obtain

$$M \nabla \phi = 0.$$

(3.12)

Hence: **one of the eigenvalues of $M$ vanishes.** Since the curvature of the wave fronts is given by $\kappa = \text{Trace}(M)$, it follows that $\kappa$ is the other eigenvalue. Therefore

$$\kappa = \frac{\kappa_0}{1 + \kappa_0 t}.$$  

(3.13)

In particular: if the initial wave front has negative curvature anywhere, then the curvature blows up at some finite positive time.

---

4 First order PDE Riemann problem #01

4.1 Statement: First order PDE Riemann problem #01

Consider the following conservation law (in a-dimensional variables)

$$u_t + \left( \frac{1}{2} u^2 \right)_x = 1, \quad \text{for} \quad -\infty < x < \infty \quad \text{and} \quad t > 0,$$

(4.1)

where $u$ is conserved, and shocks are used to avoid multiple-valued solutions.

Find the solution to the Riemann problem for this equation. That is: for the initial values

$$u(x, 0) = a \quad \text{for} \quad x < 0 \quad \text{and} \quad u(x, 0) = b \quad \text{for} \quad x > 0,$$

(4.2)

where $a$ and $b$ are arbitrary real constants $-\infty < a, b < \infty$. **Hint:** The solution involves shocks, expansion fans, and regions where $u$ depends on time only. Expansion fans are regions where all the characteristics emanate from a single point in space time.

---

$^4$ Since $M_0$ is a symmetric matrix, it has real eigenvalues, with an orthonormal set of eigenvectors.
4.2 Answer: First order PDE Riemann problem #01

The characteristic form of the equation in (4.1) is \( \frac{du}{dt} = 1 \) along \( \frac{dx}{dt} = u. \)

This has the general solution \( u = t + f(\zeta) \) and \( x = \frac{1}{2} t^2 + t f(\zeta) + \zeta. \)

where \( f(\zeta) = u(\zeta, 0) \) is given by the initial data.

In a rarefaction fan all the characteristics emanate from a single point (say \((x, t) = (0, 0))\), hence

\[
    u = t + f \quad \text{and} \quad x = \frac{1}{2} t^2 + tf, \quad \Rightarrow \quad u = \frac{1}{2} t + \frac{x}{t},
\]

valid in the region \( \frac{1}{2} t^2 + tf_1 \leq x \leq \frac{1}{2} t^2 + tf_2 \) — where \( f_1 \leq f_2 \) are constants.

At shocks the Rankine-Hugoniot jump condition, and the Entropy condition reduce to

\[
    \frac{dx}{dt} = \frac{1}{2} (u_L + u_R) \quad \text{and} \quad u_L > u_R,
\]

where \( u_L \) is the state immediately to the left of the shock, and \( u_R \) is the state immediately to the right of the shock.

Putting this all together, we arrive at the following solution to the Riemann problem.

**Case \( a \leq b. \)**

\[
    u = \begin{cases} 
    t + a & \text{for} \quad x \leq \frac{1}{2} t^2 + at, \\
    \frac{1}{2} t + \frac{x}{t} & \text{for} \quad \frac{1}{2} t^2 + at \leq x \leq \frac{1}{2} t^2 + bt, \\
    t + b & \text{for} \quad \frac{1}{2} t^2 + at \leq x.
    \end{cases}
\]

**Case \( a > b. \)**

\[
    u = \begin{cases} 
    t + a & \text{for} \quad x < \frac{1}{2} t^2 + \frac{1}{2} (a + b) t, \\
    t + b & \text{for} \quad x > \frac{1}{2} t^2 + \frac{1}{2} (a + b) t.
    \end{cases}
\]

4.3 Extras: Justification of quadratic fluxes

Here we justify the use of conservation laws of the form

\[
    u_t + \left( \frac{1}{2} u^2 \right)_x = S, \quad \text{for} \quad -\infty < x < \infty \quad \text{and} \quad t > 0,
\]

where \( S \) is some source term, \( u \) is conserved and can be both positive or negative, and shocks are used to avoid multiple-values in the solution.

Consider a scalar conservation law in 1-D, with a source term, \( \rho_t + q_x = S, \)

where \( \rho = \rho(x, t) \) is the density of some conserved quantity (hence \( \rho \geq 0 \)), \( q = Q(\rho) \) is the corresponding flux, and \( S = S(\rho, x) \) is the density of sources/sinks. **Assume now that \( S \) is “small” — this is made precise below in item 2.**

Then solutions where \( \rho \) is close to a constant should be possible. Hence let \( \rho_0 > 0 \) be some fixed (constant) density value, and proceed as follows:

1. **Expand \( Q \) near \( \rho_0 \) using Taylor’s theorem** \[
    Q = q_0 + c_0 (\rho - \rho_0) + \frac{s_1}{2 \rho_0} (\rho - \rho_0)^2 + \ldots,
\]

where \( q_0 \) is the flux for \( \rho = \rho_0 \), \( c_0 \) is the corresponding characteristic speed, and \( s_1 \) is a constant with the dimensions of a velocity. **Assume that \( s_1 \neq 0 \); in fact: \( s_1 > 0. \)**

\( ^5 \) If \( s_1 < 0 \), a similar analysis is possible. Note that \( s_1 \) is a measure of how nonlinear equation (4.8) is. The further away for zero \( s_1 \) is, the stronger the leading order nonlinear term in the equation is.
2. Let \( S_1 > 0 \) be some “typical” value for the source term size, and let \( L > 0 \) be some “typical” length scale. Then the source term is small in the sense that \( 0 < \epsilon^2 = \frac{L S_1}{S_1 \rho_0} < 1 \).

Introduce the a-dimensional variables

\[
\tilde{x} = \frac{x - c_0 t}{L} \quad \text{and} \quad \tilde{t} = \frac{\epsilon S_1}{L} t, \quad \text{with} \quad \rho = \rho_0 (1 + \epsilon u) \quad \text{and} \quad S = S_1 \tilde{S}.
\]

Then (4.8) becomes

\[
u_{\tilde{t}} + \left( \frac{1}{2} u^2 + O(\epsilon) \right) \tilde{x} = \tilde{S}.
\]

Upon neglecting the \( O(\epsilon) \) term, this has the form in (4.7).

5 Simple finite differences for a 1st order PDE

5.1 Statement: Simple finite differences for a 1st order PDE

Consider the linear PDE

\[
u_t + (c(x) \nu)_x = 0, \quad \text{where} \quad c = 1 + \frac{1}{4} \cos(x),
\]

and \( u \) is periodic of period \( 2 \pi \) — i.e.: \( u(x + 2 \pi, t) = u(x, t) \). Assume now that you are asked to calculate the solution of this p.d.e. for \( 0 \leq t \leq T = 6 \), with initial condition given by

\[
u(x, 0) = \nu_0(x) = \exp(-x^2) \quad \text{for} \quad -\pi \leq x \leq \pi,
\]

extended periodically outside the interval \([-\pi, \pi]\).

You can extract a lot of information from the solution by characteristics of the problem above, but actual numerical values are not easy to access from it. For this, the best thing to do is to integrate the problem numerically. Here we will consider a few naive numerical algorithms for this purpose.

First, introduce a numerical grid, as follows:

\[
x_n = -\pi + n h \quad \text{for} \quad 1 \leq n \leq N, \quad \text{and} \quad t_m = m k \quad \text{for} \quad 0 \leq n \leq M,
\]

where \( N \) and \( M \) are “large” integers, \( h = 2 \pi/N \), and \( k = T/M \). Let \( \nu_n^m \) be the numerical solution’s grid point values. The expectation is that these values will be related to the exact solution \( \nu(x, t) \) by \( \nu_n^m = \nu(x_n, t_m) + \text{small error} \), with the error vanishing as \( N \) and \( M \) grow.

Next, consider the following numerical discretizations of the problem, which arise upon replacing the derivatives in the equation by finite differences that approximate them up to errors of some positive order in \( h \) or \( k \). In all cases the formulas apply for \( m \geq 0 \), with \( \nu_0^0 = \nu_0(x_n) \).

A. \( 0 = \frac{1}{k} (\nu_{n+1}^m - \nu_n^m) + \frac{1}{h} \left( c(x_n) \nu_n^m - c(x_{n-1}) \nu_{n-1}^m \right). \)

B. \( 0 = \frac{1}{k} (\nu_{n+1}^m - \nu_n^m) + \frac{1}{h} \left( c(x_{n+1}) \nu_{n+1}^m - c(x_n) \nu_n^m \right). \)

C. \( 0 = \frac{1}{k} (\nu_{n+1}^m - \nu_n^m) + \frac{1}{2h} \left( c(x_{n+1}) \nu_{n+1}^m - c(x_{n-1}) \nu_{n-1}^m \right). \)
In all cases, once $u^n_m$ is known for some $m$ and all $n$, $u^{n+1}_n$ can be explicitly computed, for all $n$. **Note:** when a formula above in A, B or C, calls for a value $u^n_m$ outside the range $1 \leq n \leq N$, the periodic boundary conditions, which translate into $u^{n+N}_m = u^n_m$, must be used.

The **tasks in this problem are:**

1. **Causality and numerics.** Using arguments based solely on how the information propagates in the exact solution (characteristics), versus how it propagates in the numerical schemes above, **argue that:** (1.1) One of the schemes above cannot possibly work. (1.2) A necessary condition for the other two to work is that a restriction of the form $k \leq \lambda h$ be imposed on the time step — where $\lambda > 0$ is a constant that depends on $c = c(x)$.

2. Implement the schemes and try them out with various space resolutions, and a corresponding time resolution. Do you see convergence? Do the results agree with your analysis in item 1? Report what you see, and illustrate it with plots — a few well selected plots should be enough!

5.2 **Answer: Simple finite differences for a 1st order PDE**

**Causality.** The key question we have to address first, for both the p.d.e. as well as its discretized versions, is: **what are the domains of dependence?** Namely:

\[
\text{Given some arbitrary point } P = (x_0, t_0) \text{ in space-time, what is the region with the property that changes there affect the value of the solution at } P? \tag{5.4}
\]

For us here, this is important because **what happens outside the domain of dependence of a point } P_0, has NO EFFECT at all on the value of the solution there.** Hence

\[
\text{A necessary condition for the solution computed by a numerical algorithm to converge to the solution of a p.d.e. (as the numerical grid is refined), is: The numerical domain of dependence for any point } P \text{ should include (as the numerical grid is refined) the p.d.e. domain of dependence for } P. \tag{5.5}
\]

Proof: how can the numerical algorithm get the correct values for the solution of a p.d.e., without using the data that determines the p.d.e. solution? **Note:** restrictions on numerical algorithms that arise in this fashion are called C.F.L. conditions (C.F.L. = Courant, Friedrichs, and Lewy).

So, given an arbitrary point $P = (x_0, t_0)$, what are the domains of dependence relevant here?

1. **For equation (5.1)** the domain of dependence is given by the characteristic through $P$. Namely, by the curve determined by the o.d.e. problem:

\[
\frac{dx}{dt} = c(x), \quad \text{for } t < t_0, \quad \text{with ”initial condition” } x(t_0) = x_0.
\]

Hence, let $c_{\max} = \max_{|x| \leq \pi} c(x) = 1.25$, and $c_{\min} = \min_{|x| \leq \pi} c(x) = 0.75$. Then, the p.d.e. domain of dependence is included within the wedge:

\[
x_0 + c_{\max} (t - t_0) \leq x \leq x_0 + c_{\min} (t - t_0), \quad \text{with } t < t_0. \tag{5.6}
\]

Note that, for a “generic” $c = c(x)$, and arbitrary $P$, this wedge is the best one can do.

2. **For the scheme in item A** the domain of dependence is given by

\[
x_0 + \frac{h}{k} (t - t_0) \leq x \leq x_0, \quad \text{with } t < t_0. \tag{5.7}
\]

Note that, for any fixed $h$ and $k$, the domain is a discrete set of points. However, as the grid is refined, the whole (5.7) wedge fills up. The same applies to the wedges in (5.8 – 5.9).

---

*Note: $N$ in the range $20 \leq N \leq 200$ should be more than enough to see what happens.*
3. For the scheme in item B the domain of dependence is given by
\[x_0 \leq x \leq x_0 - \frac{h}{k} (t - t_0), \quad \text{with} \quad t < t_0.\] (5.8)

4. For the scheme in item C the domain of dependence is given by
\[x_0 + \frac{h}{k} (t - t_0) \leq x \leq x_0 - \frac{h}{k} (t - t_0), \quad \text{with} \quad t < t_0.\] (5.9)

Thus, in order for (5.5) to apply, we need:

— Scheme in item A. The wedge in (5.6) will be included within the wedge in (5.7) provided \(h/k \geq c_{\max} = 1.25\). Equivalently:
\[
k \leq \frac{1}{c_{\max}} h = \frac{1}{1.25} h \iff M \geq \frac{1.25}{2\pi} T N.\] (5.10)

— Scheme in item B. The wedge in (5.6) cannot be included within the wedge in (5.8) for any choice of \(h > 0\) and \(k > 0\). This scheme cannot/will not work.

— Scheme in item C. The wedge in (5.6) will be included within the wedge in (5.9) provided \(h/k \geq c_{\max} = 1.25\). Equivalently: (5.10) must apply.

Condition (5.10) is necessary only. It DOES NOT guarantee that either scheme (A or C) works, nor does it tell us how the scheme in item B fails. To check what happens, I implemented the schemes. The results (which do not contradict the theoretical conclusions above) follow.

Implementation of the scheme in item A.

As long as the calculation is performed satisfying the constraint in (5.10) — C.F.L. condition — the scheme appears to perform properly, with convergence as the grid is refined — see figure 5.1.

Remark 5.1 Without an exact solution to compare the answer with, one cannot (solely from numerical calculations) be sure that the observed convergence is to the actual solution — it is not hard to produce reasonable looking numerical schemes that converge to the wrong answer. But, there exist many tests (using solely numerical results) to increase the confidence level beyond “it looks right”. In the specific case of the scheme in item A, one can prove that: as \(h \to 0\), with \(k\) restricted by (5.10), the numerical solution converges to the solution to (5.1 – 5.2).

Implementation of the scheme in item B.

Grid scale oscillations (dominant wavelength is \(2h\)), growing exponentially in time (they reach huge amplitudes very quickly) appear. The oscillations’ growth rate increases as the grid is refined, with catastrophic results. This scheme is not content with simply failing to work, it fails spectacularly (which is nice: makes it easy to tell it is not working). This is illustrated by the pictures in figure 5.2.

Implementation of the scheme in item C.

This scheme does not work either, though it does not fail as spectacularly as the scheme in item B — the grid scale oscillations have (comparatively) slower growth ratios. Nevertheless: grid scale oscillations are generated, which grow exponentially with time. Furthermore, the growth rate of these oscillations becomes larger as the grid is refined. See figure 5.3.
Figure 5.1: Calculations with the scheme in item A, to solve (5.1 – 5.2). Left to write, and top to bottom: Numerical solutions as the grid is refined, enforcing the C.F.L. condition (5.10) — specifically, take $M = \text{ceil} \left( \frac{1.125}{2\pi} TN \right)$. Convergence appears to be occurring.

**Remark 5.2** If the scheme in item A is used under conditions that violate the C.F.L. condition in (5.10), then behavior similar to the one observed for the schemes in items B and C appears. Namely: grid scale oscillations that grow exponentially, with growth rates becoming larger as $h \to 0$.

*Note:* if the C.F.L. violation is “small”, then the growth rate of the oscillations is small when $N$ is moderate. Hence, the grid scale oscillations will be hard to see, unless $N$ is large enough, or the interval of computation (i.e.: $T$) is long enough.

**Remark 5.3** Not only is it possible to prove that the scheme in item A converges if the C.F.L. condition is enforced — see remark 5.1. It is also possible to explain, theoretically, why and when grid scale oscillations will appear, and to estimate their growth rate.
Figure 5.2: Calculations with the scheme in item B, to solve (5.1 – 5.2). Left to write, and top to bottom: Numerical solutions, as the grid is refined. $M$ is substantially above the value in (5.10). Larger $M$’s do not help — see top right picture. The spurious oscillations are very large. Hence, we do not plot the numerical solution directly: The plots are for $\text{sign}(u) \ln(1 + |u|)$!

6 Steady State Shallow Water #01

6.1 Statement: Steady State Shallow Water #01

The conservation form of the equations for 2-D shallow water waves over a flat bottom is

\begin{align}
0 &= h_t + (h u)_x + (h v)_y, \\
0 &= (h u)_t + (h u^2 + \frac{1}{2} g h^2)_x + (h v u)_y, \\
0 &= (h v)_t + (h u v)_x + (h v^2 + \frac{1}{2} g h^2)_y,
\end{align}
Figure 5.3: Calculations with the scheme in item C, to solve (5.1 – 5.2). Left to write, and top to bottom: Numerical solutions, as the grid is refined. \( M \) is substantially above the value in (5.10). Larger \( M \)'s do not help.

where \( h \) is the fluid depth, \( u \) is the \( x \)-flow velocity, \( v \) is the \( y \)-flow velocity, and \( g \) is the acceleration of gravity. The steady state (time independent) form of these equations is

\[
0 = (hu)_x + (hv)_y, \tag{6.4}
\]

\[
0 = (hu^2 + \frac{1}{2} gh^2)_x + (h v u)_y, \tag{6.5}
\]

\[
0 = (hu v)_x + (h v^2 + \frac{1}{2} gh^2)_y. \tag{6.6}
\]

**Answer the following questions:**

1. Under which conditions on \((h, u, v)\) is (6.4 – 6.6) strictly\(^7\) hyperbolic? Use the **Froude number**

\[
F = \sqrt{(u^2 + v^2)/(gh)}
\]

in your answer. You need \( F > 0 \) to even ask the question — WHY?

2. When the characteristic equation has a double root, the system is not hyperbolic.\(^8\) Show this.

**Hint:** The system is invariant under rotations. Hence, when computing the eigenvector(s), you can rotate the coordinate system so that \( v = 0 \) at the point of interest.

3. The system always has (at least) one characteristic, which has a Riemann invariant. Find it.

\(^7\) All the characteristic directions are distinct: the characteristic equation has three distinct roots.

\(^8\) There is only one eigenvector associated with the double root.
6.2 Answer: Steady State Shallow Water #01

First, use equation (6.4) to simplify equations (6.5 – 6.6), and re-write the system in the form

\[ 0 = u h_x + v h_y + h (u_x + v_y), \]  
\[ 0 = g h_x + u u_x + v u_y, \]  
\[ 0 = g h_y + u v_x + v v_y. \]  

Equivalently

\[ 0 = A U_x + B U_y = \begin{pmatrix} u & h \\ g & u \end{pmatrix} \begin{pmatrix} h \\ u \end{pmatrix} + \begin{pmatrix} v & 0 & h \\ 0 & v & 0 \end{pmatrix} \begin{pmatrix} h \\ v \end{pmatrix}. \]  

The characteristic equation is then

\[ 0 = \det(A dy - B dx) = (u dy - v dx) \left( (u dy - v dx)^2 - g h (dx^2 + dy^2) \right). \]  

Notice that, in order for the system to not be degenerate, we need \( u^2 + v^2 > 0 \iff F > 0. \)

In terms of the arclength \( ds^2 = dx^2 + dy^2, \) the characteristic curves are given by

\[ 0 = u y' - v x', \quad \text{thus} \quad x' = \frac{u}{\sqrt{u^2 + v^2}} \quad \text{and} \quad y' = \frac{v}{\sqrt{u^2 + v^2}}, \]  

and

\[ \pm \sqrt{g h} = u y' - v x', \quad \text{thus} \quad \begin{cases} x' = \frac{\sqrt{F^2 - 1} u \mp v}{F \sqrt{u^2 + v^2}}, \\ y' = \frac{\sqrt{F^2 - 1} v \mp u}{F \sqrt{u^2 + v^2}}. \end{cases} \]  

where primes indicate derivatives with respect to \( s \) — so that \((x')^2 + (y')^2 = 1\) — and we have selected the direction of travel so that the projection along the flow velocity is positive. Of course, (6.13) makes sense for \( F \geq 1 \) only, and gives two different directions for \( F > 1 \) only. Thus:

\[ \text{The system is strictly hyperbolic if, and only if, } F > 1. \]  

The characteristics given by (6.12) are easy to figure out: they are just the streamlines. Furthermore, the associated characteristic equation is also rather simple: multiply (6.8) by \( u, \) (6.9) by \( v, \) add, and divide everything by \( \sqrt{u^2 + v^2}. \)

The result is

\[ 0 = g h' + u u' + v v' \implies \mathcal{H} = h + \frac{1}{2g} (u^2 + v^2) \text{ is constant along streamlines}, \]

where the prime indicates derivation along the characteristics in (6.12). In particular:

\[ \mathcal{H} \text{ is a Riemann invariant for the streamlines — this applies even if } F \leq 1. \]  

Remark 6.1 \( \mathcal{H} \) has an interesting physical interpretation: Consider a fluid column.\(^{10}\) As the column moves around with the flow, its height will (generally) change. However, there is a limit to how deep the fluid column can get, anywhere: since \( \mathcal{H} \) must remain constant, it is always true that \( h \leq \mathcal{H}. \) An appropriate name for \( \mathcal{H} \) is “potential depth”, as it is the maximum depth that can be achieved — when all the kinetic energy in the fluid column is transformed into potential energy.

It is interesting to note that, when \( \mathcal{H} \) is constant throughout the flow, then the flow is potential. This follows easily from (6.8 – 6.9), which — upon elimination of \( h \) via \( g h_x = -u u_x - v v_x \) and \( g h_y = -u u_y - v v_y \) become

\[ 0 = v (u_y - v_x) = u (v_x - u_y). \]

\(^{9}\) That is: \( \det(A dy - B dx) \) does not vanish identically.

\(^{10}\) The shallow water approximation ignores vertical motions: points correspond to thin, vertical, fluid columns.
Characteristic equations in a frame aligned with the flow.

In order to understand a little better what the characteristics in (6.13) are all about, pick an arbitrary point in the $x$-$y$ plane. Then rotate the coordinate axis so that (at the selected point) $v = 0$ and $u > 0$. At the chosen point, the characteristic equation (6.11) takes the form

$$0 = u \, dy \left( (u^2 - gh) \, dy^2 - gh \, dx^2 \right).$$

(6.17)

Then, with $F = u/\sqrt{gh} > 1$, the characteristic directions are given by:

$$\frac{dy}{dx} = \pm \sqrt{F^2 - 1} \quad \text{Streamline: characteristic curve lined up with flow},$$

(6.18)

$$\frac{dx}{dy} = \pm \sqrt{F^2 - 1} \quad \text{Gravity waves: speed $\sqrt{gh}$ relative to flow — see remark 6.2.}$$

(6.19)

**Remark 6.2** Here we make sense of (6.19) in terms of gravity waves propagating at speed $\sqrt{gh}$ relative to the flow. Select the fluid column that goes through the point $(x, y)$ at the “current” time (say: $t = 0$), and ask the question: What are the positions in the plane that will be influenced by the gravity waves currently being emitted by the selected column?

Answer: the emitted signal propagates such that at time $t = \Delta t > 0$ it is on a circle of radius $\Delta t \sqrt{gh}$, centered at the position of the selected column at $t = \Delta t$, i.e.: $(x + u \Delta t, y)$. These circles fill up a wedge, whose boundary is given (precisely) by the curves in (6.19).

Consider now the (left) eigenvector corresponding to the characteristic roots in (6.19). This eigenvector must be a non-vanishing solution of

$$0 = \ell (A \, dy - B \, dx) = \ell \left( \begin{array}{ccc} u & h & \mp h \sqrt{F^2 - 1} \\ g & u & 0 \\ \mp g \sqrt{F^2 - 1} & 0 & u \end{array} \right) \, dy,$$

(6.20)

where we have used the fact that $v = 0$ at the selected point. Hence

$$\ell \propto (u, -h, \pm h \sqrt{F^2 - 1})$$

— note that there is only one eigenvector when $F^2 = 1$, thus

**The system is not hyperbolic when $F = 1$ — repeated characteristic roots case.**

(6.21)

Multiplying the equations in (6.10) by $\ell$ above then yields (see warning below):

$$\pm g \sqrt{F^2 - 1} \frac{dh}{dy} + u \frac{dv}{dy} = 0,$$

(6.22)

along $\frac{dx}{dy} = \pm \sqrt{F^2 - 1}$, and we have (again) used that $v = 0$ at the selected point. **WARNING:** This form of the characteristic equations applies only at the points where $v = 0$. For the general formulas, see equations (6.27) and (6.28).

**Characteristic equations in a general frame.**

In order to get rid of the $v = 0$ restriction, we note that (at any point $(x_0, y_0)$) the local coordinate unit vectors needed to make $u > 0$ and $v = 0$ are given by

$$\hat{i} = \frac{1}{\sqrt{u_0^2 + v_0^2}} (u_0, v_0) \quad \text{and} \quad \hat{j} = \frac{1}{\sqrt{u_0^2 + v_0^2}} (-v_0, u_0),$$

(6.23)

\[11\] Unless the flow is unidirectional, we can do this one point at the time only.
The velocity transformation is then
\[ u \rightarrow \tilde{u} = \frac{u u_0 + v v_0}{\sqrt{u_0^2 + v_0^2}} \quad \text{and} \quad v \rightarrow \tilde{v} = \frac{-u v_0 + v u_0}{\sqrt{u_0^2 + v_0^2}}, \] (6.24)

with a similar transformation for the space coordinates.\(^{12}\) Thus
\[ (\tilde{u} \tilde{v})_0 = \left( \frac{u u_0 + v v_0}{u_0^2 + v_0^2} (u_0 v - v_0 u) \right)_0 = (u d v - v d u)_0, \] (6.25)

and
\[ \frac{d\tilde{x}}{d\tilde{y}} = \frac{u_0 d x + v_0 d y}{u_0 d y - v_0 d x}. \] (6.26)

Using (6.25 – 6.26), it is easy to see that the unrestricted form of the characteristic equations in (6.22), valid even when \( v \neq 0 \), is (with \( F = \sqrt{(u^2 + v^2)/(gh)} \)):
\[ \pm g \sqrt{F^2 - 1} d h + u d v - v d u = 0, \] (6.27)

on the curves \( u d x + v d y = \pm \sqrt{F^2 - 1} (u d y - v d x) \), which we rewrite as\(^{13}\)
\[ (u \sqrt{F^2 - 1} \mp v) d y = (v \sqrt{F^2 - 1} \pm u) d x. \] (6.28)

Of course, this can be checked directly from the original equations (without using any rotations), but the derivation here is a little cleaner.

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7 Wave equations #01

7.1 Statement: Wave equations #01

Consider an elastic (homogeneous) string under tension, tied at one end, initially at rest, and forced by a (small amplitude) harmonic shaking of the other end. To simplify the situation, assume that all the motion is restricted to happen in a plane.

After a proper adimensionalization, the situation is modeled by the mathematical problem below for the wave equation in 1-D — where \( u(x, t) \) is the displacement from equilibrium of the string,\(^{14}\)
\[ u_{tt} - u_{xx} = 0, \quad 0 < x < 1, \quad \text{and} \quad t > 0, \] (7.1)

with initial data \( u(x, 0) = u_t(x, 0) = 0 \), and boundary conditions
\[ u(0, t) = 1 - \cos(\omega t) \quad \text{and} \quad u(1, t) = 0. \] (7.2)

Find the solution to this problem, for the times \( 0 < t \leq 4 \). Furthermore: note that the solution, while making sense in the classical sense (no need to invoke generalized function derivatives), is not infinitely differentiable. There are certain lines along which “singularities” occur. Find these lines of singularity, and describe what the situation is along them (nature of the singularities) — the lines are, of course, characteristics. Finally: what is special about the cases \( \omega = \pi \) and \( \omega = \pi/2 \)?

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\(^{12}\) Here we use tildes to indicate the variables in the transformed coordinates where \( v \) vanishes at \((x_0, y_0)\).

\(^{13}\) Note that this is equivalent to the formulas on the right in (6.13).

\(^{14}\) This formulation neglects dissipation in the string.
7.2 Answer: Wave equations #01

The (general) solution to the wave equation\(^{15}\) is the superposition of two waves, one traveling to the right at speed \(c = 1\), and the other to the left at speed \(-c = -1\). Specifically:

\[
u(x, t) = f(x - t) + g(x + t),
\]

where \(f\) and \(g\) are “arbitrary” functions. The task is now to find \(f\) and \(g\) for the problem posed in (7.1 – 7.2). We proceed as follows

1. Note that \(f = f(r)\) and \(g = g(s)\) are only defined for \(r = x - t \leq 1\) and \(s = x + t \geq 0\).
2. The initial conditions define \(f = f(x)\) and \(g = g(x)\) for \(0 \leq x \leq 1\).
   That is: they specify that \(f(x) + g(x) = -f'(x) + g'(x) = 0\), so that \(f = -g = \text{constant}\). Without loss of generality we can thus write \(f(x) = g(x) = 0\) for \(0 \leq x \leq 1\) — since adding a constant to \(f\), and subtracting it from \(g\), does not alter the solution in (7.3).
3. The boundary condition on the right \(u(1, t) = 0\) should be used to define the “reflected” wave (i.e.: \(g\)) in terms of the “incident” wave (i.e.: \(f\)). Thus \(g(1 + t) = -f(1 - t)\) for \(t > 0\), so that \(g(s) = -f(2 - s)\) for \(s > 1\).
4. The boundary condition on the left \(u(0, t) = \sigma(t) = 1 - \cos(\omega t)\) should be used to define the “reflected” wave (i.e.: \(f\)) in terms of the “incident” wave (i.e.: \(g\)). Thus \(f(-t) = \sigma(t) - g(t)\) for \(t > 0\), so that \(f(r) = \sigma(-r) - g(-r)\) for \(r < 0\).

Starting with the result in item 2, the results in items 3 and 4 can be used to define \(f = f(r)\) and \(g = g(s)\) for successively smaller (larger) values of \(r\) (respectively \(s\)). Putting this all together:

\[
u(x, t) = F(t - x + 2) - F(t + x),
\]

where \(F(s) = 0\) for \(0 \leq s \leq 2\), \(F(s) = \sigma(s - 2)\) for \(2 \leq s \leq 4\), \(F(s) = \sigma(s - 2) + \sigma(s - 4)\) for \(4 \leq s \leq 6\), \(F(s) = \sigma(s - 2) + \sigma(s - 4) + \sigma(s - 6)\) for \(6 \leq s \leq 8\), and so on, with \(F(s + 2) = F(s) + \sigma(s)\) and \(\sigma(s) = 1 - \cos(\omega s)\) — see Figure 7.1.

**Note 1:** Because \(\sigma(0) = \sigma'(0) = 0\), \(F = F(s)\) is continuous and has a continuous derivative at \(s = 2, 4, 6, \ldots\) However, the second derivative of \(F\) has simple jump discontinuities at \(s = 2, 4, 6, \ldots\). Thus, the solution in (7.4) satisfies the wave equation in the classical sense, but just “barely”, with singularity lines — where the second derivatives fail to be continuous — along the (right going) characteristic that originates at \(x = t = 0\), and its successive reflections at the boundaries \(x = 0, 1\). Specifically:

\[
x = t \quad \text{for} \quad 0 < t < 1,
\]

\[
x = -t + 2 \quad \text{for} \quad 1 < t < 2,
\]

\[
x = +t - 2 \quad \text{for} \quad 2 < t < 3,
\]

\[
x = -t + 4 \quad \text{for} \quad 3 < t < 4,
\]

\[
x = +t - 4 \quad \text{for} \quad 4 < t < 5,
\]

and so on. Of course, all of these are characteristics — see Figure 7.1.

**Note 2:** The input \(\sigma\) is periodic, with period \(T_\sigma = 2\pi/\omega\). On the other hand, with a wave speed of one, and a string length of one, the waves take a time of two to go back and forth across the string. Thus the set-up has a “natural” time scale \(T_n = 2\) associated with it — this scale is clearly evident in the solution above in (7.4). When \(T_\sigma = T_n = 2\), the terms in the sums defining \(F = F(s)\) add up, leading to a steadily growing amplitude for the oscillations in the solution (resonance). Of course, the same problem arises if \(\omega = n\pi\) \((n\) any natural number) — i.e.: \(T_\sigma = 2/n\).

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\(^{15}\) Obtained using the method of characteristics, for example.
The solution in the various regions is as follows:

- **R₀**: \( u = 0 \),
- **R₁**: \( u = \sigma(t - x) \),
- **R₂**: \( u = \sigma(t - x) - \sigma(t + x - 2) \),
- **R₃**: \( u = \sigma(t - x) + \sigma(t - x - 2) - \sigma(t + x - 2) \),
- **R₄**: \( u = \sigma(t - x) + \sigma(t - x - 2) - \sigma(t + x - 2) - \sigma(t + x - 4) \),

and so on, where \( \sigma(\zeta) = 1 - \cos(\omega \zeta) \).

**Note 3:** An interesting situation occurs for \( \omega = \pi/2 \), so that \( \sigma(s + 2) = 2 - \sigma(s) \), which means that \( F \) alternates between sinusoidal and constant. The left forcing cancels the oscillations it produced earlier, after they travel back and forth. After a while, there is nothing to cancel, and a new burst of oscillation is produced, which is later canceled, and so on. In this case:

- \( F(s) = 0 \) for \( 0 \leq s \leq 2 \),
- \( F(s) = 1 + \cos \left( \frac{\pi s}{2} \right) \) for \( 2 \leq s \leq 4 \),
- \( F(s) = 2 \) for \( 4 \leq s \leq 6 \),
- \( F(s) = 3 + \cos \left( \frac{\pi s}{2} \right) \) for \( 6 \leq s \leq 8 \),

and so on.

**A more “intuitive” alternative, but equivalent, answer is as follows:**

The initial conditions are \( u(x, 0) = u_t(x, 0) = 0 \), hence the solution remains \( u \equiv 0 \) till the signal from the left boundary\(^1\) arrives. Then the following happens

- **l₁** The left boundary — where the forcing is applied — produces a signal propagating to the right at speed \( c = 1 \), given by \( u_1 = 1 - \cos \omega(t - x) \) for \( t \geq x \) — of course: \( u_1 = 0 \) for \( t < x \).

  Using the Heaviside\(^1\) function, we write \( u_1 = H(t - x) \left( 1 - \cos \omega(t - x) \right) \).

- **l₂** When the signal \( u_1 \) arrives to the right boundary, it is reflected, and a signal \( u_2 \) is produced that propagates to the left at speed \( c = -1 \). The condition \( u_1 + u_2 = 0 \) on \( x = 1 \) determines \( u_2 \).

  Hence \( u_2 = H(t + x - 2) \left( -1 + \cos \omega(t + x - 2) \right) \).

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\(^{1}\) At the right boundary \( u(1, t) = 0 \). This boundary emits no signals, but it reflects signals that arrive there.

\(^{1}\) Defined by: \( H(\zeta) = 1 \) for \( \zeta \geq 0 \), and \( H(\zeta) = 0 \) for \( \zeta < 0 \).
When the signal \( u_2 \) arrives to the left boundary, it is reflected, and a signal \( u_3 \) is produced that propagates to the right at speed \( c = 1 \). The condition \( u_2 + u_3 = 0 \) on \( x = 0 \) determines \( u_3 \).

Hence \( u_3 = H(t - x - 2)(1 - \cos \omega (t - x - 2)) \).

When the signal \( u_3 \) arrives to the right boundary, it is reflected, and a signal \( u_4 \) is produced that propagates to the left at speed \( c = -1 \). The condition \( u_3 + u_4 = 0 \) on \( x = 1 \) determines \( u_4 \).

Hence \( u_4 = H(t + x - 4)(-1 + \cos \omega (t + x - 4)) \).

When the signal \( u_4 \) arrives to the left boundary, it is reflected, and a signal \( u_5 \) is produced that propagates to the right at speed \( c = 1 \). The condition \( u_4 + u_5 = 0 \) on \( x = 0 \) determines \( u_5 \).

Hence \( u_5 = H(t - x - 4)(1 - \cos \omega (t - x - 4)) \).

And so on. The pattern should be obvious by now.

We can write the full solution in the form

\[
 u(x,t) = \sum_{n=0}^{\infty} H(t - x - 2n)(1 - \cos \omega(t - x - 2n)) - \sum_{n=1}^{\infty} H(t + x - 2n)(1 - \cos \omega(t + x - 2n)). \tag{7.5}
\]

Notice that, for any \( x \) and \( t \), only a finite number of terms do not vanish, hence there are no convergence problems.

**Note 4:** All the terms in (7.5) involve the function

\[
 H(\zeta)(1 - \cos \omega \zeta). \tag{7.6}
\]

This function is continuous, and has a continuous derivative. But its second derivative has a simple jump discontinuity at \( \zeta = 0 \). Hence the solution in (7.5) is continuous and has continuous first derivatives, but the second derivatives have simple jump discontinuities along the characteristic lines \( x = \pm t + 2n \).

**Note 5:** If \( \omega = \pi \), the solution in (7.5) becomes

\[
 u = (1 - \cos \pi (t - x)) \sum_{n=0}^{\infty} H(t - x - 2n) - (1 - \cos \pi (t + x)) \sum_{n=1}^{\infty} H(t + x - 2n). \tag{7.7}
\]

This grows linearly in time, as more and more terms in the series \( \sum H(t \pm x - 2n) \) become one and kick in. Resonant case.

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\(^{18}\) The signal \( u_1 \) takes care of the boundary condition on \( x = 0 \). Any additional signals must add up to zero there.