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1 Compute a channel flow rate function #01

1.1 Statement: Compute a channel flow rate function #01

It was shown in the lectures that for a river (or a man-made channel) in the plains, under conditions that are not changing too rapidly (quasi-equilibrium), the following equation should apply

\[ A_t + q_x = 0, \]  

(1.1)

where \( A = A(x, t) \) is the cross-sectional filled area of the river bed, \( x \) measures length along the river, and \( q = Q(A) \) is a function giving the flow rate at any point.

That the flow rate \( q \) should be a function of \( A \) only follows from the assumption of quasi-equilibrium. Then \( q \) is determined by a local balance between the frictional forces and the force of gravity along the river bed.

Assume now a man-made channel, with uniform triangular cross-section and a uniform (small) downward slope, characterized by an angle \( \theta \). Assume also that the frictional forces are proportional to the product of the flow velocity \( u \) down the channel, and the wetted perimeter \( P_w \) of the channel bed \( F_f = C_f u P_w \). \textbf{Derive the form that the flow function \( Q \) should have}.

**Hints:**

1. \( Q = u A \), where \( u \) is determined by the balance of the frictional forces and gravity.
2. The wetted perimeter \( P_w \) is proportional to some power of \( A \).

1.2 Answer: Compute a channel flow rate function #01

The wetted perimeter is proportional to the square root of the filled cross-sectional area. That is:

\[ P_w = C_w \sqrt{A}, \]

where \( C_w = \sqrt{8/\sin(\phi)} \) and \( \phi \) is the bottom angle of the triangular channel bed. Thus the frictional forces (per unit length) along the river bed are given by \( F_f = C_f P_w u = C_f C_w u \sqrt{A} \) — where \( u \) is the flow velocity and \( C_f \) is a friction coefficient.

On the other hand, the component of the force of gravity (per unit length) along the channel bed is given by \( F_g = \rho g \sin(\theta) A \) — where \( \theta \) is the angle that the channel bed makes with the horizontal, \( g \) is the acceleration of gravity, and \( \rho \) is the density of the water in the channel.

From the quasi-equilibrium assumption \( F_f = F_g \), this yields \( u = \frac{\rho g \sin(\theta)}{C_f C_w} \sqrt{A} \). Hence, since \( q = u A \), it follows that:

\[ Q = \frac{\rho g \sin(\theta)}{C_f C_w} A^{3/2}. \]  

(1.2)

Note that \( q \) is a convex function of \( A \).

2 Discontinuous Coefficients in Linear 1st order pde #01

2.1 Statement: Discontinuous Coefficients in Linear 1st order pde #01

Singularities (in particular, discontinuities) in the coefficients of a pde can create ambiguities in the meaning of the equation. Sometimes these ambiguities can be easily resolved, and other times they cannot. In all cases, however, it is advisable to go back to the physical system that the pde is supposed to model, and either (a) Check that the meaning given to the solutions across the singularities in the coefficients makes physical sense, or (b) Seek for the meaning, if not clear, there.

1 Possibly also \( x \). That is: \( q = Q(x, A) \), to account for non-uniformities along the river.
2 What does it mean to be a solution?
In this exercise we consider a simple example of the situation described in the prior paragraph. Consider the initial value problem
\[
  u_t + c u_x = \text{sign}(x) \quad \text{for } t > 0 \text{ and } -\infty < x < \infty, \quad \text{with } u(x, 0) = g(x),
\]  
(2.1)
where \( c \) is some constant and \( g \) is some arbitrary function — which we will assume, for simplicity, is smooth. The task is now to give a unique, unambiguous, meaning to this I.V.P.

An important (practical) consideration is that the solution to any mathematical question must not be sensitive to small changes in the initial data. Show that the solution obtained in step 1 is not sensitive to small changes in the initial data — as long as \( c \neq 0 \). What happens for \( c = 0 \)?

\[\text{Step 1.} \quad \text{Replace (2.1) by the set of problems, parameterized by } \epsilon > 0,\]
\[
u_t + c u_x = f_\epsilon(x) \quad \text{for } t > 0 \text{ and } -\infty < x < \infty, \quad \text{with } u(x, 0) = g(x), \quad (2.3)\]
where \( f_\epsilon \) is a smooth, non-decreasing, function satisfying \( f_\epsilon(x) = \text{sign}(x) \) for \( |x| > \epsilon \). Show that the limit \( \epsilon \downarrow 0 \) of the solutions to (2.3) exists, and it is independent of the choice of the functions \( f_\epsilon \).

Thus we can use the \( \epsilon \downarrow 0 \) limit of (2.3) to give a clear meaning to (2.1).

\[\text{Hint.} \quad \text{Write the solution to (2.3) using characteristics, and then take the limit } \epsilon \downarrow 0.\]

\[\text{Step 2.} \quad \text{Show that the solution obtained in step 1 is not sensitive to small changes in the initial data } g = g(x), \text{ or to changes in the value of } c \text{ — as long as } c \neq 0. \text{ What happens for } c = 0?\]

\subsection{Answer: Discontinuous Coefficients in Linear 1st order pde \#01}

The characteristic form for (2.3) is
\[
\frac{dx}{dt} = c \quad \text{and} \quad \frac{du}{dt} = f_\epsilon(x), \quad \text{with } x = \mu \text{ and } u = g(\mu) \text{ for } t = 0, \quad (2.4)
\]
where \( -\infty < \mu < \infty \) is a constant. The solution is
\[
x = \mu + ct \quad \text{and} \quad u = g(\mu) + \int_0^t f_\epsilon(\mu + cs) \, ds, \quad (2.5)
\]
which yields
\[
u = g(x - ct) + \int_0^t f_\epsilon(x - c(t-s)) \, ds. \quad (2.6)
\]
Clearly, the \( \epsilon \downarrow 0 \) limit of this is independent of the particular choice of the functions \( f_\epsilon \). Namely
\[
u = g(x - ct) + \int_0^t \text{sign}(x - c(t-s)) \, ds = \begin{cases} 
g(x - ct) + (|x| - |x - ct|)/c & \text{if } c \neq 0, 
g(x) + t \text{sign}(x) & \text{if } c = 0. 
\end{cases} \quad (2.7)
\]
Clearly this formula for the solution \( u \) depends continuously on the initial values \( g \) — for any choice of \( c \). It is also continuous on \( c \), provided \( c \neq 0 \). What happens for \( c = 0 \)? To answer this, consider the situation where \( g \equiv 0 \), and write the solution \( u \) for \( c \neq 0 \) (and \( t > 0 \)) as follows
\[
u = t \text{sign}(c) \left( |\zeta| - |\zeta - 1| \right), \quad \text{where} \quad \zeta = \frac{x}{ct} \quad \text{and} \quad h(\zeta) = \begin{cases} 
1 & \text{for } 1 \leq \zeta \\
2\zeta - 1 & \text{for } 0 \leq \zeta \leq 1 \\
-1 & \text{for } \zeta \leq 0
\end{cases}. \quad (2.8)
\]
This approaches \( u = t \text{sign}(x) \) as \( c \to 0 \), but the limit is not uniform in \( x \). Nevertheless, it is as good as it can be, given that the solution for \( c \neq 0 \) is continuous, while the \( c = 0 \) solution is discontinuous.

\textbf{Remark 2.1} The case \( c = 0 \) is troublesome because a resonance occurs there. Namely: the discontinuity in the forcing term sits right on top of a characteristic curve.
3 Envelope of the characteristics and cusp

3.1 Statement: Envelope of the characteristics and cusp

Consider the problem
\[ c_t + c c_s = 0, \quad \text{where} \quad c(x, 0) = C(x), \quad \text{for} \quad -\infty < x < \infty. \] (3.1)

**e1. Write the characteristic curves for this problem,** each one parameterized by time, and labeled by the value of \( x = s \) at time \( t = 0 \). That is, write formulas for the characteristics of the form \( x = X(s, t) \).

**e2. Write the equation for the envelope of the characteristics, and express the envelope in parametric form**
\( x = X_e(s) \) and \( t = T_e(s) \).

**e3. Assume that \( C \) has an inflection point at \( x = 0 \). In fact, assume that\( C(0) = 0, \quad C'(0) = -a < 0, \quad C''(0) = 0, \quad \text{and} \quad C'''(0) = 2ab > 0, \) (3.2)
where the primes denote derivatives.

Let \( x_e = X_e(0) \) and \( t_e = T_e(0) \). **Show that \( T_e \) has a local minimum at \( s = 0 \), and that the envelope has a cusp at \((x_e, t_e)\).**

**HINT:** Expand the equations for \( s \) small.

**e4. Replace (3.1) by**
\[ c_t + c c_s = -c, \quad \text{where} \quad c(x, 0) = C(x), \quad \text{for} \quad -\infty < x < \infty. \] (3.3)

**Repeat steps e1 and e2 for this problem. Question: what condition is needed on \( C' \) so that the characteristics of (3.3) actually have an envelope in the upper half space-time plane? That is, so that \( T_e(s) > 0 \) somewhere.**

**Note:** Assume a one parameter set of curves in the plane, defined by an equation of the form \( F(x, t, s) = 0 \), where \( s \) is the parameter. Then the envelope is the set of points in the plane which are the intersections of two “infinitesimally close” curves. That is, points that satisfy both \( F(x, t, s) = 0 \) and \( F(x, t, s + ds) = 0 \).

Expanding the second equation we see that **the envelope is defined by**
\[ F(x, t, s) = 0 \quad \text{and} \quad F_s(x, t, s) = 0. \] [A]

Given the definition above of the envelope, it should be clear that **the envelope of characteristics is precisely the place where the derivatives of the solution by characteristics of (3.1) become infinity:** at \( t = 0, dx = ds \), and there is some \( dc \) given by the initial conditions that result in a finite \( \frac{dc}{dx} \). However, at the envelope \( dc \) is still the same, while \( dx = 0; \) hence \( c_e = \pm \infty \).

An alternative definition of the envelope is: it is a curve \( x = X(s) \) and \( y = Y(s) \) such that the point \((X(s), Y(s))\) in the curve is also in the curve with label \( s \) belonging to the family; and both curves are tangent there.\(^3\) What this means is that the “neighboring” characteristics that intersect at the envelope, are also tangent to the envelope there.

3.2 Answer: Envelope of the characteristics and cusp

**e1.** The characteristic curves for the problem in (3.1) are given by
\[ x = C(s) t + s = X(s, t), \quad \text{where} \quad -\infty < s < \infty \quad \text{and} \quad t > 0. \] (3.4)

**e2.** The envelope of the curves in (3.4) is given by the two equations
\[ x = C(s) t + s = X(s, t) \quad \text{and} \quad 0 = C'(s) t + 1. \] (3.5)

Hence
\[ t = -\frac{1}{C'(s)} = T_e(s) \quad \text{and} \quad x = s - \frac{C(s)}{C'(s)} = X_e(s). \] (3.6)

\(^3\) You can check that this also leads to the equations in [A].
e3. Expand $C = -a \left( s - \frac{1}{3} b s^3 \right) + O(s^4)$ and $C' = -a (1 - b s^2) + O(s^3)$. Thus

$$T_e = \frac{1}{a} \left( 1 + b \, s^2 \right) + O(s^3) \quad \text{and} \quad X_e = -\frac{2}{3} b s^3 + O(s^4). \quad (3.7)$$

Clearly $T_e$ has a local minimum at $s = 0$. Further, with $t_e = 1/a$ and $x_e = 0$, we see that on the envelope

$$(t - t_e) = \left( \frac{3^{2/3} b^{1/3}}{2^{2/3} a} \right) \left( (x - x_c)^2 \right)^{1/3} + O(x - x_c). \quad (3.8)$$

Hence the envelope has a cusp at $(x_c, t_c)$.

e4. The characteristics for (3.3) are given by

$$x = C(s) \left( 1 - e^{-t} \right) + s = X(s, t), \quad \text{where} \quad -\infty < s < \infty \quad \text{and} \quad t > 0. \quad (3.9)$$

The envelope of these curves is given by the equations

$$x = C(s) \left( 1 - e^{-t} \right) + s \quad \text{and} \quad 0 = C'(s) \left( 1 - e^{-t} \right) + 1. \quad (3.10)$$

The second equation has a solution with $t > 0$ if and only if $C'(s) < -1$ somewhere. Given this we can write

$$T_e(s) = -\ln \left( 1 + \frac{1}{C'(s)} \right) \quad \text{and} \quad X_e(s) = s - \frac{C(s)}{C'(s)}. \quad (3.11)$$

4 Linear 1st order PDE (problem 01)

4.1 Statement: Linear 1st order PDE (problem 01)

Part 1. Find the general solutions to the two 1st order linear scalar PDE

$$x u_x + y u_y = 0, \quad \text{and} \quad y v_x - x v_y = 0. \quad (4.1)$$

Hint: The general solutions take a particular simple form in polar coordinates.

Part 2. For $u$, find the solution such that on the circle $x^2 + y^2 = 2$, it satisfies $u = x$. Where is this solution determined by the data given?

Part 3. Is there a solution to the equation for $v$ such that $v(x, 0) = x$, for $-\infty < x < \infty$?

Part 4. How does the general solution for $u$ changes if the equation is modified to

$$x u_x + y u_y = (x^2 + y^2) \sin(x^2 + y^2)? \quad (4.2)$$

4.2 Answer: Linear 1st order PDE (problem 01)

Part 1. In terms of polar coordinates $x = r \cos(\theta)$ and $y = r \sin(\theta)$, the equations take the form

$$r u_r = 0 \quad \text{and} \quad -v_\theta = 0. \quad (4.3)$$

Hence, the general solutions are $u = U(\theta)$ and $v = V(r)$, where $U$ and $V$ are arbitrary functions of their arguments.

Alternatively, the characteristics for the first equation (namely $\frac{dr}{ds} = x$ and $\frac{d\theta}{ds} = y$, along which $\frac{du}{ds} = 0$) are the straight lines through the origin. Hence $u$ should only be a function of the angle. Similarly, the characteristics for the second equation are the circles centered at the origin, so that $v$ should depend on the radial variable only.

\[\text{Along each characteristic } c = C(s) e^{-t}.\]
Remark 4.1 Notice that the origin \( x = y = 0 \) is a “bad” point for these p.d.e.’s — where no restriction is applied on the solutions. Hence, we generally should exclude it from the domain where we seek to solve the equations.

**Part 2.** Since for \( x^2 + y^2 = 2 \) we have \( x = \sqrt{2} \cos(\theta) \) and \( y = \sqrt{2} \sin(\theta) \), from the general solution in part 1 it follows that

\[
u = \sqrt{2} \cos(\theta).
\]

(4.4)

This solution is determined for \( r > 0 \) — see remark 4.1. In fact, it is singular and becomes multiple valued for \( r \to 0 \).

**Part 3.** From part 1, it should be that \( v(x, 0) = V(|x|) = v(-x, 0) \), which the data \( v(x, 0) = x \) does not satisfy. Hence the problem in part 3 does not have a solution. Another way of seeing this is that the characteristics for this problem (circles centered at the origin) intersect the \( x \)-coordinate axis at two points each, where the data given is not consistent.

**Part 4.** The characteristic curves are still the same, namely \( \frac{dx}{ds} = x \) and \( \frac{dy}{ds} = y \), but now along then \( u \) is no longer constant and

\[
\frac{du}{ds} = (x^2 + y^2) \sin(x^2 + y^2) = -\frac{1}{2} \frac{d}{ds} \cos(x^2 + y^2).
\]

(4.5)

Hence, the general solution is \( u = U(\theta) - \frac{1}{2} \cos(r^2) = \text{“general solution of the homogeneous problem” + “particular solution” — since the problem is linear.}

Alternatively: in polar coordinates the equation is \( ru_r = r^2 \sin(r^2) \), which leads to the same answer.

---

5 Linear 1st order PDE (problem 10)

5.1 Statement: Linear 1st order PDE (problem 10)

**Integrating factors.** Show that the pde

\[
(a(x, y) \mu)_y = (b(x, y) \mu)_x
\]

is a necessary and sufficient condition guaranteeing that \( \mu = \mu(x, y) \neq 0 \) is an integrating factor for the ode

\[
a(x, y) dx + b(x, y) dy = 0,
\]

(5.2)

in any open subset of the plane without holes.

**Part II.** Assume that \( a = 3x y + 2y^2 \) and \( b = 3x y + 2x^2 \).

Find an integrating factor for (5.2) — i.e.: obtain a nontrivial solution of (5.1). Use it to integrate (5.2), and write (5.2) in the form

\[
\Phi(x, y) = \text{constant}, \quad \text{for some function } \Phi.
\]

(5.3)

**Hint 5.1** Solving by characteristics (5.1) leads to (5.2), or equivalent (as part of the process — check this!) To get out of this circular situation, note that: for \( a \) and \( b \) as above, \( \mu = F(x, y) \) solves (5.1) iff \( \mu = F(y, x) \) does. This suggests that you should look for solutions\(^5\) invariant under this symmetry; that is: \( \mu(x, y) = \mu(y, x) \). Hence write \( \mu = \mu(u, v) \), with \( u = x + y \) and \( v = x y \), since solutions that satisfy \( \mu(x, y) = \mu(y, x) \) must have this form — see remark 5.1.

**Remark 5.1** The transformation \( (x, y) \to (u, v) \) is not one to one: it maps the whole \( xy \)-plane into the region \( v \leq \frac{1}{2} u^2 \) of the \( uv \)-plane, with double valued inverse \( x = \frac{1}{2} (u \pm \sqrt{u^2 - 4v}) \) and \( y = \frac{1}{2} (u \mp \sqrt{u^2 - 4v}) \). Furthermore:

(a) The two inverses are related by the \( x \leftrightarrow y \) switch. (b) The singular line \( u^2 = 4v \) corresponds to the line \( x = y \).

(c) The map is a bijection between the regions \( x \leq y \) and \( 4v \leq u^2 \). (d) The map is a bijection between the regions \( x \geq y \) and \( 4v \leq u^2 \).

From (c - d) we see that: for any \( \mu = \mu(x, y) \), \( \mu = f(u, v) \) for \( x \leq y \) and \( \mu = g(u, v) \) for \( x \geq y \), for some \( f \) and \( g \). Then, if \( \mu(x, y) = \mu(y, x) \), \( f = g \), so that \( \mu \) has the form \( \mu = \mu(u, v) \).

---

\(^{5}\) Note that you only need one nontrivial solution.
Part III. Why is it that this approach cannot be generalized to three variables? That is, to find integrating factors for equations of the form
\[ a(x, y, z) \, dx + b(x, y, z) \, dy + c(x, y, z) \, dz = 0. \] (5.4)

5.2 Answer: Linear 1st order PDE (problem 10)

If (5.1) is satisfied in an open subset without holes, then there exists a function \( \Phi = \Phi(x, y) \) such that
\[ \Phi_x = a \mu \quad \text{and} \quad \Phi_y = b \mu. \] (5.5)

In fact, pick a fixed point in the set, say \( Q \). Then define
\[ \Phi(x, y) = \int_{\Gamma} \mu \, dx + \mu \, dy, \] (5.6)
where \( \Gamma \) is any curve (in the set) connecting \( Q \) to \( (x, y) \). The value of this integral does not depend on the curve \( \Gamma \), as follows from Green’s theorem and (5.1) — i.e.: \( \oint_{\Lambda} \mu \, dx + \mu \, dy = 0 \) for any closed curve \( \Lambda \) in the set. Hence (5.6) does define a function — which (obviously) satisfies (5.5).

Given (5.5), equation (5.2) yields
\[ 0 = \mu a \, dx + \mu b \, dy = \Phi_x \, dx + \Phi_y \, dy = d\Phi \quad \iff \quad \Phi = \text{constant}. \] (5.7)

Of course, for (5.7) to be of any use, \( \mu \) must be non-trivial: \( \mu = 0 \) always works, but it also leads to the useless function \( \Phi(x, y) \equiv \text{constant} \).

Vice-versa, if \( \mu \) is an integrating factor, there is some function \( \Phi \) such that
\[ \mu a \, dx + \mu b \, dy = d\Phi = \Phi_x \, dx + \Phi_y \, dy. \] (5.8)

Hence (5.5) applies, from which (5.1) follows.

Part II. With \( a = 3 \, x \, y + 2 \, y^2 \), and \( b = 3 \, x \, y + 2 \, x^2 \), equation (5.1) takes the form
\[ a \, \mu_y - b \, \mu_x = (b_x - a_y) \, \mu = (x - y) \, \mu. \] (5.9)

In terms of the coordinates \( u = x + y \) and \( v = x \, y \), a little bit of algebra reduces this equation to
\[ (y - x) \left( 2 \, u \, \mu_u - v \, \mu_v + \mu \right) = 0. \] (5.10)

That is, for \( x \neq y \) we have
\[ 2 \, u \, \mu_u - v \, \mu_v + \mu = 0. \] (5.11)

We only need one non-trivial solution for this equation to find an integrating factor. Nevertheless, next we find all the solutions, using characteristics.

The characteristic equations for (5.11) can be written in the form
\[ \frac{du}{2 \, u} = - \frac{dv}{v} = - \frac{d\mu}{\mu} = ds, \] (5.12)
where \( s \) is a parameter along each characteristic. From the first equality it follows that \( u \, v^2 = \zeta \), where \( \zeta \) is a constant on each characteristic — which we use as a label for the characteristic curve. From the second equality it follows that \( \mu/v = f \), where \( f \) is also a constant along each characteristic — hence \( f = f(\zeta) \) must be some function of the characteristic label. Thus the general solution to (5.11) has the form
\[ \mu = v \, f(u \, v^2) = x \, y \, f((x + y) \, x^2 \, y^2), \] (5.13)
where \( f \) is some arbitrary function (with, at least, one derivative).

\(^6\) Here is where having a set without holes matters: if there are holes, then \( \oint_{\Lambda} \mu \, dx + \mu \, dy = 0 \) is not guaranteed, and (5.6) cannot be used to define a function.
Remark 5.2  The formula in (5.13) follows from solving (5.11), which is equivalent to (5.9) only for \( x < y \), or \( x > y \). Hence, at this stage, the only thing that we can say about the general solution to (5.9) is that it has the form
\[
\mu = x y f \left( (x + y)x^2 y^2 \right) \quad \text{for } x > y, \quad \text{and} \quad \mu = x y g \left( (x + y)x^2 y^2 \right) \quad \text{for } x < y,
\]
for some arbitrary function \( f \) and \( g \). However, evaluating along \( x = y \) yields
\[
x^2 f \left( 2x^5 \right) = x^2 g \left( 2x^5 \right).
\]
Thus, we conclude that:

All the solutions to (5.9) satisfy \( \mu(x, y) = \mu(y, x) \), and have the form in (5.13), with the same function \( f \) for all \( x \) and \( y \). This is rather interesting, since (generally) the fact that a pde. has a symmetry does not imply that all the solutions have it too. For example: the heat equation \( T_t = T_{xx} \) is invariant under \( x \leftrightarrow -x \), but \( T = 2 + \sin(x) e^{-t} \) is a solution that is not invariant under \( x \leftrightarrow -x \).

We now take the simplest of the solutions in (5.13), \( \mu = v = x y \), and plug it into equation (5.5). This yields
\[
\Phi_x = a x y = 3 x^2 y^2 + 2 x y^3 \quad \text{and} \quad \Phi_y = b x y = 3 x^2 y^2 + 2 x^3 y. \tag{5.14}
\]
Thus \( \Phi = x^3 y^2 + x^2 y^3 + \Phi_0 \) — where \( \Phi_0 \) is a constant. Hence, from (5.7), it follows that: For the case \( a = 3 x y + 2 y^2 \) and \( b = 3 x y + 2 x^2 \), (5.2) can be integrated to
\[
x^3 y^2 + x^2 y^3 = u v^2 = \text{constant}. \tag{5.15}
\]

Part III. If \( \mu \) is a non-trivial integrating factor for (5.4), then
\[
\mu a dx + \mu b dy + \mu c dz = d\Phi, \tag{5.16}
\]
for some function \( \Phi = \Phi(x, y, z) \). This is equivalent to
\[
\mu a = \Phi_x, \quad \mu b = \Phi_y, \quad \text{and} \quad \mu c = \Phi_z. \tag{5.17}
\]
However, this then implies the equations
\[
(\mu a)_y = (\mu b)_x, \quad (\mu c)_x = (\mu a)_z, \quad \text{and} \quad (\mu b)_z = (\mu c)_y. \tag{5.18}
\]
Thus, we get three equations for a single unknown \( \mu \). This is an over-determined system that (generally) has only one solution: \( \mu = 0 \). In fact, in (5.18), multiply the first equation by \( c \), the second equation by \( b \), the third equation by \( a \), add, and use the fact that we want \( \mu \neq 0 \). This then yields
\[
0 = a (b_z - c_y) + b (a_x - c_z) + c (a_y - b_x). \tag{5.19}
\]
Equivalently
\[
0 = w \cdot (\nabla \times w), \quad \text{where} \quad w = \begin{bmatrix} a \\ b \\ c \end{bmatrix}. \tag{5.20}
\]
Hence, if \( a, b, \) and \( c \) do not satisfy equation (5.20), (5.4) has no integrating factor.

**Task left to the reader:** Show that, at least locally, (5.20) is sufficient to guarantee that (5.18) has a non-trivial solution.
Hint 5.2 If \( w \equiv 0 \), then any \( \mu \) solves (5.18). Hence, in any sufficiently small cube, we can assume that one of the components of \( w \) is never zero. Thus, without loss of generality, assume \( a \geq \delta > 0 \) in the cube \(-\epsilon < x, y, z < \epsilon\), where \( \delta \) and \( \epsilon > 0 \) are constants. Then: I. Use the first equation in (5.18) to construct \( \mu_0 = \mu_0(x, y) \) for \(-\epsilon < x, y < \epsilon\). II. Use the second equation in (5.18), with \( \mu(x, y, 0) = \mu_0(x, y) \), to define \( \mu \) in a neighborhood in \( \mathbb{R}^3 \) of the square \(-\epsilon < x, y < \epsilon\). III. Show that the function \( \mu \) that you just constructed solves (5.18). To do this: Define \( \phi = (\mu a) y - (\mu b) x \) and \( \psi = (\mu c) y - (\mu b) z \). Then: (i) Use (5.19) to find an algebraic relationship between \( \phi \) and \( \psi \). (ii) Use that \( \mu \) satisfies the middle equation in (5.18), to derive a p.d.e. that \( \phi \) satisfies. (iii) By construction \( \phi = 0 \) for \( z = 0 \). Use the pde in (ii) to conclude that \( \phi \equiv 0 \). (iv) Use (i) and (iii), conclude that \( \psi \equiv 0 \). Since \( \mu \) satisfies the the middle equation in (5.18) by construction, this ends the proof.

Note: point out where the assumption \( a > 0 \) comes into play in your arguments.

Remark 5.3 Of course, once a \( \mu \) satisfying (5.18) is obtained, a \( \Phi \) satisfying (5.17) is given by

\[
\Phi(x, y, z) = \int_{\Gamma} \mu a \, dx + \mu b \, dy,
\]

where \( \Gamma \) is any curve connecting some fixed point \( Q \) with \((x, y, z)\).

Why is it that the integral in (5.21) depends ONLY on the endpoints of the curve \( \Gamma \)?

6 Nondimensionalization and modulation for Short Waves

6.1 Statement: Nondimensionalization and modulation for Short Waves

6.1.1 Introduction: Dispersive waves and modulation

Consider the following situation

1. There is some media, where waves can propagate. What exactly these waves are will not matter here. We will not even define what we mean by “waves”, and rely on intuition to give the concept a meaning. Examples are: the waves on the surface of the sea, sound propagating through air, seismic waves on the earth, light on a transparent media, etc. For simplicity here we restrict our attention to one dimension in space only.

2. We can associate a wavelength \( \lambda \) and a frequency \( f \) to each wave. Alternatively, a wave number \( k = \frac{2\pi}{\lambda} \) and an angular frequency \( \omega = 2\pi f \).

Note 1: While the wavelength and frequency have a direct, intuitive physical meaning, this is not so for the wave number and angular frequency. So, why are they important? A first approximation to the answer is: In the context of mathematical models for wave phenomena, it is generally true that small enough amplitude waves can be expressed in terms of harmonic functions. That is, the solutions of the equations depend on space and time via terms of the form

\[
\sin(k x - \omega t + \theta_0) = \sin \theta,
\]

where \( \theta_0 \) is some constant and \( \theta \) is the wave phase. It is then more convenient to deal with wave numbers and angular frequencies, rather than with wavelengths and frequencies.

There are many more reasons (some deeper, but harder to explain shortly) than the above. This very problem offers more reasons to deal with \( k \) and \( \omega \), instead of \( \lambda \) and \( f \).

Note 2: The speed at which the “bumps” in the waves propagate is called the phase speed — in the expression above in (6.1), it is the speed at which any constant value of the phase \( \theta = \text{constant} \) propagates. The phase speed is given by

\[
c_p = \frac{\omega}{k} = \lambda f.
\]
3. There is some relationship between the wavelength and frequency, expressed by a

$$\text{Dispersion Relation: } \omega = \Omega(k), \quad (6.3)$$

where $\Omega$ is some given function. This function can follow from some mathematical theory for the waves, or it can be something that is obtained observationally.

- What (6.3) says is that, once the wavelength is known, the wave frequency (hence the phase speed) follows. The reason it is called a “dispersion relation” is that (except for the trivial case when $\Omega$ depends linearly on $k$) it generally implies that waves with different $\lambda$’s propagate at different phase speeds. Thus, if a wave packet is made by combining several waves, it will eventually disperse (because — to keep it as a wave packet — it is necessary that all the component waves stay with their phases properly “lined up”, so that the waves “add up” in some small region only).

Imagine now that we have a “sea” of these waves, where (at each point in space and time) the waves are (essentially) “monochromatic”, so that a single frequency and wavelength can be associated with them. However, the whole field of waves does not have a single frequency (or wavelength). To be precise, what we are saying is that it is possible to define the wave number and angular frequency as functions of position and time:

$$k = k(x, t) \quad \text{and} \quad \omega = \omega(x, t), \quad (6.4)$$

and that these functions completely describe the waves (up to scaling by an amplitude, that can also change in space and time).

**Remark 6.1** Notice that, to even begin to talk about a wave frequency and a wavelength at each point, the waves must be reasonably periodic in both space and time, over lengths and times much larger than their wavelength and period. This is what we mean by saying that they are essentially monochromatic (by analogy with light, where a single frequency means a single color).

Now we make the assumption that the “bumps” in the waves are conserved — the idea is that, if $k$ and $\omega$ vary “slowly” then nothing as drastic as a chunk of a wave vanishing can happen. With this assumption we can construct a mathematical model to describe the evolution in time of $k$ and $\omega$ above in (6.4). The equations can be derived without any knowledge of the underlying detailed mechanisms governing the waves, only the dispersion relation (6.3) is needed — i.e.: it is a “phenomenological” model, not based on any discussion of the detailed mechanisms underlying the phenomena being studied. The equations for the model are:

$$k_t + \omega_x = 0, \quad \text{where} \quad \omega = \Omega(k). \quad (6.5)$$

**What we want to discuss next is how the “phenomenological” derivation can be justified within the context of a mathematical model for the wave behavior.** It is important to notice, however, that such a detailed model is not needed to be able to use (6.5). In principle, we could simply measure $\Omega = \Omega(k)$ experimentally and then use (6.5) without any knowledge of the detailed equations that govern the waves.

### 6.1.2 Justification of the phenomenological approach (example)

In this subsection we use both dimensional and nondimensional variables. To avoid confusion, we indicate which variables are dimensional by using a tilde over their symbols. We do this only for the variables, where we want to use the same symbol for both the dimensional and the nondimensional versions, not the constants — where confusion cannot not arise.

The idea is to introduce the notion of a phase\(^7\) $\theta = \theta(\tilde{x}, \tilde{t})$, which should behave like the phase $\Theta$ in (6.1) near any point in space and time. That is, in some appropriate sense, we want:

$$\theta(\tilde{x} + \Delta\tilde{x}, \tilde{t} + \Delta\tilde{t}) \approx \tilde{k}(\tilde{x}, \tilde{t}) \Delta\tilde{x} - \tilde{\omega}(\tilde{x}, \tilde{t}) \Delta\tilde{t} + \theta_0, \quad (6.6)$$

\(^7\) Note that phases are always nondimensional.
whenever $\Delta \tilde{x}$ and $\Delta \tilde{t}$ are small, where now we must allow $\theta_0$ to depend on space and time. To do this properly, we must first nondimensionalize the problem (since otherwise “large” and “small” do not have a precise meaning). Then we will use this phase $\theta$ to justify equation (6.5) above.

Consider the propagation of the waves governed by the single scalar equation:

$$\ddot{\tilde{u}} - \beta \tilde{u}_{\tilde{x} \tilde{x}} = 0,$$

(6.7)

where $\beta > 0$ is a constant with dimensions $\text{length}^3 \text{time}$. This has solutions of the form (6.1), given by

$$\tilde{u} = \tilde{a} \sin(\tilde{k} \tilde{x} - \tilde{\omega} \tilde{t} + \theta_0),$$

(6.8)

and $\tilde{a}$ is an arbitrary constant. Now assume that we have a solution of the type described in equation (6.4) and remark 6.1. Then, nondimensionalize using (for distances and times, respectively):

- $L = \text{distance over which } O(1) \text{ variations in the wave number } \tilde{k} \text{ and angular frequency } \tilde{\omega} \text{ occur. That is, over distances of size } L, \frac{\Delta \tilde{k}}{k_*} \text{ and } \frac{\Delta \tilde{\omega}}{\omega_*} \text{ can be } O(1)$ — where $k_*$ and $\omega_*$ are typical wave numbers and angular frequencies.
- $T = \text{time over which } O(1) \text{ variations in the wave number } \tilde{k} \text{ and angular frequency } \tilde{\omega} \text{ occur. That is, over times of size } T, \frac{\Delta \tilde{k}}{k_*} \text{ and } \frac{\Delta \tilde{\omega}}{\omega_*} \text{ can be } O(1)$.

Then the assumptions explained in remark 6.1 are equivalent to the statement that

$$L \gg \frac{2\pi}{k_*} = \lambda_* \quad \text{and} \quad T \gg \frac{2\pi}{\omega_*} = \frac{1}{f_*}.$$

(6.9)

It is important to note, however, that $L$ and $T$ are not “independent”. In fact, they must be related by a “typical” speed $c_*$ of wave propagation:

$$L = c_* T,$$

(6.10)

since any variations in the wave parameters can be expected to move at some velocity which cannot be too different from the wave speeds in the problem. So, the question is now:

**How do we find a “typical” speed of propagation?**

The answer is fairly simple: this speed can only depend on the equation constant $\beta$ and the typical wave number $k_*$, since there are no other parameters in the problem. Thus it is clear that we must have:

$$c_* = \beta k_*^2 \quad \Rightarrow \quad T = \frac{L}{\beta k_*^2},$$

(6.11)

since this is the only way a quantity with the dimensions of a speed can be produced from the constants $\beta$ and $k_*$. We stress here that equation (6.11) can be justified entirely on dimensional grounds. It follows from the requirement that a velocity for a given wavelength $\lambda_* = 2\pi/k_*$, must obtain its dimensions from the wavelength and the only dimensional parameter in the equation (nothing else around having dimensions). On the other hand, to double check that this is the correct equation, notice that $c_*$ above is precisely the phase speed for the wave number $k_*$ — see (6.2) and (6.8).

$$\left\{ \begin{array}{l}
\frac{L}{T} = c_* = c_p(k_*) = \frac{\omega_*}{k_*}, \text{ we can write } \epsilon = \frac{1}{k_* L} = \frac{1}{\omega_* T}. \text{ Thus the two conditions in (6.9) are actually the same. That is } \epsilon \ll 1.
\end{array} \right.$$

(6.12)

Since $\frac{L}{T} = c_* = c_p(k_*) = \frac{\omega_*}{k_*}$, we can write $\epsilon = \frac{1}{k_* L} = \frac{1}{\omega_* T}$. Thus the two conditions in (6.9) are actually the same. That is $\epsilon \ll 1$.

We now nondimensionalize now as follows:

$$\tilde{x} = L x, \quad \tilde{t} = T t, \quad \tilde{u} = a_* u, \quad \tilde{k} = k_* k \quad \text{and} \quad \tilde{\omega} = \omega_* \omega,$$

(6.13)

where $a_*$ is a typical wave amplitude. Note that:
The nondimensionalization for \( u, k, \) and \( \omega \) is such that all these quantities are \( O(1) \).

The nondimensionalization for \( x \) and \( t \) is such that significant variations in the wave properties occur over \( O(1) \) distances and times.

Then the **nondimensional form** of the wave’s equation (6.7) is

\[
 u_t - \epsilon^2 u_{xxx} = 0, \tag{6.15}
\]

where

\[
 \epsilon = \sqrt{\frac{\beta T}{L^3}} = \frac{1}{k_x L} = \frac{1}{\omega_x T} \ll 1
\]

is as before — in (6.13).

**Remark 6.2** Notice that we have used different units to nondimensionalize \( x \) and \( k \) (respectively: \( t \) and \( \omega \)). This means that, in the nondimensional units, the relationship between wavelength and wavenumber has changed from \( \tilde{k} \lambda = 2\pi \) (respectively: the relationship between frequency and angular frequency has changed from \( \tilde{\omega} = 2\pi \tilde{f} \)) to

\[
 k \lambda = 2\pi \epsilon \quad \text{(respectively: } \omega = 2\pi \epsilon f \text{),}
\]

as follows easily from \( \epsilon \tilde{k} \lambda = \frac{1}{k_x L} k_x k L \lambda = k \lambda \) — with a similar calculation for the frequencies.

Thus, equation (6.6) for the phase takes the form (in the nondimensional units)

\[
 \theta(x + \Delta x, t + \Delta t) \approx \frac{1}{\epsilon} k(x, t) \Delta x - \frac{1}{\epsilon} \omega(x, t) \Delta t + \theta_0(x, t). \tag{6.17}
\]

Furthermore, the solutions in equation (6.8) now take the form

\[
 u = a \sin \left( \frac{1}{\epsilon} k x - \frac{1}{\epsilon} \omega t + \theta_0 \right), \quad \text{where } \omega = k^3, \tag{6.18}
\]

and \( a \) is an arbitrary constant. Regarding this last equation, notice that the fact that the small coefficient \( \epsilon^2 \) is in front of the third space derivative in (6.15) is important in making a solution of the form above possible (with both \( k \) and \( \omega \) order one quantities). This is related to having made the correct choice for \( T \) in (6.11).

**Remark 6.3** Equation (6.16) shows that: In the nondimensional units, the wavelength is small (i.e.: \( O(\epsilon) \)) and the frequency is large (i.e.: \( O(\epsilon^{-1}) \)) (as follows from the fact that \( k \) and \( \omega \) are \( O(1) \)). Thus, in these units, the situation that can be described by: Short waves, whose parameters vary over \( O(1) \) space and time scales.

Equations (6.17) and (6.18) suggest that we define the phase (and approximate solution) in the following way:

\[
 \theta = \frac{\Theta(x, t)}{\epsilon}, \quad \text{and} \quad u = a(x, t) \sin \theta, \tag{6.19}
\]

where \( \Theta \) and \( a \) are functions with \( O(1) \) derivatives — and \( a \) has order one size. Then, if we take:

\[
 k = \Theta_x \quad \text{and} \quad \omega = -\Theta_t, \tag{6.20}
\]

equation (6.17) is just the Taylor expansion for \( \theta \), with \( \theta_0 = \theta(x, t) \). Furthermore, the equation

\[
 k_t + \omega_x = 0,
\]

is just the statement that \( \Theta_{xt} = \Theta_{tx} \). Finally, \( \omega = k^3 \) follows because (for \( u \) as in (6.19)) we have:

\[
 u_t = -\frac{\omega}{\epsilon} a \cos \theta + O(1) \quad \text{and} \quad \epsilon^2 u_{xxx} = -\frac{k^3}{\epsilon} a \cos \theta + O(1).
\]

Substituting this into (6.15), and requiring that the equation be satisfied at leading order, yields the required result.
6.1.3 Justification of the phenomenological approach

The general justification of the phenomenological approach follows the same basic steps as in the example in §6.1.2. The key idea is still the introduction of a phase \( \theta = \frac{1}{\epsilon} \Theta(x, t) \) — as in (6.19) — so that (6.17) applies and the solutions can be expressed as proportional to sines and cosines of the phase, with space and time dependent coefficients — as in (6.19).

For this to work, it is necessary that the equations be first properly nondimensionalized. In particular the space scale \( L \) and time scale \( T \) for the wave parameters variations (modulations) must be properly related to each other (by a typical speed of wave propagation). Then (6.13) will apply, and everything else will work pretty much in the same way as it did in §6.1.2 — except for the fact that the actual calculations may be much more involved.

6.1.4 The problem to be done

Consider the equation

\[
\frac{\partial \tilde{u}}{\partial \tilde{t}} + \alpha \frac{\partial \tilde{u}}{\partial \tilde{x}} + \beta \frac{\partial^5 \tilde{u}}{\partial \tilde{x}^5} = 0, \tag{6.21}
\]

where \( \alpha > 0 \) and \( \beta > 0 \) are dimensional constants. As before we use tildes to denote dimensional variables. The questions are now:

1. **What are the dimensions of the constant \( \alpha \)?**
2. **What are the dimensions of the constant \( \beta \)?**
3. **What form will the nondimensional equations have in this case, in a situation like the one described in the prior subsections?**

**Hint 6.3** Notice that in this case waves (solutions) of the form

\[
\tilde{u} = \tilde{a} \sin(\tilde{k} \tilde{x} - \tilde{\omega} \tilde{t} + \theta_0), \tag{6.22}
\]

exist (where \( \tilde{a} \) is an arbitrary constant), provided that the Dispersion Relation

\[
\tilde{\omega} = \alpha \tilde{k} + \beta \tilde{k}^5, \tag{6.23}
\]

holds.

**Now assume that the wavelength \( \lambda \) of the waves under consideration is such that BOTH terms in the dispersion relation (6.22) are important.** This means that \( \beta k^4 \) and \( \alpha \) must both have roughly the same size (note that they have the same dimensions). Thus, introducing the typical wave number \( k_* \), it should be that

\[
k_* = (\alpha/\beta)^{1/4} \implies \lambda_* = \frac{2\pi}{k_*} = 2\pi (\beta/\alpha)^{1/4}, \tag{6.24}
\]

where \( \lambda_* \) is the typical wavelength. Introduce — just as before — the typical angular frequency \( \omega_* \), and let \( L \) and \( T \) be the distance and time over which \( \tilde{k} \) and \( \tilde{\omega} \) vary by \( O(1) \).

Then, using (6.24) and the fact that (again) it should be \( \epsilon = 1/(k_* L) \ll 1 \) — with \( 1/(\omega_* T) \) also \( O(\epsilon) \), you should be able to do this problem using the same arguments in the prior sections. Basically what is left for you to do is to figure out what the typical velocity that relates \( T \) and \( L \) is, and then to carry the algebra to express the coefficients of the nondimensional equation in terms of \( \epsilon \) alone.

6.2 Answer: Nondimensionalization and modulation for Short Waves

6.2.1 Solution to the problem to be done

1. The constant \( \alpha \) has the dimensions of a velocity, i.e.: \( \frac{\text{length}}{\text{time}} \).
2. The constant $\beta$ has dimensions \(\text{length}^5 \text{time}^{-1}\).

3. Given a wave number $k_*$ (since the equation has two dimensional constants) two velocities can be produced, namely: $\alpha$ and $\beta k_4^4$. However, with the choice (6.24), these have the same magnitude. Thus, we take

$$c_* = \alpha \implies T = \frac{L}{\alpha}. \quad (6.25)$$

Nondimensionalizing as before — see (6.14) — equation (6.21) takes the form:

$$u_t + \frac{\alpha T}{L} u_x + \frac{\beta T}{L^5} u_{xxxx} = 0.$$  

However: \(\frac{\alpha T}{L} = 1\) and \(\frac{\beta T}{L^5} = \frac{1}{k_4^4 L^4} = \epsilon^4\). Thus, the final form of the equation is:

$$u_t + u_x + \epsilon^4 u_{xxxx} = 0. \quad (6.26)$$

This equation has solutions of the form:

$$u = a \sin \left( \frac{1}{\epsilon} k x - \frac{1}{\epsilon} \omega t + \theta_0 \right), \quad \text{where} \quad \omega = k + k^4, \quad (6.27)$$

and $a$ is an arbitrary constant. Thus, it has the right form needed for the process described at the end of § 6.1.2 — see (6.19) and what follows.

### 6.2.2 Amplitude equation

It is interesting to note that the approach in the prior subsections can be extended so as to get an equation for the propagation of the wave amplitude $a$. For example, consider equation (6.15) and the expression in (6.19) for its solution. Then:

$$u_t = -\frac{1}{\epsilon} \omega a \cos \theta + a_t \sin \theta,$$

$$\epsilon^2 u_{xxx} = -\frac{1}{\epsilon} k^3 a \cos \theta - \frac{1}{2 a} (3k^2 a^2)_x \sin \theta + O(\epsilon).$$

Thus, neglecting the $O(\epsilon)$ terms and substituting into equation (6.15), we obtain both the dispersion relation $\omega = \Omega(k) = k^3$ and the equation for the amplitude

$$\left( \frac{1}{2} a^2 \right)_t + \left( \frac{1}{2} k^2 a^2 \right)_x = 0. \quad (6.28)$$

Notice that we can rewrite this — using the group speed $c_g = \frac{d\Omega}{dk}$ — in the form

$$\left( \frac{1}{2} a^2 \right)_t + \left( \frac{1}{2} c_g(k) a^2 \right)_x = 0. \quad (6.29)$$

This form is general (e.g.: it also holds for the example in §6.2.1.) It is a conservation form: the square of the wave amplitude is conserved by the wave motion, with a flow velocity given by the group speed. Since the square of the wave amplitude can be identified with the wave energy density, this equation states the conservation of the wave energy.

### 6.2.3 Nonlinear waves

The prior examples deal with **Linear Waves**. The theory can be extended to nonlinear waves, but the situation is more complicated. Equation (6.5) still applies, except that the dispersion relation depends also on the wave amplitude $\omega = \Omega(k, a)$. An analog of equation (6.29) applies, but the conserved object has a more complicated form (and it is not the energy, but the wave action).
7 Semi-Linear 1st order PDE (problem 02)

7.1 Statement: Semi-Linear 1st order PDE (problem 02)

Discuss the problems

\[ u_x + 2xu_y = 2xu^2, \quad \text{with} \quad \begin{cases} 
(a) \ u(x, x^2) = 1 & \text{for } -1 < x < 1, \\
(b) \ u(x, x^2) = -\pi/(1 + \pi x^2) & \text{for } -1 < x < 1, \\
(c) \ u(x, x^2) = (1 - x^2/4)^{-1}/4 & \text{for } -1 < x < 1.
\] (7.1)

How many solutions exist in each case? Where are they uniquely defined?

Note that the data in these problems is prescribed along a characteristic!

7.2 Answer: Semi-Linear 1st order PDE (problem 02)

Using \( x \) to parametrize the characteristic equations for the problem in (7.1), we obtain

\[ dy/dx = 2x \quad \text{and} \quad du/dx = 2xu^2. \] (7.2)

These equations have the general solution

\[ y = \tau + x^2 \quad \text{and} \quad u = \mu \left(1 - x^2 \mu\right)^{-1}. \] (7.3)

where \( \tau \) and \( \mu \) are arbitrary constants. For \( \tau = 0 \) this yields \( u(x, x^2) = \mu \left(1 - x^2 \mu\right)^{-1} \), which is inconsistent with (a) in (7.1), consistent with (b) in (7.1) — provided that \( \mu = -\pi \), and consistent with (c) in (7.1) — provided that \( \mu = 1/4 \). Thus

**Part 1.** Problem (a) in (7.1) has no solutions.

**Part 2.** Problem (b) in (7.1) has infinitely many solutions. These are obtained by first using the first expression in (7.3) to obtain \( \tau = y - x^2 \), and then taking \( \mu \) as a function of \( \tau \) in the second expression (with \( \mu = -\pi \) for \( \tau = 0 \)). In other words

\[ u = f(y - x^2) \left(1 - x^2 f(y - x^2)\right)^{-1}, \] (7.4)

where \( f \) is any function such that \( f(0) = -\pi \). These solutions are uniquely determined along the parabola \( y = x^2 \), for \( -\infty < x < \infty \), and nowhere else!

**Part 3.** Problem (c) in (7.1) has infinitely many solutions. Obtained in the same fashion as for problem (b), these are given by

\[ u = f(y - x^2) \left(1 - x^2 f(y - x^2)\right)^{-1}, \] (7.5)

where \( f \) is any function such that \( f(0) = 1/4 \). These solutions are uniquely determined along the parabola \( y = x^2 \), for \( -2 < x < 2 \), and nowhere else!

Note: The reason that \( x \) is restricted to the interval \( -2 < x < 2 \), is that the solution blows up along the \( y = x^2 \) characteristic when \( x \) reaches \( \pm 2 \), hence it cannot be (uniquely) extended past these points.

THE END.